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On the uniqueness and structure of solutions to a coupled elliptic system

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ABSTRACT

In this paper, we consider a nonlinear elliptic system which is an extension of the single equation derived by investigating the stationary states of the nonlinear Schrödinger equation. We establish the existence and uniqueness of solutions to the Dirichlet problem on the ball. In addition, the nonexistence of the ground state solutions under certain conditions on the nonlinearities and the complete structure of different types of solutions to the shooting problem are proved.

1. Introduction

In this article, we study the solutions of the nonlinear elliptic system

$$\begin{cases} \Delta u - u + v^p = 0, \\ \Delta v - v + u^q = 0, \end{cases} \quad \text{in } \Omega, \tag{1.1}$$

where $p, q > 0$, $\Delta = \sum_{i=1}^n \partial/\partial x_i^2$ is the Laplacian operator in $\mathbf{R}^n$, $n \geqslant 1$, Ω is either a ball of $\mathbf{R}^n$ or the entire space $\mathbf{R}^n$.

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(1.1) can be viewed as a high-dimensional counterpart of a well-known equation:

$$\Delta u - u + u^p = 0, \quad \text{in } \mathbf{R}^n, \quad u(x) \to 0, \quad |x| \to \infty, \tag{1.2}$$

where $p > 0$. The uniqueness of positive solutions of (1.2) for $p \in (1, (n+2)/(n-2))$ was first proved in [20], and simplifications of proof and generalizations can be found in, for example, [3,6,21,29,30].

To cite a source of (1.2), we consider the following stationary Keller–Segel system, which is related to a model to describe the chemotactic aggregation stage of cellular slime mold (see, e.g., [11,24]):

$$\begin{cases} D_1 \Delta u - \xi \nabla(u \nabla \log v) = 0, \\ D_2 \Delta v - av + bu = 0, \end{cases} \quad \text{in } \Omega, \tag{KS}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \\ \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u(x)\, dx = \eta, \end{cases} \tag{BC}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $|\Omega|$ denotes the Lebesgue measure of Ω, D_1, D_2, a, b, ξ and η are positive constants. From (KS) and (BC), we have $u = \lambda v^{\xi/D_1}$ for some constant $\lambda > 0$. Then (KS) can be rewritten as

$$D_2 \Delta v - av + b\lambda v^{\xi/D_1} = 0 \quad \text{in } \Omega. \tag{KS*}$$

Furthermore, if we consider $\xi > D_1$ and put $p = \xi/D_1$, $d = D_2/a$ and

$$w(x) = \left(\frac{b}{a}\lambda\right)^{1/(p-1)} v(x),$$

(KS*) can be reduced to (1.2) on a bounded domain.

Another motivation for studying the scalar equation related to (1.1) arises from the nonlinear Schrödinger systems (see, e.g., [1,4,15,25])

$$\begin{cases} \Delta u - \lambda_1 u + (\mu_1 u^2 + \beta v^2) u = 0, \\ \Delta v - \lambda_2 v + (\beta u^2 + \mu_2 v^2) v = 0, \end{cases} \quad \text{in } \mathbf{R}^n, \tag{NS}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ and $\beta \in \mathbf{R}$. If we treat specific solutions $(u_0, 0)$ and $(0, v_0)$ of (NS), then u_0 and v_0 must satisfy

$$\Delta u_0 - \lambda_1 u_0 + \mu_1 u_0^3 = 0 \quad \text{and} \quad \Delta v_0 - \lambda_2 v_0 + \mu_2 v_0^3 = 0.$$

The uniqueness of solutions to systems of nonlinear partial differential equations has been an important research topic in recent years. For example, considered as a natural extension of the celebrated Lane–Emden equation, the Dirichlet problem to the Lane–Emden system

$$\begin{cases} \Delta u + v^p = 0 & \text{in } B_R(\mathbf{0}), \\ \Delta v + u^q = 0 & \text{in } B_R(\mathbf{0}), \\ u = v = 0 & \text{on } \partial B_R(\mathbf{0}), \end{cases} \tag{LE}$$

where $B_R(\mathbf{0}) \subset \mathbf{R}^n$ is the ball centered at the origin with radius R, has been investigated by many mathematicians. The existence of solutions of (LE) or more general type was established in [10,12,13,18,31,35] for $p, q \geqslant 1$, $pq \neq 1$, $n \geqslant 3$ and satisfying the subcritical condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.$$

On the other hand, if $p, q > 0$, $n \geqslant 3$ and are supercritical, *i.e.*,

$$\frac{1}{p+1} + \frac{1}{q+1} \leqslant \frac{n-2}{n},$$

the nonexistence of solutions of (LE) was obtained in [27,37]. Moreover in [13,19], the uniqueness of positive radial solutions to (LE) has been derived for $p, q > 0$ with $pq \neq 1$ as well. We remark that, in [12,13,19], the scaling argument is a key to the proof of the uniqueness due to the homogeneity of the nonlinearities in (LE). The existence of positive solutions to (1.1) for subcritical p, q was considered in [16], but the uniqueness of solutions to (1.1) is still not known. For the nonlinear Schrödinger systems (NS), a few uniqueness results were obtained recently for some special cases [23,26].

In this paper, we apply linearization techniques and the implicit function theorem to prove the uniqueness of positive solutions to (1.1). While such techniques have been widely used in proving the uniqueness of solutions of scalar equations (see [6,20,21,29,30]), the generalization to systems of equations is not straightforward due to the coupling. Here we use some ideas appearing in our earlier works [7,8].

By virtue of the method of moving planes, the Dirichlet problem of (1.1) in $B_R(\mathbf{0})$ with $(u, v) = (0, 0)$ on $\partial B_R(\mathbf{0})$ can be reduced to a corresponding system of ordinary differential equations with $u(R) = v(R) = 0$, see for example [2,5,8,14,22,36]. While the proof is standard, for the reader's convenience, we will present it in detail in Appendix A.

Throughout this paper, we investigate the radial solutions of (1.1) by considering the following initial value problem (here we extend the definition of the nonlinearities so solutions can be defined for $u, v \leqslant 0$):

$$\begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) - \big[u(r) - f(v)\big] = 0, & r > 0, \\ v''(r) + \dfrac{n-1}{r} v'(r) - \big[v(r) - g(u)\big] = 0, & r > 0, \\ u(0) = \alpha_1, \quad v(0) = \alpha_2, \\ u'(0) = 0, \quad v'(0) = 0, \end{cases} \tag{1.3}$$

where

$$f(v) = \begin{cases} v^p, & v > 0, \\ 0, & v \leqslant 0, \end{cases} \qquad g(u) = \begin{cases} u^q, & u > 0, \\ 0, & u \leqslant 0, \end{cases} \tag{1.4}$$

with $p, q > 0$; (α_1, α_2) is the initial data, $\alpha_i > 0$. The local existence and uniqueness of the solution can be proved via a standard application of the contraction mapping principle, see, for example, Lemma 2.1 in [34]. We denote the solution of (1.3) by $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ or simply $(u(r), v(r))$ when there is no confusion. The solution $(u(r), v(r))$ can be extended to $r \in [0, \infty)$ by comparison arguments (see Lemma 2.1 below).

Remark 1.1. We note that if $(u(r), v(r))$ is a solution of (1.3) satisfying $u(r_0) = 0$ and $u'(r_0) < 0$ for some $r_0 > 0$, then $u(r) < 0$ for $r > r_0$ by the maximum principle. The same result also holds for $v(r)$.

Now, we state the main result on the existence and uniqueness of positive solutions to the Dirichlet problem of (1.1).

Theorem 1.1 *(Existence and uniqueness). Let $p, q > 0$ and $pq < 1$. Then for any $R > 0$, (1.1) possesses one and only one solution $(u(x), v(x))$ satisfying*

$$\begin{cases} u(x) > 0, \quad v(x) > 0, & |x| < R, \\ u(x) = v(x) = 0, & |x| = R. \end{cases}$$

Furthermore, the corresponding initial data of such solutions at the origin, denoted by $\alpha_1(R)$ and $\alpha_2(R)$, are increasing in $R > 0$.

In addition to the existence and uniqueness of solutions to the Dirichlet problem for (1.1) stated above, we also present the nonexistence of the ground state solutions and describe a complete structure of solutions of (1.3) in terms of initial data under certain conditions on the nonlinearities. The results in this part require a stronger condition on p and q: $0 < p < 1$ and $0 < q < 1$ instead of $0 < pq < 1$. Before we introduce other main results, various types of solutions for (1.3) are introduced as follows.

Definition 1.1. A solution $(u(r), v(r))$ of (1.3) is classified into the following types:

Type C_u: $u, v > 0$ in $(0, R)$, $v(R) > u(R) = 0$ for some $R > 0$.
Type C_v: $u, v > 0$ in $(0, R)$, $u(R) > v(R) = 0$ for some $R > 0$.
Type C: $u, v > 0$ in $(0, R)$, $u(R) = v(R) = 0$ for some $R > 0$.
Type G: $u, v > 0$ in $(0, \infty)$, $(u(r), v(r)) \to (0, 0)$ as $r \to \infty$.
Type B: $u, v > 0$ in $(0, \infty)$, $(u(r), v(r)) \to (\infty, \infty)$ as $r \to \infty$.

We use the following notations for the regions of initial data corresponding to various types of solutions for (1.3):

$$\begin{cases} T = \{(1, 1)\}, \\ \Omega_C = \{(\alpha_1, \alpha_2)\colon \ (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2)) \text{ is of Type C}\}, \\ \Omega_G = \{(\alpha_1, \alpha_2)\colon \ (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2)) \text{ is of Type G}\}, \\ \Omega_B = \{(\alpha_1, \alpha_2)\colon \ (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2)) \text{ is of Type B}\}, \\ S_u = \{(\alpha_1, \alpha_2)\colon \ (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2)) \text{ is of Type C}_u\}, \\ S_v = \{(\alpha_1, \alpha_2)\colon \ (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2)) \text{ is of Type C}_v\}. \end{cases} \tag{1.5}$$

Our next result is the nonexistence of ground state solutions of (1.3).

Theorem 1.2. *If $0 < p < 1$ and $0 < q < 1$, then $\Omega_G = \emptyset$, that is, (1.3) does not possess a solution of Type G.*

With no Type G solution of (1.3), the following result completely characterizes the structure of initial data sets corresponding to solutions of remaining Types B, C, C_u and C_v.

Theorem 1.3. *Suppose that $0 < p < 1$ and $0 < q < 1$. Then:*

(a) *There exists a strictly increasing function $\gamma : (0, 1) \to (0, 1)$ satisfying*

$$\lim_{\alpha_1 \to 0^+} \gamma(\alpha_1) = 0 \quad \textit{and} \quad \lim_{\alpha_1 \to 1^-} \gamma(\alpha_1) = 1$$

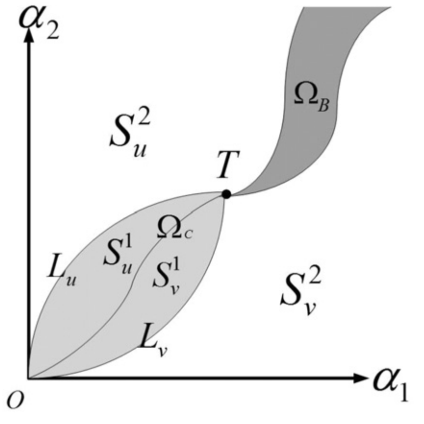

Fig. 1. Structure of solutions for (1.3).

such that

$$\Omega_C = \left\{\left(\alpha_1, \gamma(\alpha_1)\right)\colon \alpha \in (0,1)\right\}.$$

(b) *There exist strictly increasing functions* $\rho_u, \rho_v : (1,\infty) \to (1,\infty)$ *satisfying* $\rho_u(\alpha_1) \geqslant \rho_v(\alpha_1)$, *and*

$$\lim_{\alpha_1 \to 1^+} \rho_i(\alpha_1) = 1 \quad \textit{and} \quad \lim_{\alpha_1 \to \infty} \rho_i(\alpha_1) = \infty, \quad i = u, v,$$

such that

$$\Omega_B = \left\{(\alpha_1, \alpha_2)\colon \alpha_1 > 1,\ \rho_v(\alpha_1) \leqslant \alpha_2 \leqslant \rho_u(\alpha_1)\right\}.$$

(c) *Define*

$$\theta_u(\alpha_1) = \begin{cases} \gamma(\alpha_1), & 0 < \alpha_1 < 1, \\ 1, & \alpha_1 = 1, \\ \rho_u(\alpha_1), & \alpha_1 > 1, \end{cases} \qquad \theta_v(\alpha_1) = \begin{cases} \gamma(\alpha_1), & 0 < \alpha_1 < 1, \\ 1, & \alpha_1 = 1, \\ \rho_v(\alpha_1), & \alpha_1 > 1. \end{cases}$$

Then

$$S_u = \left\{(\alpha_1, \alpha_2)\colon \alpha_1 > 0,\ \theta_u(\alpha_1) < \alpha_2 < \infty\right\}$$

and

$$S_v = \left\{(\alpha_1, \alpha_2)\colon \alpha_1 > 0,\ 0 < \alpha_2 < \theta_v(\alpha_1)\right\}.$$

Geometrically the set $\Omega_C \cup T \cup \Omega_B$ separates the two open subsets S_u and S_v, see Fig. 1 for the illustration of the structure of Ω_C, Ω_B, S_u and S_v. While the set Ω_C is proved to be a curve, it is still unclear whether Ω_B is also a curve. Most results in Theorem 1.3 still hold if we replace the assumption $0 < p < 1$, $0 < q < 1$ by merely $p, q > 0$ and $0 < pq < 1$, see Remark 3.1 for details.

It is possible to obtain an even refiner description of S_u and S_v. For that purpose, we define

Definition 1.2. Any solution $(u(r), v(r))$ of Type C_u or Type C_v for (1.3) is classified further into the following types:

Type C_{uv}: $u(R_u) = v(R_v) = 0$ for some $R_v > R_u > 0$.
Type C_{vu}: $u(R_u) = v(R_v) = 0$ for some $R_u > R_v > 0$.
Type G_u: $\lim_{r\to\infty} v(r) = 0$ and $u(R) = 0$ for some $R > 0$.
Type G_v: $\lim_{r\to\infty} u(r) = 0$ and $v(R) = 0$ for some $R > 0$.
Type B_u: $\lim_{r\to\infty} v(r) = \infty$ and $u(R) = 0$ for some $R > 0$.
Type B_v: $\lim_{r\to\infty} u(r) = \infty$ and $v(R) = 0$ for some $R > 0$.

Accordingly we also define

$$S_u^1 = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{C}_{uv}\},$$
$$S_v^1 = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{C}_{vu}\},$$
$$L_u = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{G}_u\},$$
$$L_v = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{G}_v\},$$
$$S_u^2 = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{B}_u\},$$
$$S_v^2 = \{(\alpha_1,\alpha_2)\colon \big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\big) \text{ is of Type } \mathrm{B}_v\}.$$

Then the following results hold for the further partitions of S_u and S_v (see Fig. 1):

Theorem 1.4. *Suppose that $0 < p < 1$, $0 < q < 1$ and let $\gamma(\cdot)$, $\rho_u(\cdot)$ and $\rho_v(\cdot)$ be defined as in Theorem 1.3. Then there exist strictly increasing functions $\gamma_u, \gamma_v : (0,1) \to (0,1)$ satisfying*

$$\lim_{\alpha_1\to 0^+} \gamma_i(\alpha_1) = 0, \qquad \lim_{\alpha_1\to 1^-} \gamma_i(\alpha_1) = 1, \quad i = u, v,$$

and

$$\gamma_u(\alpha_1) > \gamma(\alpha_1) > \gamma_v(\alpha_1), \quad \alpha_1 \in (0,1),$$

such that

$$S_u^1 = \{(\alpha_1,\alpha_2)\colon 0 < \alpha_1 < 1,\ \gamma(\alpha_1) < \alpha_2 < \gamma_u(\alpha_1)\},$$
$$L_u = \{(\alpha_1,\alpha_2)\colon 0 < \alpha_1 < 1,\ \alpha_2 = \gamma_u(\alpha_1)\},$$
$$S_u^2 = S_u \setminus \big(S_u^1 \cup L_u\big);$$

and

$$S_v^1 = \{(\alpha_1,\alpha_2)\colon 0 < \alpha_1 < 1,\ \gamma_v(\alpha_1) < \alpha_2 < \gamma(\alpha_1)\},$$
$$L_v = \{(\alpha_1,\alpha_2)\colon 0 < \alpha_1 < 1,\ \alpha_2 = \gamma_v(\alpha_1)\},$$
$$S_v^2 = S_v \setminus \big(S_v^1 \cup L_v\big).$$

We believe that our method in this paper will be useful for some other more general nonlinear elliptic systems on bounded or unbounded domains of general dimensions. We also mention that result like Theorem 1.1 can be proved using other method even for a general bounded domain, see [9], but our main emphasis here is to present a systematic approach for radially symmetric solutions and obtain a complete structure of solutions to the shooting problem (1.3), that is information not provided in [9]. We also mention that the shooting problem like (1.3) can be solved numerically, and bifurcation diagrams showing the regions of different types of solutions as in Theorems 1.3 and 1.4 can be numerically obtained [17].

This article is organized as follows. In Section 2, some preliminaries are prepared. In Section 3, we give proofs of Theorems 1.2, 1.3 and 1.4. By investigating the corresponding linearized systems and employing the implicit function theorem, a complete demonstration of Theorem 1.1 will be offered in Section 4. Finally, Section 5 is devoted to some generalizations which can be done by similar methods, and after that, a detailed proof of the symmetry of solutions to the Dirichlet problem is provided in Appendix A.

2. Preliminaries

First we show the global existence of the solutions of (1.3).

Lemma 2.1. *For any $\alpha_1, \alpha_2 > 0$, the solution $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ of (1.3) is defined for all $r \in [0,\infty)$.*

Proof. Suppose that $(u(r), v(r)) = (u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ exists on an interval $[0, R)$ for some $R > 0$, and $w(r)$ is the positive entire solution of

$$\begin{cases} w'' + \dfrac{n-1}{r}w' - w = 0, \quad r > 0, \\ w(0) = c, \quad w'(0) = 0, \end{cases}$$

where $c > \alpha_1$ is a fixed constant. We define $z_1(r) = w(r) - u(r)$. Then $z_1(r)$ satisfies

$$\begin{cases} z_1'' + \dfrac{n-1}{r}z_1' - z_1 = f(v) \geqslant 0, \quad 0 < r < R, \\ z_1(0) = c - \alpha_1, \quad z_1'(0) = 0. \end{cases}$$

Then the maximum principle implies that

$$\lim_{r\to R^-} z_1(r) = \lim_{n\to\infty} z_1(R-1/n) = \lim_{n\to\infty}\Big[\max_{r\in[0,R-1/n]} z_1(r)\Big] \geqslant c - \alpha_1 > 0.$$

Hence $u(r)$ cannot blow up to $+\infty$ at $r = R$. Similarly, $v(r)$ cannot blow up to $+\infty$ at $r = R$ as well.

If $u(r)$ and $v(r)$ vanish at some points (not necessary to be the same), then from Remark 1.1, we see

$$u'' + \frac{n-1}{r}u' - u = 0, \qquad v'' + \frac{n-1}{r}v' - v = 0, \quad r \in [R_0, R),$$

for some $R_0 \in (0, R)$. Then by standard arguments, $u(r)$ and $v(r)$ cannot blow up to $-\infty$ at $r = R$. If $u(r) > 0$ on $[0, R)$, $v(r) > 0$ on $[0, R_1)$ and $v(r) < 0$ on (R_1, R) for some $R_1 \in (0, R)$, then $u(r)$ is bounded on $[0, R)$. Let $z_2(r) = -e^{kr} - v(r)$ for $r \in [R_1, R)$, where $k > 0$. Then $z_2(r)$ satisfies

$$\begin{cases} z_2'' + \dfrac{n-1}{r}z_2' - z_2 = -e^{kr}\Big[k^2 - 1 + \dfrac{(n-1)k}{r}\Big] + u^q, \quad R_1 \leqslant r < R, \\ z_2(R_1) = -e^{kR_1}, \quad z_2'(R_1) = -ke^{kR_1} - v'(R_1), \end{cases}$$

which implies $z_2(r) < 0$ on $[R_1, R)$ if k is sufficiently large. Therefore $v(r)$ cannot blow up to $-\infty$ at $r = R$ for this case. This also holds if we exchange the roles of $u(r)$ and $v(r)$. This completes the proof of Lemma 2.1. □

Next we introduce the corresponding linearized systems for (1.3). Let $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ be a solution of (1.3) and put, for $i=1,2$,

$$\begin{cases} \varphi_i(r)=\varphi_i(r;\alpha_1,\alpha_2)=\dfrac{\partial u(r;\alpha_1,\alpha_2)}{\partial\alpha_i}, \\ \psi_i(r)=\psi_i(r;\alpha_1,\alpha_2)=\dfrac{\partial v(r;\alpha_1,\alpha_2)}{\partial\alpha_i}. \end{cases} \tag{2.1}$$

Then (φ_i,ψ_i), $i=1,2$, satisfy the following linearized systems

$$\begin{cases} {\varphi_i}''(r)+\dfrac{n-1}{r}{\varphi_i}'(r)-\left[\varphi_i(r)-\dfrac{df}{dv}(v)\psi_i(r)\right]=0, \quad r>0, \\ {\psi_i}''(r)+\dfrac{n-1}{r}{\psi_i}'(r)-\left[\psi_i(r)-\dfrac{dg}{du}(u)\varphi_i(r)\right]=0, \quad r>0, \\ \varphi_1(0)=1, \quad \varphi_2(0)=0; \quad \psi_1(0)=0, \quad \psi_2(0)=1, \\ \varphi_i'(0)=\psi_i'(0)=0. \end{cases} \tag{2.2}$$

The following lemma gives the monotone properties of φ_i and ψ_i $(i=1,2)$.

Lemma 2.2. *Suppose that $(u(r),v(r))=(u(r;\alpha_1,\alpha_2),v(r;\alpha_1,\alpha_2))$ is a solution of (1.3). Then for any $r>0$,*

$$\begin{cases} \varphi_1(r)>0, \quad \varphi_1'(r)>0; \quad \psi_1(r)<0, \quad \psi_1'(r)<0; \\ \varphi_2(r)<0, \quad \varphi_2'(r)<0; \quad \psi_2(r)>0, \quad \psi_2'(r)>0. \end{cases} \tag{2.3}$$

Proof. Since $\varphi_1(0)=1$, $\psi_1(0)=0$ and by (2.2), we have

$$\begin{aligned} r^{n-1}\varphi_1'(r) &= \int_0^r s^{n-1}\big(\varphi_1(s)-pv^{p-1}(s)\psi_1(s)\big)\,ds \\ &\geqslant c_1\int_0^r s^{n-1}\,ds=\frac{c_1r^n}{n} \quad \text{near } r=0, \end{aligned} \tag{2.4}$$

for some $c_1>0$, which implies

$$\varphi_1(r)>0, \qquad \varphi_1'(r)>0 \quad \text{near } r=0.$$

Similarly we have

$$\begin{aligned} r^{n-1}\psi_1'(r) &= \int_0^r s^{n-1}\big(\psi_1(s)-qu^{q-1}(s)\varphi_1(s)\big)\,ds \\ &\leqslant -c_2\int_0^r s^{n-1}\,ds=-\frac{c_2r^n}{n} \quad \text{near } r=0, \end{aligned} \tag{2.5}$$

for some $c_2>0$, and hence

$$\psi_1(r)<0, \qquad \psi_1'(r)<0 \quad \text{near } r=0.$$

Since the inequalities in (2.4) and (2.5) hold whenever $\varphi_1(r) > 0$ and $\psi_1(r) < 0$, we deduce that $\varphi_1(r) > 0$, $\varphi_1'(r) > 0$ and $\psi_1(r) < 0$, $\psi_1'(r) < 0$ for all $r > 0$. The situations for $\varphi_2(r)$ and $\psi_2(r)$ are similar, and we complete this proof. □

To present our results more concisely, we next show that the classification in (1.5) exhausts all possible situations.

Lemma 2.3. *Let $p, q > 0$ and $pq < 1$. If $(u(r), v(r)) = (u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ is a positive entire solution of* (1.3) *on $[0,\infty)$, then $(u(r), v(r))$ is only possibly of Type G, Type B or the equilibrium solution $(u(r), v(r)) \equiv (1,1)$. Moreover,*

$$S_u \cup \Omega_G \cup \Omega_C \cup T \cup \Omega_B \cup S_v = (0,\infty)\times(0,\infty).$$

Proof. From Lemma 2.1, $(u(r), v(r))$ is defined on $[0,\infty)$. We assume that $u(r) > 0$, $v(r) > 0$ for all $r > 0$, and it does not belong to Type G, Type B, or $(1,1)$. We claim that there exists $r_0 \geqslant 0$ such that either

$$\begin{cases} u'(r_0) \geqslant 0, & u(r_0) - v^p(r_0) > 0,\\ v'(r_0) \leqslant 0, & v(r_0) - u^q(r_0) < 0,\end{cases}$$

or

$$\begin{cases} u'(r_0) \leqslant 0, & u(r_0) - v^p(r_0) < 0,\\ v'(r_0) \geqslant 0, & v(r_0) - u^q(r_0) > 0.\end{cases}$$

Proof of Claim. We prove it for all possible (α_1,α_2). First $\alpha_1 - \alpha_2^p \neq 0$ or $\alpha_2 - \alpha_1^q \neq 0$ since, otherwise, $\alpha_1 = \alpha_2 = 1$ which implies $(u(r), v(r)) \equiv (1,1)$.

Case 1. If $(\alpha_1 - \alpha_2^p)(\alpha_2 - \alpha_1^q) < 0$, then we may take $r_0 = 0$.

Case 2. If $(\alpha_1 - \alpha_2^p)(\alpha_2 - \alpha_1^q) = 0$, then from comment above, at least one of $\alpha_1 - \alpha_2^p$ and $\alpha_2 - \alpha_1^q$ is not zero. Assume that $\alpha_1 - \alpha_2^p > 0$ and $\alpha_2 - \alpha_1^q = 0$. Then (1.3) implies $u''(0) > 0$, $v''(0) = 0$ and $u(r) - v^p(r) > 0$, $u'(r) > 0$ near $r = 0$. Hence, we have

$$\big(v - u^q\big)(0) = 0, \qquad \big(v - u^q\big)'(0) = 0 \quad \text{and} \quad \big(v - u^q\big)''(0) < 0,$$

which implies that

$$v(r) - u^q(r) < 0 \quad \text{near } r = 0,$$

and consequently $v'(r) < 0$ near $r = 0$. Thus the claim holds if we take $r_0 > 0$ to be small. The other cases of α_1 and α_2 satisfying $(\alpha_1 - \alpha_2^p)(\alpha_2 - \alpha_1^q) = 0$ can be proved similarly.

Case 3. Suppose that $(\alpha_1 - \alpha_2^p)(\alpha_2 - \alpha_1^q) > 0$. First we consider the case that $\alpha_1 - \alpha_2^p > 0$ and $\alpha_2 - \alpha_1^q > 0$, then $\alpha_1, \alpha_2 > 1$ and

$$u(r) - v^p(r) > 0, \qquad v(r) - u^q(r) > 0 \quad \text{near } r = 0.$$

Hence $u'(r) > 0$ and $v'(r) > 0$ for r near 0. If $u'(r) > 0$ and $v'(r) > 0$ for $r \in [0,\infty)$, then both $u(r)$ and $v(r)$ increase to some finite constants u_∞ and v_∞ which are greater than 1 as $r \to \infty$ respectively, and (u_∞, v_∞) satisfies $u_\infty - v_\infty^p = 0$, $v_\infty - u_\infty^q = 0$. That is a contradiction since $(1,1)$ is the only point in the first quadrant so that $u - v^p = 0$ and $v - u^q = 0$.

Hence either $u'(r)$ or $v'(r)$ must change sign. We define

$$r_1 = \sup\{r\colon\ u'(s) > 0,\ v'(s) > 0 \text{ for all } s \in (0,r)\} < \infty.$$

Without loss of generality, we assume that $v'(r_1) = 0$ and $u'(r_1) \geqslant 0$, then $v''(r_1) \leqslant 0$ and $(v - u^q)(r_1) \leqslant 0$. This implies that $(u - v^p)(r_1) > 0$. Otherwise $(u - v^p)(r_1) \leqslant 0$ and $(v - u^q)(r_1) \leqslant 0$ imply that $u(r_1) \leqslant 1$, which contradicts with $\alpha_1 > 1$ and $u'(r) > 0$ in $(0, r_1)$. If $(v - u^q)(r_1) < 0$, then $r_0 = r_1$ fulfills the requirements in the claim. If $(v - u^q)(r_1) = 0$, then we notice that

$$(v - u^q)(r_1) = 0 \quad \text{and} \quad (v - u^q)'(r_1) \leqslant 0,$$

and if $(v - u^q)'(r_1) = 0$, then $(v - u^q)''(r_1) < 0$ as $(u - v^p)(r_1) > 0$. Hence $(v - u^q)(r) < 0$, $v'(r) < 0$ and $u'(r) > 0$ for $r \in (r_1, r_1 + \delta)$ for some small $\delta > 0$, and we choose $r_0 \in (r_1, r_1 + \delta)$ which will satisfy the requirements in the claim. This completes the proof of the claim for $\alpha_1 - \alpha_2^p > 0$ and $\alpha_2 - \alpha_1^q > 0$. The case for $\alpha_1 - \alpha_2^p < 0$ and $\alpha_2 - \alpha_1^q < 0$ is similar, and we omit the details. Hence the claim is proved. □

By combining the above claim and (1.3), we conclude that either

$$u'(r) > 0, \qquad v'(r) < 0, \quad r > r_0,$$

or

$$u'(r) < 0, \qquad v'(r) > 0, \quad r > r_0,$$

and hence either $u(r) \to \infty$ or $v(r) \to \infty$ as $r \to \infty$. Assume $(u(r), v(r)) \to (\infty, c)$ as $r \to \infty$ for some $0 \leqslant c < \infty$. By (1.3) again, we obtain

$$\left(r^{n-1} v'(r)\right)' = r^{n-1}\left(v(r) - u^q(r)\right) < -Cr^{n-1} \quad \text{for large } r,$$

for some $C > 0$. This yields a contradiction since $v(r)$ is positive on $[0, \infty)$. We complete the proof of this lemma. □

From the proof of Lemma 2.3, the regions S_u and S_v are nonempty.

Corollary 2.1. *Let S_u and S_v be defined in* (1.5). *Then*:

(a) *For any given $\alpha_1 > 0$ (resp., $\alpha_2 > 0$), $(\alpha_1, \alpha_2) \in S_u$ (resp., $(\alpha_1, \alpha_2) \in S_v$) for large α_2 (resp., for large α_1).*
(b) *For any given $\alpha_1 > 0$ (resp., $\alpha_2 > 0$), $(\alpha_1, \alpha_2) \in S_v$ (resp., $(\alpha_1, \alpha_2) \in S_u$) for small α_2 (resp., for small α_1).*

Remark 2.1. By the arguments described in the proof of Lemma 2.3, we see that if $pq < 1$ and $\alpha_1, \alpha_2 \geqslant 1$, then $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ cannot be a solution of Type C or Type G. In addition, if $(\alpha_1, \alpha_2) \in \Omega_B$, then $\alpha_1, \alpha_2 > 1$.

3. Nonexistence and structure of solutions

For the proof of Theorem 1.2, the following lemma is a key step.

Lemma 3.1. *Suppose that $0 < p < 1$ and $0 < q < 1$. If $(u(r), v(r)) = (u(r; \alpha_1, \alpha_2),\ v(r; \alpha_1, \alpha_2))$ is a solution of Type G, then*

$$u(r) - v^p(r) < 0, \qquad v(r) - u^q(r) < 0, \quad r \in [0, \infty). \tag{3.1}$$

Furthermore, $u'(r) < 0$ *and* $v'(r) < 0$ *on* $(0, \infty)$*.* (3.1) *also holds for solution* $(u(r), v(r))$ *of Type C on the interval where* $u(r)$ *and* $v(r)$ *are positive.*

Proof. Since $(u(r), v(r))$ is of Type G, we have $\alpha_1, \alpha_2 < 1$, $\alpha_1 - \alpha_2^p < 0$ and $\alpha_2 - \alpha_1^q < 0$ by the proof of Lemma 2.3. Then

$$u(r) - v^p(r) < 0, \qquad v(r) - u^q(r) < 0 \quad \text{near } r = 0.$$

In fact, from the proof of Lemma 2.3 case 3, if $(\alpha_1, \alpha_2) \in \Omega_G$, either $u'(r) < 0$, $v'(r) < 0$ for all $r > 0$ or the claim in the proof of Lemma 2.3 holds which implies that $(\alpha_1, \alpha_2) \notin \Omega_G$. Hence $u'(r) < 0$, $v'(r) < 0$ for all $r > 0$, and consequently $0 < u(r) < \alpha_1 < 1$ and $0 < v(r) < \alpha_2 < 1$ for all $r > 0$.

We prove the lemma by contradiction. Suppose (3.1) is not true, then

$$r_0 = \sup\bigl\{r > 0\colon\ u(s) - v^p(s) < 0,\ v(s) - u^q(s) < 0 \text{ for all } s \in (0, r)\bigr\} < \infty.$$

Without loss of generality, suppose that $u(r_0) - v^p(r_0) = 0$. For $r \in (0, r_0]$, we have

$$r^{n-1}u'(r) = \int_0^r s^{n-1}\bigl(u(s) - v^p(s)\bigr)\,ds < 0,$$

and

$$r^{n-1}v'(r) = \int_0^r s^{n-1}\bigl(v(s) - u^q(s)\bigr)\,ds < 0.$$

Hence $0 < u(r_0) < \alpha_1 < 1$ and $0 < v(r_0) < \alpha_2 < 1$, and we must have $v(r_0) - u^q(r_0) < 0$. Clearly $(u - v^p)'(r_0) \geqslant 0$. If $(u - v^p)'(r_0) = 0$, then $(u - v^p)''(r_0) = -pv^{p-1}(r_0)[v(r_0) - u^q(r_0)] - p(p-1)v^{p-2}(r_0)[v'(r_0)]^2 > 0$ since $p < 1$. Hence $(u - v^p)(r) > 0$ for $(r_0, r_0 + \delta)$ for some small $\delta > 0$, and we can also assume that $(v - u^q)(r) < 0$ for $(r_0, r_0 + \delta)$.

We claim that $(u - v^p)(r) > 0$ and $(v - u^q)(r) < 0$ for all $r > r_0$. If not, define

$$r_1 = \sup\bigl\{r > r_0\colon\ u(s) - v^p(s) > 0,\ v(s) - u^q(s) < 0 \text{ for all } s \in (r_0, r)\bigr\} < \infty.$$

At $r = r_1$, either $(u - v^p)(r_1) = 0$ or $(v - u^q)(r_1) = 0$. Suppose that $(u - v^p)(r_1) = 0$, then for $x \in \{r_0 < |x| < r_1\}$,

$$\Delta\bigl(u - v^p\bigr) = u - v^p - pv^{p-1}\bigl(v - u^q\bigr) - p(p-1)v^{p-2}|\nabla v|^2 > 0, \tag{3.2}$$

and $u - v^p = 0$ when $|x| = r_0$ or $|x| = r_1$. Hence from the maximum principle, $(u - v^p)(x) < 0$ for $x \in \{r_0 < |x| < r_1\}$, that is a contradiction. Thus $(u - v^p)(r_1) > 0$, then we must have $(v - u^q)(r_1) = 0$. But that will violate the fact $0 < v(r_1), v(r_1) < 1$. This proves the claim.

Now we have proved that $(u - v^p)(r) > 0$ and $(v - u^q)(r) < 0$ for all $r > r_0$. On the other hand, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 0$. Hence $(u - v^p)(r)$ achieves the maximum in $[r_0, \infty)$ at some $r = r_2 > r_0$. But Eq. (3.2) holds at $|x| = r_2$, that is a contradiction. Therefore (3.1) holds and $u'(r) < 0$, $v'(r) < 0$ on $(0, \infty)$. The case for solutions of Type C is similar, and we omit the details. Hence Lemma 3.1 is proved. $\square$

By virtue of Lemma 3.1, the proof of Theorem 1.2 can be given now.

Proof of Theorem 1.2. Suppose that $(u(r), v(r))$ is a solution of Type G. Then from (1.3) and Lemma 3.1,

$$u'(r) \leqslant -c_1 r^{1-n}, \quad r \geqslant 1, \tag{3.3}$$

where

$$c_1 = -\int_0^1 s^{n-1}\big(u(s) - v^p(s)\big)\,ds > 0.$$

If $n = 1$, then (3.3) leads to a contradiction since $u'(r) \to 0$ as $r \to \infty$. If $n = 2$, then by integrating both sides in (3.3) from r to r^2 for $r > 1$, we have

$$u(r) \geqslant u\big(r^2\big) + c_1 \log r, \quad r > 1,$$

which contradicts with the fact that $u(r)$ must be small for r near ∞. Hence Theorem 1.2 holds for $n = 1, 2$.

If $n \geqslant 3$, then from (3.3) we have

$$u(r) \geqslant u(r_1) + \frac{c_1}{n-2}\big(r^{2-n} - r_1^{2-n}\big), \quad 1 \leqslant r < r_1.$$

Letting $r_1 \to \infty$ in the above inequality, we get

$$u(r) \geqslant C_1 r^{2-n}, \quad r \geqslant 1,$$

where $C_1 = c_1/(n-2)$. The case for $v(r)$ is similar, and hence we obtain

$$cr^{2-n} \leqslant u(r), \qquad v(r) < 1 \quad \text{for large } r, \tag{3.4}$$

with some $c > 0$.

Without loss of generality, we assume that $q \geqslant p$. Then (1.3) implies

$$\begin{aligned}\big\{r^{n-1}\big(u(r)+v(r)\big)'\big\}' &= r^{n-1}\big\{u(r)+v(r) - \big(u^q(r)+v^p(r)\big)\big\}\\ &\leqslant r^{n-1}\big\{u(r)+v(r) - \big(u^q(r)+v^q(r)\big)\big\}\\ &\leqslant r^{n-1}\big\{u(r)+v(r) - \big(u(r)+v(r)\big)^q\big\}\\ &\leqslant -\frac{r^{n-1}}{2}\big(u(r)+v(r)\big)^q \quad \text{for large } r,\end{aligned}$$

since $p \leqslant q < 1$ and $(u(r), v(r)) \to (0,0)$ as $r \to \infty$. By using (3.4) and integrating the above inequality, we obtain $u(r) + v(r) \geqslant Cr^{2+(2-n)q}$ for large r. And repeating this process inductively with new estimates of $u + v$, we obtain that

$$u(r) + v(r) \geqslant Cr^{(2-n)q^k + 2\sum_{i=0}^{k-1} q^i} \quad \text{for large } r,$$

for any $k \in \mathbf{N}$ and some $C > 0$. This is impossible because $q < 1$ which deduces

$$(2-n)q^k + 2\sum_{i=0}^{k-1} q^i > 0 \quad \text{for large } k.$$

This completes the proof of Theorem 1.2. □

In order to prove Theorems 1.3 and 1.4, we derive a geometric property of the regions defined in (1.5) which is an essential element to clarify the structure of solutions on (α_1, α_2)-plane.

Proposition 3.1. *Let $p, q > 0$. If $(\alpha_1^*, \alpha_2^*) \in S_u \cup \Omega_C \cup \Omega_G$, then $(\alpha_1, \alpha_2^*) \in S_u$ for any $0 < \alpha_1 < \alpha_1^*$, and $(\alpha_1^*, \alpha_2) \in S_u$ for any $\alpha_2 > \alpha_2^*$; similarly if $(\alpha_1^*, \alpha_2^*) \in S_v \cup \Omega_C \cup \Omega_G$, then $(\alpha_1, \alpha_2^*) \in S_v$ for any $\alpha_1 > \alpha_1^*$, and $(\alpha_1^*, \alpha_2) \in S_v$ any for $0 < \alpha_2 < \alpha_2^*$.*

Proof. We only prove the results involving S_u. The others involving S_v are similar, and we omit the details.

First, if $(\alpha_1^*, \alpha_2^*) \in \Omega_C$, then $u(r; \alpha_1^*, \alpha_2^*) > 0$, $v(r; \alpha_1^*, \alpha_2^*) > 0$ for $r \in [0, R^*)$ and $u(R^*; \alpha_1^*, \alpha_2^*) = v(R^*; \alpha_1^*, \alpha_2^*) = 0$ for some $R^* > 0$. We prove that for $0 < \alpha_1 < \alpha_1^*$, $(\alpha_1, \alpha_2^*) \in S_u$. Define

$$R = \sup\{r > 0\colon\ u(r; \alpha_1, \alpha_2^*) > 0,\ v(r; \alpha_1, \alpha_2^*) > 0\} \tag{3.5}$$

and let

$$\phi(r) = u(r; \alpha_1^*, \alpha_2^*) - u(r; \alpha_1, \alpha_2^*), \qquad \psi(r) = v(r; \alpha_1^*, \alpha_2^*) - v(r; \alpha_1, \alpha_2^*). \tag{3.6}$$

Then (ϕ, ψ) satisfies

$$\begin{cases} \phi'' + \dfrac{n-1}{r}\phi' - \phi + p v_*^{p-1}\psi = 0, & r \in (0, R_1), \\ \psi'' + \dfrac{n-1}{r}\psi' - \psi + q u_*^{q-1}\phi = 0, & r \in (0, R_1), \\ \phi(0) = \alpha_1^* - \alpha_1 > 0, \quad \psi(0) = 0, \\ \phi'(0) = \psi'(0) = 0, \end{cases} \tag{3.7}$$

where $R_1 = \min\{R, R^*\}$, $u_*(r) = \xi(r)u(r; \alpha_1^*, \alpha_2^*) + (1 - \xi(r))u(r; \alpha_1, \alpha_2^*) > 0$ and $v_*(r) = \zeta(r)v(r; \alpha_1^*, \alpha_2^*) + (1 - \zeta(r))v(r; \alpha_1, \alpha_2^*) > 0$ for some functions $\xi, \zeta : (0, R_1) \to [0, 1]$. By the same arguments in the proof of Lemma 2.2, we have that $\phi(r) > 0$ and $\psi(r) < 0$ for $r \in (0, R_1]$. In particular, $\phi(R_1) > 0$ and $\psi(R_1) < 0$ which implies that $u(R_1; \alpha_1^*, \alpha_2^*) > u(R_1; \alpha_1, \alpha_2^*)$ and $v(R_1; \alpha_1^*, \alpha_2^*) < v(R_1; \alpha_1, \alpha_2^*)$. Since $R_1 \leqslant R$, $u(R_1; \alpha_1, \alpha_2^*) \geqslant 0$ and then $R_1 = R < R^*$. Hence $u(R; \alpha_1, \alpha_2^*) = 0$ and $v(R; \alpha_1, \alpha_2^*) > 0$, *i.e.*, $(\alpha_1, \alpha_2^*) \in S_u$.

Next if $(\alpha_1^*, \alpha_2^*) \in S_u$, then there exists a finite $R^* > 0$ such that $u(r; \alpha_1^*, \alpha_2^*) > 0$, $v(r; \alpha_1^*, \alpha_2^*) > 0$ for $r \in [0, R^*)$ and $v(R^*; \alpha_1^*, \alpha_2^*) > u(R^*; \alpha_1^*, \alpha_2^*) = 0$. Hence for $0 < \alpha_1 < \alpha_1^*$, we can also conclude that $(\alpha_1, \alpha_2^*) \in S_u$ by following the same arguments as above.

Finally if $(\alpha_1^*, \alpha_2^*) \in \Omega_G$, then $R^* = \infty$. Let $0 < \alpha_1 < \alpha_1^*$, and we prove that $(\alpha_1, \alpha_2^*) \in S_u$. If $R = \infty$ (R is defined in (3.5)), then $(\alpha_1, \alpha_2^*) \in \Omega_G$ too. The functions (ϕ, ψ) defined in (3.6) and (3.7) are also defined on $(0, \infty)$, and $\phi(r) > 0$ and $\psi(r) < 0$ for $r \in (0, \infty)$. Since $u(r; \alpha_1^*, \alpha_2^*) \to 0$ and $u(r; \alpha_1, \alpha_2^*) \to 0$ as $r \to \infty$, then $\phi(r) \to 0$ as $r \to \infty$. Hence ϕ satisfies

$$\begin{cases} \Delta\phi - \phi = -p v_*^{p-1}\psi \geqslant 0, & x \in \mathbf{R}^n, \\ \phi(x) > 0, \quad x \in \mathbf{R}^n, & \lim\limits_{|x| \to \infty} \phi(x) = 0. \end{cases} \tag{3.8}$$

Then $\phi(x)$ must achieve its positive maximum value at $x_0 \in \mathbf{R}^n$ and $\Delta\phi(x_0) \leqslant 0$, which contradicts with $\Delta\phi - \phi \geqslant 0$ in $\mathbf{R}^n$. Hence $R = \infty$ is impossible and R is finite, then similar to the proof above, we have $v(R; \alpha_1, \alpha_2^*) > 0$ then $(\alpha_1, \alpha_2^*) \in S_u$. Other cases involving S_u can be shown similarly. This completes the proof of Proposition 3.1. $\square$

Now we are in the position to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Define

$$\gamma_1(\alpha_1)=\inf\{\alpha_2\colon (\alpha_1,\alpha_2)\in S_u\},\qquad \gamma_2(\alpha_1)=\sup\{\alpha_2\colon (\alpha_1,\alpha_2)\in S_v\},\quad \alpha_1>0.$$

Then from Corollary 2.1, Proposition 3.1 and Theorem 1.2, we obtain that

$$\Omega_C\cup T\cup\Omega_B=\{(\alpha_1,\alpha_2)\colon \alpha_1>0,\ \gamma_2(\alpha_1)\leqslant\alpha_2\leqslant\gamma_1(\alpha_1)\},$$

which is a nonempty simply connected closed subset of $\mathbf{R}^2_+$, and

$$S_u=\{(\alpha_1,\alpha_2)\colon \alpha_1>0,\ \alpha_2>\gamma_1(\alpha_1)\},\qquad S_v=\{(\alpha_1,\alpha_2)\colon \alpha_1>0,\ \alpha_2<\gamma_2(\alpha_1)\},$$

which are nonempty simply connected open subsets of $\mathbf{R}^2_+$. Hence all statements in Theorem 1.3 hold except proving $\gamma_1(\alpha_1)=\gamma_2(\alpha_1)=\gamma(\alpha_1)$ for $\alpha_1\in(0,1)$.

To prove that, we first note that $\gamma_1(1)=\gamma_2(1)=1$. Indeed, if $\alpha_1=1$ and $0<\alpha_2<1$, then $\alpha_1-\alpha_2^p>0$ and $\alpha_2-\alpha_1^q<0$ which implies that $u'(r)>0$ and $v'(r)<0$ for all $r>0$. Hence $(1,\alpha_2)\in S_v$ by Lemma 2.3. Similarly, if $\alpha_2>1$, then $(1,\alpha_2)\in S_u$. Moreover, by continuity, $\lim_{\alpha_1\to 0^+}\gamma_1(\alpha_1)=\lim_{\alpha_1\to 0^+}\gamma_2(\alpha_1)=0$, and from Remark 2.1, we obtain

$$\Omega_C=\{(\alpha_1,\alpha_2)\colon \gamma_2(\alpha_1)\leqslant\alpha_2\leqslant\gamma_1(\alpha_1),\ \alpha_1\in(0,1)\}.$$

From Proposition 3.1, it is easy to see that $\gamma_1=\gamma_2$ which is strictly increasing on $(0,1)$. So results in Theorem 1.3 hold if we define $\gamma(\alpha_1)=\gamma_1(\alpha_1)=\gamma_2(\alpha_1)$, and $\rho_u(\alpha_1)=\gamma_1(\alpha_1)$, $\rho_v(\alpha_1)=\gamma_2(\alpha_1)$ for $\alpha_1>1$. □

Proof of Theorem 1.4. We first note that $(u(r;\alpha_1,\gamma(\alpha_1)),v(r;\alpha_1,\gamma(\alpha_1)))$ is of Type C for $\alpha_1\in(0,1)$ by Theorem 1.3(a), where $\gamma(\alpha_1)$ is defined in Theorem 1.3(a). For $\alpha_1\in(0,1)$, we have $(\alpha_1,\alpha_2)\in S_u$ for $\alpha_2>\gamma(\alpha_1)$ from Proposition 3.1, and $R(\alpha_1,\alpha_2)<R(\alpha_1,\gamma(\alpha_1))$ where $R=R(\alpha_1,\alpha_2)$ is defined in Definition 1.1 related to Type C_u. From the continuity of solutions with respect to initial data and Lemma 2.2, S_u^1 is nonempty. On the other hand, for fixed $\alpha_1\in(0,1)$, for large enough α_2, $(\alpha_1,\alpha_2)\in S_u^2$ since $\alpha_2-\alpha_1^q>0$ and $v'(0)=0$, then $v'(r)>0$ for all $r>0$. Moreover, $(1,\alpha_2)\in S_u^2$ for all $\alpha_2>1$ since $u'(r;1,\alpha_2)>0$ for all $r>0$. If $\alpha_1>1$ and $\alpha_2>\rho_u(\alpha_1)$, where $\rho_u(\alpha_1)$ is defined in Theorem 1.3(b), then Lemma 2.2 implies that $(\alpha_1,\alpha_2)\in S_u^2$. From Lemma 2.2 and arguments similar to the one in the proof of Proposition 3.1, S_u^1 and S_u^2 are nonempty simply connected open subsets of $(0,\infty)\times(0,\infty)$. The case of S_v^i is similar.

By using similar arguments as the ones in the proof of Proposition 3.1, one can show that if $(\alpha_1,\alpha_2^*)\in S_u^1$, then for any $\gamma(\alpha_1)<\alpha_2<\alpha_2^*$, $(\alpha_1,\alpha_2)\in S_u^1$; on the other hand, if $(\alpha_1,\alpha_2^*)\in S_u^2$, then for any $\alpha_2>\alpha_2^*$, $(\alpha_1,\alpha_2)\in S_u^2$. In fact for such a fixed α_1, as long as $\alpha_2>\gamma(\alpha_1)$, $u(r;\alpha_1,\alpha_2)<0$ for large enough $r>r_0(\alpha_2)$, and $r_0(\alpha_2)$ can be chosen as a fixed $r_0>0$ for $\alpha_2\geqslant\gamma(\alpha_1)+\delta$ for some small $\delta>0$ (this can be seen from the proof of Proposition 3.1). Then $v(r;\alpha_1,\alpha_2)$ satisfies

$$\left(r^{n-1}v'\right)'-r^{n-1}v=0,\quad r\in[r_0,\infty),\tag{3.9}$$

where $\alpha_2\geqslant\gamma(\alpha_1)+\delta$. From the theory of second order linear ODEs, (3.9) has only one solution satisfying $\lim_{r\to\infty}v(r)=0$, and all other solutions satisfy $\lim_{r\to\infty}v(r)=\pm\infty$. Hence we see that $(u(r;\alpha_1,\alpha_2),v(r;\alpha_1,\alpha_2))$ is of Type G_u if (α_1,α_2) lies on $S_u\setminus(S_u^1\cup S_u^2)$. Furthermore, by applying the same arguments as in the proof of Proposition 3.1, we deduce that $S_u\setminus(S_u^1\cup S_u^2)$ is the graph of some curve $(\alpha_1,\gamma_u(\alpha_1))$ over $(0,1)$ such that $\gamma_u(\alpha_1)$ satisfies the properties listed in Theorem 1.4. The proof of S_v is similar and hence the proof of Theorem 1.4 is completed. □

Remark 3.1. We note that Proposition 3.1 holds for all $p, q > 0$. In the case of $p, q > 0$ and $pq < 1$, the arguments in the proof of Theorem 1.3 yield

$$\Omega_C \cup T \cup \Omega_B \cup \Omega_G = \big\{(\alpha_1, \alpha_2)\colon\ \alpha_1 > 0,\ \gamma_2(\alpha_1) \leqslant \alpha_2 \leqslant \gamma_1(\alpha_1)\big\},$$

and parts (b) and (c) in Theorem 1.3 still hold. On the other hand, Proposition 3.1 implies that part (a) in Theorem 1.3 now becomes

$$\Omega_C \cup \Omega_G = \big\{\big(\alpha_1, \gamma(\alpha_1)\big)\colon\ \alpha \in (0, 1)\big\}.$$

Theorem 1.2 excludes the possibility of ground states when $p < 1$, $q < 1$, and it is unclear whether the ground state exists when only $pq < 1$ holds.

4. Existence and uniqueness

This section is devoted to the existence and uniqueness of solutions to the Dirichlet problem (1.1) in the ball with zero boundary condition:

$$\begin{cases} \Delta u - u + v^p = 0 & \text{in } B_R(\mathbf{0}), \\ \Delta v - v + u^q = 0 & \text{in } B_R(\mathbf{0}), \\ u, v > 0 & \text{in } B_R(\mathbf{0}), \\ u = v = 0 & \text{on } \partial B_R(\mathbf{0}). \end{cases} \tag{4.1}$$

From Theorem A in Appendix A, we only need to deal with the radially symmetric solutions of (4.1). That is, for any given $R > 0$, we consider

$$(1.3) \quad \text{with } u(r), v(r) > 0 \text{ for } r \in [0, R) \quad \text{and} \quad u(R) = v(R) = 0. \tag{4.2}$$

To attain our uniqueness result, we introduce the following functions:

$$\begin{cases} \Phi(r; \alpha_1, \alpha_2, C) = \varphi_1(r; \alpha_1, \alpha_2) + C\varphi_2(r; \alpha_1, \alpha_2), & r > 0, \\ \Psi(r; \alpha_1, \alpha_2, C) = \psi_1(r; \alpha_1, \alpha_2) + C\psi_2(r; \alpha_1, \alpha_2), & r > 0, \end{cases} \tag{4.3}$$

and

$$\begin{cases} C_\Phi(r; \alpha_1, \alpha_2) = -\dfrac{\varphi_1}{\varphi_2}(r; \alpha_1, \alpha_2), & r > 0, \\ C_\Psi(r; \alpha_1, \alpha_2) = -\dfrac{\psi_1}{\psi_2}(r; \alpha_1, \alpha_2), & r > 0, \end{cases} \tag{4.4}$$

where φ_i and ψ_i $(i = 1, 2)$ are defined in (2.1) with respect to the solution $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ of (4.2) and $C > 0$. For simplification, we leave out the symbol of initial data (α_1, α_2) in the functions defined in (4.3) and (4.4) if no confusion arises. Then $\Phi(r; C)$ and $\Psi(r; C)$ satisfy

$$\begin{cases} \Phi''(r; C) + \dfrac{n-1}{r}\Phi'(r; C) - \left[\Phi(r; C) - \dfrac{df}{dv}(v)\Psi(r; C)\right] = 0, \\ \Psi''(r; C) + \dfrac{n-1}{r}\Psi'(r; C) - \left[\Psi(r; C) - \dfrac{dg}{du}(u)\Phi(r; C)\right] = 0, \\ \Phi(0; C) = 1, \quad \Psi(0; C) = C, \\ \Phi'(0; C) = \Psi'(0; C) = 0, \end{cases} \tag{4.5}$$

for $r > 0$.

Remark 4.1. (i) By Lemma 2.2, it is easy to see $C_\Phi(r) \to +\infty$ and $C_\Psi(r) \to 0$ as $r \to 0$.

(ii) $C_\Phi(r)$ and $C_\Psi(r)$ cannot be constant on an interval. Indeed, if $C_\Phi(r) \equiv K$ for $r \in [a,b]$, then $\Phi(r;K) = 0$ for $r \in [a,b]$ which is impossible by (4.5).

The following assertions play a significant role in proving the uniqueness of solutions of (4.2).

Lemma 4.1. *Let $p, q > 0$ and $pq < 1$. If $(u(r), v(r))$ is a solution of (4.2), then $C_\Phi(r)$ is strictly decreasing and $C_\Psi(r)$ is strictly increasing on $(0, R]$. Furthermore, $C_\Phi(R) > C_\Psi(R)$.*

Proof. First, by Remark 4.1(i), we have $C_\Phi(r) > C_\Psi(r)$ for $r \in (0, r_0)$ for some $0 < r_0 \leqslant R$. We claim that $C'_\Phi(r) < 0$ and $C'_\Psi(r) > 0$ for $r \in (0, r_0)$.

Proof of Claim. Suppose that $C_\Phi(r)$ is not strictly decreasing on $(0, r_0)$. Then by Remark 4.1(ii), there exist $0 < r_1 < r_2 \leqslant r_0$ such that

$$C'_\Phi(r_1) < 0, \qquad C'_\Phi(r_2) > 0, \qquad C_\Phi(r_1) = C_\Phi(r_2) \equiv C_0$$

and

$$0 < C_\Psi(r) < C_\Phi(r) < C_0, \quad r \in (r_1, r_2).$$

By combining (4.3), (4.4) and Lemma 2.2, we obtain

$$\begin{cases} \Phi(r; C_0) < 0 < \Psi(r; C_0), & r \in (r_1, r_2), \\ \Phi(r_1; C_0) = \Phi(r_2; C_0) = 0, \end{cases} \tag{4.6}$$

which implies that $\Phi(r; C_0)$ has a local minimum at some $\bar{r} \in (r_1, r_2)$ and $\Phi''(\bar{r}; C_0) \geqslant 0$. However, from (4.5) and (4.6), we have

$$\Phi''(\bar{r}; C_0) = \Phi(\bar{r}; C_0) - pv^{p-1}(\bar{r})\Psi(\bar{r}; C_0) < 0.$$

This is a contradiction. The proof for $C_\Psi(r)$ is similar and we complete the proof of this claim. □

Now, suppose there exists $R_0 \in (0, R]$ such that $C_\Phi(R_0) = C_\Psi(R_0) \equiv C$ and $C_\Phi(r) > C_\Psi(r) > 0$ for $r \in (0, R_0)$. Then from the claim above, we obtain

$$\begin{cases} \Phi(r; C) > 0, \quad \Psi(r; C) > 0, \quad r \in (0, R_0), \\ \Phi(R_0; C) = \Psi(R_0; C) = 0, \\ \Phi'(R_0; C) < 0, \quad \Psi'(R_0; C) < 0. \end{cases} \tag{4.7}$$

Moreover, from (1.3) and (4.5), we get

$$\begin{aligned} \big[r^{n-1}\Phi'(r;C)v - r^{n-1}\Phi(r;C)v'\big]' &= \big[r^{n-1}\Phi'(r;C)\big]'v - \big[r^{n-1}v'\big]'\Phi(r;C) \\ &= r^{n-1}\big[\Phi(r;C)v - pv^p\Psi(r;C)\big] - r^{n-1}\big(v - u^q\big)\Phi(r;C) \\ &= r^{n-1}\big[u^q\Phi(r;C) - pv^p\Psi(r;C)\big]. \end{aligned}$$

Hence, by integrating the above equality from 0 to R_0, we obtain

$$R_0^{n-1}\Phi'(R_0;C)v(R_0) = \int_0^{R_0} r^{n-1}\big[u^q\Phi(r;C) - pv^p\Psi(r;C)\big]\,dr. \tag{4.8}$$

Similarly, we also have

$$R_0^{n-1}\Psi'(R_0;C)u(R_0)=\int_0^{R_0} r^{n-1}\big[v^p\Psi(r;C)u-qu^q\Phi(r;C)\big]\,dr. \tag{4.9}$$

Since $pq<1$, $(1-pq)u^q(r)>0$. Therefore from (4.7), (4.8) and (4.9), we deduce

$$\begin{aligned}0&\geqslant R_0^{n-1}\Phi'(R_0;C)v(R_0)+pR_0^{n-1}\Psi'(R_0;C)u(R_0)\\&=\int_0^{R_0} r^{n-1}\Phi(r;C)(1-pq)u^q(r)\,dr>0,\end{aligned}$$

which is impossible. Hence the graphs of C_Φ and C_Ψ do not intersect on $[0,R]$. The proof of this lemma is complete. □

Based on Lemma 4.1, it is easy to obtain the following consequences.

Lemma 4.2. *Let $p,q>0$ and $pq<1$. If $(u(r),v(r))$ is a solution of (4.2) and define*

$$C^*=-\frac{\varphi_1(R)}{\varphi_2(R)},\qquad C_*=-\frac{\psi_1(R)}{\psi_2(R)},$$

then $\Phi(r;C)$ and $\Psi(r;C)$ satisfy the following properties.

(a) *If $C>C^*$, then $\Psi(r;C)>0$ on $[0,R]$ and $\Phi(R;C)<0$.*
(b) *If $C=C^*$, then $\Phi(r;C),\Psi(r;C)>0$ on $[0,R)$, $\Phi(R;C)=0$ and $\Psi(R;C)>0$.*
(c) *If $C_*<C<C^*$, then $\Phi(r;C),\Psi(r;C)>0$ on $[0,R]$.*
(d) *If $C=C_*$, then $\Phi(r;C),\Psi(r;C)>0$ on $[0,R)$, $\Phi(R;C)>0$ and $\Psi(R;C)=0$.*
(e) *If $0<C<C_*$, then $\Phi(r;C)>0$ on $[0,R]$ and $\Psi(R;C)<0$.*

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We split the proof into several steps.

Step 1. (*Non-degeneracy*) We show that each solution of (4.2) is non-degenerate, hence the solution set of (4.2) is locally a smooth curve. Let $(u(r;\alpha_1,\alpha_2),v(r;\alpha_1,\alpha_2))$ be a solution of (4.2). We define

$$G(r;\alpha_1,\alpha_2)=\begin{pmatrix}u(r;\alpha_1,\alpha_2)\\v(r;\alpha_1,\alpha_2)\end{pmatrix}\equiv\begin{pmatrix}G_1(r;\alpha_1,\alpha_2)\\G_2(r;\alpha_1,\alpha_2)\end{pmatrix},\quad r>0, \tag{4.10}$$

and

$$\Lambda=\left\{(r,\alpha_1,\alpha_2)\colon\ G(r;\alpha_1,\alpha_2)=\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

Then for $(R_0,\alpha_{10},\alpha_{20})\in\Lambda$, we obtain that

$$\det\left(\frac{\partial G_i}{\partial\alpha_j}(R_0;\alpha_{10},\alpha_{20})\right)=\det\begin{pmatrix}\varphi_1(R_0)&\varphi_2(R_0)\\\psi_1(R_0)&\psi_2(R_0)\end{pmatrix}\neq 0,$$

where $\varphi_i(r)$ and $\psi_i(r)$, $i = 1, 2$, are defined in (2.1) associated with $(u(r; \alpha_{10}, \alpha_{20}), v(r; \alpha_{10}, \alpha_{20}))$. Indeed, otherwise, $\varphi_1(R_0) + C\varphi_2(R_0) = \psi_1(R_0) + C\psi_2(R_0) = 0$ for some $C > 0$ and hence $C_\Phi(R_0) = C_\Psi(R_0) = C$ which contradicts the fact in Lemma 4.1.

Now, by applying the implicit function theorem, we have that there exist $\varepsilon = \varepsilon(R_0) > 0$ and a unique C^1 curve $(\beta_1, \beta_2) : (R_0 - \varepsilon, R_0 + \varepsilon) \to (0, \infty) \times (0, \infty)$ such that $(R_0, \beta_1(R_0), \beta_2(R_0)) = (R_0, \alpha_{10}, \alpha_{20})$ and $(R, \beta_1(R), \beta_2(R)) \in \Lambda$ for $R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. Hence, $(\beta_1(R), \beta_2(R))$ is the initial data corresponding to solutions of (4.2) for $R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. Next, we claim $\beta_1'(R) > 0$ and $\beta_2'(R) > 0$ for $R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. By (1.3) and Lemma 3.1, we get

$$u'\big(R; \beta_1(R), \beta_2(R)\big) < 0, \qquad v'\big(R; \beta_1(R), \beta_2(R)\big) < 0 \tag{4.11}$$

and

$$\begin{cases} \dfrac{\partial G_1}{\partial R} = u'\big(R; \beta_1(R), \beta_2(R)\big) + \varphi_1(R)\beta_1'(R) + \varphi_2(R)\beta_2'(R) = 0, \\ \dfrac{\partial G_2}{\partial R} = v'\big(R; \beta_1(R), \beta_2(R)\big) + \psi_1(R)\beta_1'(R) + \psi_2(R)\beta_2'(R) = 0 \end{cases} \tag{4.12}$$

for $R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. Rearranging (4.12), we attain

$$\begin{cases} \beta_1'(R) + \dfrac{\varphi_2(R)}{\varphi_1(R)}\beta_2'(R) = -\dfrac{u'(R; \beta_1(R), \beta_2(R))}{\varphi_1(R)}, \\ \beta_1'(R) + \dfrac{\psi_2(R)}{\psi_1(R)}\beta_2'(R) = -\dfrac{v'(R; \beta_1(R), \beta_2(R))}{\psi_1(R)}. \end{cases} \tag{4.13}$$

By combining (4.11) and (4.13), we assure that $\beta_1'(R)$ and $\beta_2'(R)$ have the same sign. Moreover, (4.11) and (4.12) also imply

$$\varphi_1(R)\beta_1'(R) + \varphi_2(R)\beta_2'(R) > 0, \qquad \psi_1(R)\beta_1'(R) + \psi_2(R)\beta_2'(R) > 0.$$

Hence

$$\Phi\big(R; \beta_2'(R)/\beta_1'(R)\big) < 0, \qquad \Psi\big(R; \beta_2'(R)/\beta_1'(R)\big) < 0$$

if $\beta_1'(R) < 0$, which is impossible by Lemma 4.2. Therefore, we get

$$\beta_1'(R) > 0, \qquad \beta_2'(R) > 0, \quad R \in (R_0 - \varepsilon, R_0 + \varepsilon).$$

This proves that each connected component of the solution set of (4.1) is a curve which can be parameterized by R.

Step 2. (*Existence*) We show the existence result by applying the monotone iteration method in [32,33]. First, let

$$u^*(x) \equiv 1, \qquad v^*(x) \equiv 1, \quad x \in B_R(\mathbf{0}).$$

Then it is easy to see that $(u^*(x), v^*(x))$ is a super-solution of (4.1). Since $pq < 1$, then without loss of generality, we can assume that $0 < p \leqslant q$ and in particular $0 < p < 1$. For a sub-solution, we define

$$u_*(x) = \varepsilon_1\phi_1^2(x), \qquad v_*(x) = \varepsilon_2\phi_1^{2/p}(x), \quad x \in B_R(\mathbf{0}),$$

where $\phi_1 > 0$ is the eigenfunction corresponding to the principal eigenvalue $\lambda_1 > 0$ of

$$\begin{cases} -\Delta\phi(x) = \lambda\phi(x), & x \in B_R(\mathbf{0}), \\ \phi(x) = 0, & x \in \partial B_R(\mathbf{0}), \end{cases} \tag{4.14}$$

and $\varepsilon_1, \varepsilon_2$ satisfy

$$\begin{cases} 0 < \varepsilon_1 \leqslant (2\lambda_1 + 1)^{1/(pq-1)} \left(\dfrac{2\lambda_1}{p} + M\right)^{p/(pq-1)}, \\ 0 < \varepsilon_2 \leqslant \left(\dfrac{2\lambda_1}{p} + M\right)^{-1} \varepsilon_1^q \end{cases} \tag{4.15}$$

with

$$M = \sup_{x \in B_R(\mathbf{0})} \phi_1^{2/p-2q}(x).$$

Then from (4.14) and (4.15), we have

$$\Delta u_* - u_* + v_*^p = 2\varepsilon_1 |\nabla\phi_1|^2 + \phi_1^2\left[-(2\lambda_1 + 1)\varepsilon_1 + \varepsilon_2^p\right] \geqslant 0,$$

and

$$\begin{aligned} \Delta v_* - v_* + u_*^q = {} & 2\varepsilon_2\left[2\left(\frac{1}{p}\right)^2 - \left(\frac{1}{p}\right)\right]\phi_1^{2/p-2}|\nabla\phi_1|^2 \\ & + \phi_1^{2q}\left\{\varepsilon_1^q - \left[2\left(\frac{1}{p}\right)\lambda_1 + \phi_1^{2/p-2q}\right]\varepsilon_2\right\} \\ \geqslant {} & 0. \end{aligned}$$

Hence $(u_*(x), v_*(x))$ is a sub-solution of (4.1). By choosing ε_1 and ε_2 sufficiently small such that

$$u_*(x) \leqslant u^*(x), \qquad v_*(x) \leqslant v^*(x) \quad \text{on } B_R(\mathbf{0}),$$

and applying the extension of monotone methods to systems in [33], we prove the existence of a positive solution of (4.1) for any $R > 0$.

Step 3. (*Uniqueness*) From Step 2, for any fixed $R^* > 0$, there exists a positive solution $(u(x; R^*), v(x; R^*))$ of (4.1). Let $(u(0; R^*), v(0; R^*)) = (\beta_1^*, \beta_2^*)$. Then from Step 1, $(R^*, \beta_1^*, \beta_2^*)$ belongs to a monotone curve $\{(R, \beta_1(R), \beta_2(R))\colon R \in (R^* - \epsilon, R^* + \epsilon)\}$ so that $\beta_1(R^*) = \beta_1^*$ and $\beta_2(R^*) = \beta_2^*$. This curve can be extended to a maximum curve Γ in $\mathbf{R}_+^3$ by repeatedly applying Step 1.

Let $\Gamma = \{(R, \beta_1(R), \beta_2(R))\colon R \in (R_1, R_2)\}$ for $0 \leqslant R_1 < R_2 \leqslant \infty$. Since $\beta_1'(R) > 0$ and $\beta_2'(R) > 0$, then $\lim_{R \to R_1^+} \beta_i(R) = \hat{\beta}_i \geqslant 0$ exists. If $\hat{\beta}_i = 0$ for $i = 1$ or 2, then $R_1 = 0$ from the definition of $\hat{\beta}_i$. If $\hat{\beta}_i > 0$ for $i = 1, 2$ and $R_1 > 0$, then $(R_1, \hat{\beta}_1, \hat{\beta}_2)$ still represents a positive solution of (4.1) so Step 1 can be applied to extend Γ further left, which contradicts with the maximality of Γ. Hence $R_1 = 0$ and $\hat{\beta}_i = 0$ for $i = 1, 2$. That is, the left endpoint of Γ is at $(0, 0, 0)$. On the other hand, $\lim_{R \to R_2^-} \beta_i(R) = \tilde{\beta}_i \leqslant 1$ since from the maximum principle, $\beta_i(R) \leqslant 1$. If $R_2 < \infty$, then again $(R_2, \tilde{\beta}_1, \tilde{\beta}_2)$ still represents a positive solution of (4.1) so Step 1 can be applied to extend Γ further right, which contradicts with the maximality of Γ. Therefore $R_2 = \infty$ and $\Gamma = \{(R, \beta_1(R), \beta_2(R))\colon R \in (0, \infty)\}$.

Suppose that (4.1) has positive solutions other than the ones on Γ, then using the exact same proof, one can show that such solutions are on another branch $\tilde{\gamma} = \{(R, \bar{\beta}_1(R), \bar{\beta}_2(R))\colon R \in (0, \infty)\}$

satisfying $\beta_i'(R) > 0$ and $\lim_{R\to 0^+} \bar{\beta}_i(R) = 0$, $i = 1, 2$. Due to the monotonicity, both γ and $\tilde{\gamma}$ can be re-parameterized by α_1 (the initial value of u). Hence $\gamma = \{(R(\alpha_1), \alpha_1, \alpha_2(\alpha_1))\colon \alpha_1 \in (0, 1)\}$ and $\tilde{\gamma} = \{(\tilde{R}(\alpha_1), \alpha_1, \tilde{\alpha}_2(\alpha_1))\colon \alpha_1 \in (0, 1)\}$. But from Theorem 1.3 and Remark 3.1, Ω_C is a unique monotone curve $\{(\alpha_1, \gamma(\alpha_1))\colon \alpha \in (0, 1)\}$. Thus $\alpha_2(\alpha_1) = \tilde{\alpha}_2(\alpha_1) = \gamma(\alpha_1)$ which deduces that γ and $\tilde{\gamma}$ are identical. This proves the uniqueness of positive solutions to (4.1). □

Remark 4.2. While the monotone iteration method (see [32,33]) is well known to be used to prove the existence of solutions, the technique applied to the proof of Theorem 1.1 can provide us with a new way to construct a desired sub-solution or super-solution involving the power of nonlinearities for more general systems of PDEs.

Remark 4.3. If we can show that $\Omega_G = \emptyset$ for the case considered in Theorem 1.1, then $\lim_{R\to\infty} \beta_i(R) = \tilde{\beta}_i = 1$ in the proof of Theorem 1.1 (that is the case if $0 < p < 1$ and $0 < q < 1$). Otherwise $(\tilde{\beta}_1, \tilde{\beta}_2)$ could be an element in Ω_G if $\tilde{\beta}_i < 1$ for at least one of $i = 1, 2$.

5. Generalizations

The methods presented above can be used to prove existence, uniqueness of solutions in other similar situations. As an example other than (1.1), we consider

$$\begin{cases} \Delta u + f_1(v) = 0, \\ \Delta v + g_1(u) = 0, \end{cases} \quad \text{in } \mathbf{R}^n, \tag{5.1}$$

where f_1 and g_1 are non-negative, unbounded C^1 functions on $\mathbf{R}_+$ which satisfy

$$f_1(0) = 0, \qquad g_1(0) = 0, \qquad f_1'(v) \geqslant 0, \qquad g_1'(u) \geqslant 0, \tag{5.2}$$

and we also extend f_1 and g_1 to $C(\mathbf{R})$ so that

$$f_1(v) \begin{cases} > 0, & v > 0, \\ = 0, & v \leqslant 0, \end{cases} \qquad g_1(u) \begin{cases} > 0, & u > 0, \\ = 0, & u \leqslant 0. \end{cases} \tag{5.3}$$

We investigate the initial value problem, *i.e.*, the radial case of (5.1):

$$\begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) + f_1(v) = 0, & r > 0, \\ v''(r) + \dfrac{n-1}{r} v'(r) + g_1(u) = 0, & r > 0, \\ u(0) = \alpha_1, \quad v(0) = \alpha_2, \quad u'(0) = v'(0) = 0, \end{cases} \tag{5.4}$$

where $\alpha_1, \alpha_2 > 0$ is the initial data. We note that the solution of (5.4) is globally defined on $[0, \infty)$, and denote it by $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ conventionally.

Similar to Theorem 1.1, we have the existence/uniqueness of positive solutions to the Dirichlet problem of (5.4) as follows:

Theorem 5.1. *Suppose that f_1 and g_1 satisfy* (5.2), (5.3) *and for any $u > 0$, $v > 0$,*

$$v f_1'(v) \leqslant p f_1(v), \qquad u g_1'(u) \leqslant q g_1(u) \tag{5.5}$$

for some $p, q > 0$ and $pq < 1$. Then for any $R > 0$, there exists a unique $(\alpha_1(R), \alpha_2(R)) \in \mathbf{R}_+^2$ such that the corresponding solution $(u(r; \alpha_1(R), \alpha_2(R)),\ v(r; \alpha_1(R), \alpha_2(R)))$ of (5.4) *is positive on $[0, R)$ and vanishes at $r = R$. Furthermore, both $\alpha_1(R)$ and $\alpha_2(R)$ are increasing in $R > 0$.*

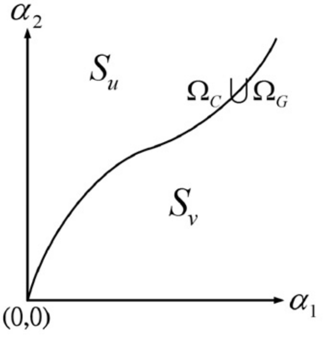

Fig. 2. Structure of solutions for (5.4).

We call the solution stated in Theorem 5.1 the *Dirichlet-type* solution. In addition, we can also clarify the structure of all solutions to (5.4). Similar to before, we call the solution (u, v) of (5.4) is a *ground state* if $(u(r), v(r)) \to (0, 0)$ as $r \to \infty$. Also, a *u-crossing* (resp., *v-crossing*) solution (u, v) is that u (resp., v) vanishes at some finite point where v (resp., u) is still positive.

Theorem 5.2. *Suppose that f_1 and g_1 satisfy* (5.2) *and* (5.3).

(a) *There exists a strictly increasing continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that for any $\alpha_1 > 0$, $(u(r; \alpha_1, \gamma(\alpha_1)), v(r; \alpha_1, \gamma(\alpha_1)))$ is either a ground state or a Dirichlet-type solution. If $0 < p < 1$ and $0 < q < 1$, then the ground state solution does not exist.*
(b) *If (α_1, α_2) lies on the region bounded from below by the curve $\{(\alpha_1, \gamma(\alpha_1))\colon \alpha_1 \geqslant 0\}$ and bounded from left by α_2^+-axis, then the corresponding solution is a u-crossing solution.*
(c) *If (α_1, α_2) lies on the region bounded from above by the curve $\{(\alpha_1, \gamma(\alpha_1))\colon \alpha_1 \geqslant 0\}$ and bounded from below by α_1^+-axis, then the corresponding solution is a v-crossing solution.*

Remark 5.1. Theorems 5.1 and 5.2 can be proved via the similar arguments introduced in previous sections. We omit the details here.

Remark 5.2. As stated in Theorem 5.2(a), the solution type on the curve $\{(\alpha_1, \gamma(\alpha_1))\colon \alpha_1 > 0\}$ is determined by the nonlinearities f_1 and g_1. For instance, under conditions of Theorems 2.1 and 4.1 in [28], we have that such curve is the collection of all initial data corresponding to Dirichlet-type solutions and ground state solutions, respectively.

Remark 5.3. Under many circumstances, it is not easy to deal with the uniqueness of solutions for (5.4) by applying the scaling arguments (see, e.g., [13]). For example, consider $f_1(v) = v^m + v^n$ and $g_1(u) = u^t + u^\gamma$, where $u, v \geqslant 0$, $m > n > 0$, $t > \gamma > 0$ and $mt < 1$. Nevertheless, the concept of linearization provided here gives us another approach, as in Theorem 5.1, to manage such issue in more general situations including the case of f_1 and g_1 presented above.

Based on Theorem 5.2, we illustrate the structure of solutions for (5.4) in Fig. 2. Here, Ω_C, Ω_G, S_u and S_v denote the regions of initial data corresponding to Dirichlet-type, ground state, u-crossing and v-crossing solutions respectively.

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Appendix A

In this appendix, we provide a proof of the radial symmetry of solutions for

$$\begin{cases} \Delta u - u + f(v) = 0 & \text{in } B_R, \\ \Delta v - v + g(u) = 0 & \text{in } B_R, \\ u(x) = v(x) = 0 & \text{on } \partial B_R, \end{cases} \tag{A.1}$$

where B_R is the ball of $\mathbf{R}^n$ centered at the origin with radius $R > 0$ and f, g are defined as in (1.4). It is similar to that in [8] based on the arguments used in [22] for scalar equations.

Theorem A. *For each $R > 0$, every nontrivial classical solution (u_R, v_R) of* (A.1) *is radially symmetric and satisfies $u_R > 0$, $v_R > 0$ in B_R.*

Proof. Let $R > 0$ be given. Suppose that (A.1) possesses a nontrivial solution pair (u_R, v_R). First, we prove $u_R > 0$ and $v_R > 0$ in B_R. Suppose $u_R(x_0) = \min_{x \in B_R} u_R(x) < 0$. Then $\Delta u_R(x_0) \geqslant 0$ and thus $\Delta u_R(x_0) - u_R(x_0) + f(v_R(x_0)) > 0$, which yields a contradiction. Hence $u_R \geqslant 0$ in B_R. The strong maximum principle implies $u_R > 0$ in B_R. Similarly, it also holds for v_R.

Now, we show that $(u, v) \equiv (u_R, v_R)$ is a radially symmetric pair. We will apply the method of moving planes with some modifications based on [22]. It suffices to prove that both $u(x)$ and $v(x)$ decrease when the point $x = (x_1, x_2, \ldots, x_n)$ changes its position along the x_1-axis from the origin to the point $(R, 0, \ldots, 0)$. For $0 < \varrho < R$, define $\Sigma_\varrho = \{x \in B_R \colon x_1 > \varrho\}$, $T_\varrho = \{x \in B_R \colon x_1 = \varrho\}$ and $u_\varrho(x) = u(x^\varrho)$, $v_\varrho(x) = v(x^\varrho)$ for $x \in \Sigma_\varrho$, where x^ϱ is the reflection of x with respect to the line $x_1 = \varrho$, *i.e.*, $x^\varrho = (2\varrho - x_1, x_2, \ldots, x_n)$.

Let $w_\varrho(x) = u(x) - u_\varrho(x)$ and $z_\varrho(x) = v(x) - v_\varrho(x)$ for $x \in \Sigma_\varrho$. Then w_ϱ and z_ϱ satisfy the following equations respectively:

$$\Delta w_\varrho - w_\varrho = v^p\big(x^\varrho\big) - v^p(x), \quad x \in \Sigma_\varrho; \qquad w_\varrho \leqslant 0 \quad \text{on } \partial \Sigma_\varrho, \tag{A.2}$$

and

$$\Delta z_\varrho - z_\varrho = u^q\big(x^\varrho\big) - u^q(x), \quad x \in \Sigma_\varrho; \qquad z_\varrho \leqslant 0 \quad \text{on } \partial \Sigma_\varrho.$$

Define

$$Y_u = \big\{\eta \in (0, R) \colon w_\varrho < 0 \text{ in } \Sigma_\varrho \text{ for } \varrho \in (\eta, R)\big\}, \qquad \eta_u = \inf Y_u,$$

and

$$Y_v = \big\{\eta \in (0, R) \colon z_\varrho < 0 \text{ in } \Sigma_\varrho \text{ for } \varrho \in (\eta, R)\big\}, \qquad \eta_v = \inf Y_v.$$

First, by the Hopf's lemma, we have

$$\frac{\partial u}{\partial \nu}(x) < 0, \qquad \frac{\partial v}{\partial \nu}(x) < 0, \quad |x| = R,$$

where $\nu(x)$ is the unit outer normal to ∂B_R at x. In particular, if ϱ is sufficiently close to R, then

$$u(x) < u\big(x^\varrho\big) = u_\varrho(x), \qquad v(x) < v\big(x^\varrho\big) = v_\varrho(x), \quad x \in \Sigma_\varrho,$$

and thus $w_\varrho(x) < 0$ and $z_\varrho(x) < 0$ in Σ_ϱ. Hence the sets Y_u and Y_v are nonempty.

Next, to prove $\eta_u = \eta_v = 0$, suppose that this is not true. Then, without loss of generality, we can assume $0 \leqslant \eta_v \leqslant \eta_u$ and $\eta_u > 0$. Hence $v(x) \leqslant v_{\eta_u}(x)$ in Σ_{η_u} and, by continuity, we have $w_{\eta_u}(x) \leqslant 0$ in Σ_{η_u}. Moreover, by (A.2), it is easy to see that

$$\Delta w_{\eta_u} - w_{\eta_u} \geqslant 0 \quad \text{in } \Sigma_{\eta_u}; \qquad w_{\eta_u} \leqslant 0 \quad \text{in } \Sigma_{\eta_u} \cup \partial\Sigma_{\eta_u}. \tag{A.3}$$

Thus, if $w_{\eta_u}(x_1) = 0$ for some $x_1 \in \Sigma_{\eta_u}$, then by (A.3) and the strong maximum principle, we have $w_{\eta_u} \equiv 0$ in $\overline{\Sigma}_{\eta_u}$. However, this contradicts the fact that $w_{\eta_u}(x) = u(x) - u(x^{\eta_u}) = -u(x^{\eta_u}) < 0$ for $x \in \partial\Sigma_{\eta_u} \setminus \{x_1 = \eta_u\}$. Therefore, we obtain

$$w_{\eta_u}(x) < 0, \quad x \in \overline{\Sigma}_{\eta_u} \setminus T_{\eta_u}; \qquad w_{\eta_u} = 0 \quad \text{on } \partial\Sigma_{\eta_u} \cap T_{\eta_u}. \tag{A.4}$$

By (A.3), (A.4) and the Hopf's lemma again, we obtain

$$\frac{\partial w_{\eta_u}}{\partial x_1} < 0 \quad \text{on } \partial\Sigma_{\eta_u} \cap T_{\eta_u}. \tag{A.5}$$

In addition, since $\eta_u > 0$, there exists a positive sequence $\{\varepsilon_k\}$ such that $\eta_u - \varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$. By the definition of η_u, for each ε_k, we obtain that $w_{\eta_u - \varepsilon_k}$ is non-negative somewhere in $\Sigma_{\eta_u - \varepsilon_k}$. Also, we have $w_{\eta_u - \varepsilon_k} < 0$ on $\partial\Sigma_{\eta_u - \varepsilon_k} \setminus T_{\eta_u - \varepsilon_k}$ and $w_{\eta_u - \varepsilon_k} = 0$ on $\partial\Sigma_{\eta_u - \varepsilon_k} \cap T_{\eta_u - \varepsilon_k}$. Hence, for each ε_k, there exists $x_k \in \Sigma_{\eta_u - \varepsilon_k}$ such that

$$w_{\eta_u - \varepsilon_k}(x_k) \geqslant 0; \qquad \nabla w_{\eta_u - \varepsilon_k}(x_k) = \mathbf{0}. \tag{A.6}$$

Since $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence, which is still denoted by $\{x_k\}$, such that $x_k \to x_0$ for some x_0. By (A.6), we obtain

$$0 \leqslant \lim_{k\to\infty} w_{\eta_u - \varepsilon_k}(x_k) = \lim_{k\to\infty} \left[u(x_k) - u\left(x_k^{\eta_u - \varepsilon_k}\right)\right] = u(x_0) - u\left(x_0^{\eta_u}\right) = w_{\eta_u}(x_0).$$

Hence, by the above inequality and (A.4), we conclude $x_0 \in \partial\Sigma_{\eta_u} \cap T_{\eta_u}$ and, by (A.6),

$$0 = \lim_{k\to\infty} \frac{\partial w_{\eta_u - \varepsilon_k}}{\partial x_1}(x_k) = \frac{\partial u}{\partial x_1}(x_0) - \frac{\partial u}{\partial x_1}\left(x_0^{\eta_u}\right) = \frac{\partial w_{\eta_u}}{\partial x_1}(x_0).$$

This is in contradiction with (A.5). Therefore, $\eta_u = \eta_v = 0$, and we deduce that both $u(x)$ and $v(x)$ are radially symmetric. □

References

[1] A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc. (2) 75 (2007) 67–82.

[2] F. Brock, Continuous rearrangement and symmetry of solutions of elliptic problems, Proc. Indian Acad. Sci. Math. Sci. 110 (2000) 157–204.

[3] P.W. Bates, J.-P. Shi, Existence and instability of spike layer solutions to singular perturbation problems, J. Funct. Anal. 196 (2002) 429–482.

[4] T. Bartsch, Z.-Q. Wang, J.-C. Wei, Bound states for a coupled Schrödinger system, J. Fixed Point Theory Appl. 2 (2007) 353–367.

[5] J. Busca, B. Sirakov, Symmetry results for semilinear elliptic systems in the whole space, J. Differential Equations 163 (2000) 41–56.

[6] C.-C. Chen, C.-S. Lin, Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geqslant 3$, Comm. Partial Differential Equations 16 (1991) 1549–1572.

[7] J.-L. Chern, Z.-Y. Chen, C.-S. Lin, Uniqueness of topological solutions and the structure of solutions for the Chern–Simons system with two Higgs particles, Comm. Math. Phys. 296 (2010) 323–351.

[8] J.-L. Chern, Z.-Y. Chen, Y.-L. Tang, C.-S. Lin, Uniqueness and structure of solutions to the Dirichlet problem for an elliptic system, J. Differential Equations 246 (2009) 3704–3714.

[9] J.-L. Chern, C.-S. Lin, J.-P. Shi, Y.-L. Tang, Existence, uniqueness and stability of positive solutions to sublinear elliptic systems, Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), in press.
[10] Ph. Clément, D.G. de Figueiredo, E. Mitidieri, Positive solutions of semilinear elliptic systems, Comm. Partial Differential Equations 17 (1992) 923–940.
[11] J.-L. Chern, C.-S. Lin, The symmetry of least-energy solutions for semilinear elliptic equations, J. Differential Equations 187 (2003) 240–268.
[12] R. Dalmasso, Existence and uniqueness of positive solutions of semilinear elliptic systems, Nonlinear Anal. 39 (2000) 559–568.
[13] R. Dalmasso, Existence and uniqueness of positive radial solutions for the Lane–Emden systems, Nonlinear Anal. 57 (2004) 341–348.
[14] J. Dolbeault, P. Felmer, Symmetry and monotonicity properties for positive solutions of semi-linear elliptic PDE's, Comm. Partial Differential Equations 25 (2000) 1153–1169.
[15] D.G. de Figueiredo, O. Lopes, Solitary waves for some nonlinear Schrödinger systems, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 149–161.
[16] D.G. De Figueiredo, J.-F. Yang, Decay, symmetry and existence of solutions of semilinear elliptic systems, Nonlinear Anal. 33 (1998) 211–234.
[17] M. Essman, J.-P. Shi, Bifurcation diagrams of coupled Schrödinger equations, submitted for publication.
[18] J. Hulshof, R. Van der Vorst, Differential systems with strongly indefinite variational structure, J. Funct. Anal. 114 (1993) 32–58.
[19] P. Korman, J.-P. Shi, On Lane–Emden type systems, Discrete Contin. Dyn. Syst. (Suppl.) (2005) 510–517.
[20] M.-K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.
[21] M.-K. Kwong, L.-Q. Zhang, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus, Differential Integral Equations 4 (1991) 583–599.
[22] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math. 123 (1996) 221–231.
[23] C.-M. Li, L. Ma, Uniqueness of positive bound states to Schrödinger systems with critical exponents, SIAM J. Math. Anal. 40 (2008) 1049–1057.
[24] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.
[25] T.-C. Lin, J.-C. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^n$, $n \leqslant 3$, Comm. Math. Phys. 255 (2005) 629–653.
[26] L. Ma, L. Zhao, Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application, J. Differential Equations 245 (2008) 2551–2565.
[27] E. Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1993) 125–151.
[28] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$, Differential Integral Equations 9 (1996) 465–479.
[29] T.-C. Ouyang, J.-P. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, J. Differential Equations 146 (1998) 121–156.
[30] T.-C. Ouyang, J.-P. Shi, Exact multiplicity of positive solutions for a class of semilinear problem: II, J. Differential Equations 158 (1999) 94–151.
[31] L.A. Peletier, R. Van der Vorst, Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation, Differential Integral Equations 5 (1992) 747–767.
[32] D.H. Sattinger, Topics in Stability and Bifurcation, Lecture Notes in Math., vol. 309, Springer, Berlin, 1973.
[33] D.H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 21 (1971/72) 979–1000.
[34] J. Serrin, H.-H. Zou, Existence of positive solutions of the Lane–Emden system, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 46 (Suppl.) (1998) 369–380.
[35] J. Serrin, H.-H. Zou, Existence of positive entire solutions of elliptic Hamiltonian systems, Comm. Partial Differential Equations 23 (1998) 577–599.
[36] W.C. Troy, Symmetry properties in systems of semilinear elliptic equations, J. Differential Equations 42 (1981) 400–413.
[37] R. Van der Vorst, Variational identities and applications to differential systems, Arch. Ration. Mech. Anal. 116 (1991) 375–398.