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# Hopf bifurcations in a reaction–diffusion population model with delay effect <sup>☆</sup>

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## ABSTRACT

A reaction–diffusion population model with a general time-delayed growth rate per capita is considered. The growth rate per capita can be logistic or weak Allee effect type. From a careful analysis of the characteristic equation, the stability of the positive steady state solution and the existence of forward Hopf bifurcation from the positive steady state solution are obtained via the implicit function theorem, where the time delay is used as the bifurcation parameter. The general results are applied to a “food-limited” population model with diffusion and delay effects as well as a weak Allee effect population model.

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## 1. Introduction

Reaction–diffusion equations with time delay effect have been proposed as models for the population ecology, the cell biology and the control theory in recent years (see e.g. [13,23,39]). In this paper, we consider the following reaction–diffusion equation

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$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= d \frac{\partial^2 u^2(x, t)}{\partial x^2} + \lambda u(x, t) f(u(x, t - \tau)), \quad x \in (0, l), t > 0, \\ u(0, t) = u(l, t) &= 0, \quad t \geq 0, \end{aligned} \tag{1.1}$$

where  $d > 0$  is the diffusion coefficient,  $\tau > 0$  is the time delay, and  $\lambda > 0$  is a scaling constant; the spatial domain is the interval  $(0, l)$ , and Dirichlet boundary condition is imposed so the exterior environment is hostile. We consider Eq. (1.1) with the following initial value:

$$u(x, s) = \eta(x, s), \quad x \in [0, l], s \in [-\tau, 0], \tag{1.2}$$

where  $\eta \in C \stackrel{\text{def}}{=} C([-\tau, 0], Y)$  and  $Y = L^2((0, l))$ . This particular form of the equation arises from the study of population biology models, and the nonlinear function  $f$  is the growth rate per capita (see e.g. [4,29]). In (1.1) the growth rate per capita depends on the historical population density, which occurs naturally for many populations. Such population models with dispersal and delay effect have been studied with proper choices of  $f$  (see e.g. [7,23]).

We consider two types of growth rates per capita: the logistic growth (with  $f$  decreasing) or weak Allee effect growth (with  $f$  initially increasing from a positive value then decreasing). Logistic growth is well known, and it mainly models the crowding effect of the population. Logistic growth is also widely utilized by biologists and scientists in other research fields. While the most popular choice of the logistic growth function  $f$  is  $f(u) = a - u$ , others have also been used (see below or [4,29] for other examples). On the other hand, Allee effect describes the phenomenon that for smaller populations, the reproduction and survival of individuals decrease, see more detailed discussions in [1,4,29,35]. Weak Allee effect growth function  $f$  is always positive, while in the strong Allee effect, there is a critical density below which the growth is negative. The reaction–diffusion model without delay effect but with logistic growth or Allee effect has been considered in [29], see also the recent survey [25].

The goal of this paper is to determine the long time dynamical behavior of the system (1.1)–(1.2). We employ the method applied by Busenberg and Huang [3] to investigate the stability of the steady state solutions and Hopf bifurcation near the spatially nonhomogeneous positive steady state solution for model (1.1)–(1.2). In the present paper, we define the Hopf bifurcation to be *forward* (resp. *backward*) if the periodic solutions exist for parameter values in a right-hand side neighborhood  $(\tau_*, \tau_* + \varepsilon)$  (resp. left-hand side neighborhood  $(\tau_* - \varepsilon, \tau_*)$ ) of the bifurcation value  $\tau_*$ . There are only a few discussions about Hopf bifurcation near the spatial nonhomogeneous positive steady state solution, because the analysis of the characteristic equation is difficult [3,26,44]. Define  $\lambda_* = d(\frac{\pi}{l})^2$ . Our main results for the logistic growth case can be summarized as follows:

- (i) If  $\lambda < \lambda_*$ , then the zero solution is the global attractor of all nonnegative solutions to Eq. (1.1) for any  $\tau \geq 0$ .
- (ii) For any fixed  $\lambda$  satisfying  $0 < \lambda - \lambda_* \ll 1$ , Eq. (1.1) has a positive steady state solution  $u_\lambda$  and there is a constant  $\tau_0$  such that  $u_\lambda$  is locally asymptotically stable when  $\tau \in [0, \tau_0)$  and is unstable when  $\tau \in (\tau_0, \infty)$ . Moreover, there exists a sequence of values  $\{\tau_n\}_{n=0}^\infty$ , such that Eq. (1.1) undergoes a forward Hopf bifurcation at the positive steady state solution  $u = u_\lambda$  when  $\tau = \tau_n$ .

The results tell us that the direction of the Hopf bifurcation is independent of  $d$  and  $l$ , and is always forward. Similar results hold for weak Allee effect case, see details in Sections 5 and 6.

As an example, we apply the main results to the “food-limited” population model:

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial u^2(x, t)}{\partial x^2} + \lambda u(x, t) \frac{1 - u(x, t - \tau)}{1 + cu(x, t - \tau)}, \quad x \in (0, \pi), t > 0, \\ u(0, t) = u(\pi, t) &= 0, \quad t \geq 0. \end{aligned} \tag{1.3}$$

The existence and stability of positive steady state solutions and the existence of forward Hopf bifurcations from the positive steady state solution for Eq. (1.3) are obtained. We notice that the

food-limited model is of logistic growth type. Similarly, we can apply the results to the diffusive Hutchinson equation, which have been obtained in Busenberg and Huang [3]. Our results for the weak Allee effect case appear to be completely new.

Our analysis follows the lines of Busenberg and Huang [3], but we consider the problem in a more general setting. We show that the basic framework of [3] works for more general cases even the weak Allee effect one, but the generalization is non-trivial as calculations for general nonlinearities are much more difficult. Numerical simulations suggest that similar bifurcations occur for steady states away from bifurcation points as well, and it is an interesting open question how that can be proved. Other studies on steady states and bifurcation of delayed reaction–diffusion models can be found in [7,8,11,12,18–20,22,27,31,33,34,42,43].

The rest of this paper is organized as follows. In Section 2, the eigenvalue problem is investigated. In Section 3, we describe the stability of the steady state solutions. The existence of forward Hopf bifurcations is established in Section 4. In Section 5, we consider the weak Allee effect case, and in Section 6 we discuss the global dynamics and present some numerical results motivated by our theoretical studies, and in particular we apply the general results to a “food-limited” population model.

Throughout the paper, we use standard notation  $L^2$ ,  $H^k$ ,  $H_0^k$  for the real-valued Sobolev spaces based on  $L^2$  spaces, and the underlying spatial domain is always the interval  $(0, l)$ . Moreover we denote  $X = H^2 \cap H_0^1$ ,  $Y = L^2$  and, for any subspace  $Z$  of  $X$  or  $Y$  (including  $X$  and  $Y$ ), we also define the complexification of  $Z$  to be  $Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 \mid x_1, x_2 \in Z\}$ . For the complex-valued Hilbert space  $Y_{\mathbb{C}}$ , we use the standard inner product  $\langle u, v \rangle = \int_0^l u(x)\bar{v}(x) dx$ . We also use  $\mathcal{C}_{\mathbb{C}} = \mathcal{C} \oplus i\mathcal{C}$ .

**2. Eigenvalue problems**

In this section we study the existence and properties of the positive steady state solutions of Eq. (1.1), which satisfy the following boundary value problem

$$\begin{aligned} \frac{d^2u(x)}{dx^2} + \lambda u(x)f(u(x)) &= 0, \quad x \in (0, l), \\ u(0) = u(l) &= 0. \end{aligned} \tag{2.1}$$

In Sections 2–4, we always assume the following assumptions:

- (A1) There exists  $\delta > 0$  such that  $f$  is a  $C^4$  function on  $[0, \delta]$ .
- (A2)  $f(0) = 1$ , and  $f'(u) < 0$  for  $u \in [0, \delta]$ .

It is well known that

$$Y = \mathcal{N}(dD^2 + \lambda_*) \oplus \mathcal{R}(dD^2 + \lambda_*),$$

where

$$D^2 = \frac{\partial^2}{\partial x^2}, \quad \mathcal{N}(dD^2 + \lambda_*) = \text{Span} \left\{ \sin\left(\frac{\pi}{l}(\cdot)\right) \right\}$$

and

$$\mathcal{R}(dD^2 + \lambda_*) = \left\{ y \in Y : \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), y \right\rangle = \int_0^l \sin\left(\frac{\pi}{l}x\right)y(x) dx = 0 \right\}.$$

Now we give a result on the existence of positive steady state solutions as follows.

**Theorem 2.1.** *There exist  $\lambda^* > \lambda_*$  and a continuously differentiable mapping  $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$  from  $[\lambda_*, \lambda^*]$  to  $(X \cap \mathcal{R}(dD^2 + \lambda_*)) \times \mathbb{R}^+$  such that Eq. (1.1) has a positive steady state solution given by*

$$u_\lambda = \alpha_\lambda(\lambda - \lambda_*) \left[ \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi_\lambda \right], \quad \lambda \in [\lambda_*, \lambda^*]. \tag{2.2}$$

Moreover,

$$\alpha_{\lambda_*} = \frac{-\int_0^l \sin^2\left(\frac{\pi}{l}x\right) dx}{\lambda_* f'(0) \int_0^l \sin^3\left(\frac{\pi}{l}x\right) dx}$$

and  $\xi_{\lambda_*} \in X$  is the unique solution of the equation

$$(dD^2 + \lambda_*)\xi + \left[ 1 + \lambda_* \alpha_{\lambda_*} f'(0) \sin\left(\frac{\pi}{l}(\cdot)\right) \right] \sin\left(\frac{\pi}{l}(\cdot)\right) = 0, \quad \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \xi \right\rangle = 0.$$

**Proof.** Since  $dD^2 + \lambda_*$  is bijective from  $X \cap \mathcal{R}(dD^2 + \lambda_*)$  to  $\mathcal{R}(dD^2 + \lambda_*)$  we know that  $\xi_{\lambda_*}$  is well defined. Let  $m : X \times \mathbb{R} \times \mathbb{R} \rightarrow Y \times \mathbb{R}$  be defined as

$$m(\xi, \alpha, \lambda) = \left( (dD^2 + \lambda_*)\xi + \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi \right. \\ \left. + \lambda \left[ \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi \right] m_1(\xi, \alpha, \lambda), \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \xi \right\rangle \right),$$

where

$$m_1(\xi, \alpha, \lambda) = \begin{cases} \frac{f(\alpha(\lambda - \lambda_*)[\sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)\xi]) - 1}{\lambda - \lambda_*}, & \text{if } \lambda \neq \lambda_*, \\ f'(0)\alpha \sin(\frac{\pi}{l}(\cdot)), & \text{if } \lambda = \lambda_*. \end{cases} \tag{2.3}$$

From the definition of  $\xi_{\lambda_*}$ , we have that

$$m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*) \\ = \left( (dD^2 + \lambda_*)\xi_{\lambda_*} + \left[ 1 + \lambda_* \alpha_{\lambda_*} f'(0) \sin\left(\frac{\pi}{l}(\cdot)\right) \right] \sin\left(\frac{\pi}{l}(\cdot)\right), \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \xi_{\lambda_*} \right\rangle \right) = 0$$

and the partial derivative of  $m$  is given by

$$D_{(\xi, \alpha)} m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*)(\eta, \epsilon) = \left( (dD^2 + \lambda_*)\eta + \lambda_* \epsilon f'(0) \sin^2\left(\frac{\pi}{l}(\cdot)\right), \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \eta \right\rangle \right).$$

From  $\sin^2(\frac{\pi}{l}(\cdot)) \notin \mathcal{R}(dD^2 + \lambda_*)$ , it follows that  $D_{(\xi, \alpha)} m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*)$  is bijective from  $X \times \mathbb{R}$  to  $Y \times \mathbb{R}$ . Therefore, the implicit function theorem implies that there exist  $\lambda^* > \lambda_*$  and a continuously differentiable mapping  $\lambda \mapsto (\xi_\lambda, \alpha_\lambda) \in X \times \mathbb{R}^+$  such that

$$m(\xi_\lambda, \alpha_\lambda, \lambda) = 0, \quad \lambda \in [\lambda_*, \lambda^*].$$

An easy calculation shows that  $\alpha_\lambda(\lambda - \lambda_*)[\sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)\xi_\lambda]$  solves Eq. (2.1).  $\square$

Throughout Sections 2–4, we will always assume  $\lambda \in [\lambda_*, \lambda^*]$  unless otherwise specified, and  $0 < \lambda^* - \lambda_* \ll 1$ . But the value of  $\lambda^*$  may change from one place to another when further perturbation arguments are used. We also remark that the solution  $u_\lambda$  is small in  $W^{2,p}$  norm, thus it is also small in  $C^\alpha$  norm for  $\alpha \in (0, 1)$  from standard elliptic estimates. Hence by taking  $|\lambda^* - \lambda_*|$  small enough, we can assume that  $\max_{x \in [0, l]} u_\lambda(x) < \delta/2$ , where  $\delta$  is defined in (A1). In Sections 2–4, we only consider local bifurcations near the steady state  $u_\lambda$ , hence involved solutions of (1.1) only take values in  $[0, \delta]$ .

The linearization of (1.1)–(1.2) at  $u_\lambda$  is given by

$$\begin{aligned} \frac{\partial v(x, t)}{\partial t} &= d \frac{\partial^2 v(x, t)}{\partial x^2} + \lambda f(u_\lambda) v(x, t) + \lambda u_\lambda f'(u_\lambda) v(x, t - \tau), \quad t > 0, \\ v(0, t) &= v(l, t) = 0, \quad t \geq 0, \\ v(x, t) &= \eta(x, t), \quad (x, t) \in [0, l] \times [-\tau, 0], \end{aligned} \tag{2.4}$$

where  $\eta \in \mathcal{C}$ .

We introduce the operator  $A(\lambda) : \mathcal{D}(A(\lambda)) \rightarrow Y_{\mathbb{C}}$  defined by

$$A(\lambda) = dD^2 + \lambda f(u_\lambda),$$

with domain

$$\mathcal{D}(A(\lambda)) = \{y \in Y_{\mathbb{C}} : \dot{y}, \ddot{y} \in Y_{\mathbb{C}}, y(0) = y(l) = 0\} = X_{\mathbb{C}},$$

and set  $v(t) = v(\cdot, t)$ ,  $\eta(t) = \eta(\cdot, t)$ . Then Eq. (2.4) can be rewritten as

$$\begin{aligned} \frac{dv(t)}{dt} &= A(\lambda)v(t) + \lambda u_\lambda f'(u_\lambda)v(t - \tau), \quad t > 0, \\ v(t) &= \eta(t), \quad t \in [-\tau, 0], \quad \eta \in \mathcal{C}, \end{aligned} \tag{2.5}$$

with  $A(\lambda)$  an infinitesimal generator of a compact  $C_0$ -semigroup [28]. From [37] (or [42]), the semigroup induced by the solutions of Eq. (2.5) has the infinitesimal generator  $A_\tau(\lambda)$  given by

$$\begin{aligned} A_\tau(\lambda)\phi &= \dot{\phi}, \\ \mathcal{D}(A_\tau(\lambda)) &= \{\phi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}_{\mathbb{C}}^1 : \phi(0) \in X_{\mathbb{C}}, \dot{\phi}(0) = A(\lambda)\phi(0) + \lambda u_\lambda f'(u_\lambda)\phi(-\tau)\}, \end{aligned}$$

where  $\mathcal{C}_{\mathbb{C}}^1 = C^1([-\tau, 0], Y_{\mathbb{C}})$ . The spectral set  $\sigma(A_\tau(\lambda)) = \{\mu \in \mathbb{C} : \Delta(\lambda, \mu, \tau)y = 0, \text{ for some } y \in X_{\mathbb{C}} \setminus \{0\}\}$ , and

$$\Delta(\lambda, \mu, \tau) = A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-\mu\tau} - \mu.$$

The eigenvalues of  $A_\tau(\lambda)$  depend continuously on  $\tau$  (see e.g. [6]). It is clear that  $A_\tau(\lambda)$  has an imaginary eigenvalue  $\mu = iv$  ( $v \neq 0$ ) for some  $\tau > 0$  if and only if

$$[A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-i\theta} - iv]y = 0, \quad y(\neq 0) \in X_{\mathbb{C}}, \tag{2.6}$$

is solvable for some value of  $v > 0$ ,  $\theta \in [0, 2\pi)$ . One can see that if we find a pair of  $(v, \theta)$  such that Eq. (2.6) has a solution  $y$ , then

$$\Delta(\lambda, iv, \tau_n)y = 0, \quad \tau_n = \frac{\theta + 2n\pi}{v}, \quad n = 0, 1, 2, \dots$$

Next we shall show that, if  $0 < \lambda^* - \lambda_* \ll 1$ , then there is a unique pair  $(\nu, \theta)$  which solves Eq. (2.6). Now we give two lemmas which will be used to conclude our assertion.

**Lemma 2.2.** *If  $z \in X_{\mathbb{C}}$  and  $\langle \sin(\frac{\pi}{l}(\cdot)), z \rangle = 0$ , then  $|\langle (dD^2 + \lambda_*)z, z \rangle| \geq 3\lambda_* \|z\|_{Y_{\mathbb{C}}}^2$ .*

This is exactly Lemma 2.3 of [3] and we omit its proof here.

**Lemma 2.3.** *If  $0 < \lambda^* - \lambda_* \ll 1$  and  $(\nu, \theta, y)$  solves Eq. (2.6) with  $y (\neq 0) \in X_{\mathbb{C}}$ , then  $\frac{\nu}{\lambda - \lambda_*}$  is uniformly bounded for  $\lambda \in (\lambda_*, \lambda^*]$ .*

**Proof.** Noting that

$$\langle [A(\lambda) + \lambda u_{\lambda} f'(u_{\lambda})e^{-i\theta} - i\nu]y, y \rangle = 0,$$

and also  $A(\lambda)$  is self-adjoint, then separating the real and imaginary parts of the above equality, we obtain

$$\nu \langle y, y \rangle = -\langle \lambda \sin \theta u_{\lambda} f'(u_{\lambda})y, y \rangle.$$

Hence

$$\frac{|\nu|}{\lambda - \lambda_*} = \lambda \alpha_{\lambda} |\sin \theta| \left\langle f'(u_{\lambda}) \left[ \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi_{\lambda} \right] y, y \right\rangle / \|y\|_{Y_{\mathbb{C}}}^2.$$

It follows from the boundedness of  $f'$  that there is a constant  $M > 0$  such that

$$\frac{|\nu|}{\lambda - \lambda_*} \leq \lambda \alpha_{\lambda} M [1 + (\lambda - \lambda_*) \|\xi_{\lambda}\|_{\infty}], \quad \lambda \in (\lambda_*, \lambda^*].$$

The boundedness of  $\nu/(\lambda - \lambda_*)$  follows from the continuity of  $\lambda \mapsto (\|\xi_{\lambda}\|_{\infty}, \alpha_{\lambda})$ .  $\square$

Now, for  $\lambda \in (\lambda_*, \lambda^*]$ , suppose that  $(\nu, \theta, y)$  is a solution of Eq. (2.6) with  $y (\neq 0) \in X_{\mathbb{C}}$ . If we ignore a scalar factor,  $y$  can be represented as

$$\begin{aligned} y &= \beta \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)z, \quad \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), z \right\rangle = 0, \quad \beta \geq 0, \\ \|y\|_{Y_{\mathbb{C}}}^2 &= \beta^2 \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_{\mathbb{C}}}^2 + (\lambda - \lambda_*)^2 \|z\|_{Y_{\mathbb{C}}}^2 = \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_{\mathbb{C}}}^2. \end{aligned} \tag{2.7}$$

Substituting (2.2), (2.7) and  $\nu = (\lambda - \lambda_*)h$  into Eq. (2.6), we obtain the equivalent system to Eq. (2.6):

$$\begin{aligned} g_1(z, \beta, h, \theta, \lambda) &\stackrel{\text{def}}{=} (dD^2 + \lambda_*)z + \left[ \beta \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)z \right] \\ &\cdot \left[ 1 + \lambda m_1(\xi_{\lambda}, \alpha_{\lambda}, \lambda) + \lambda \alpha_{\lambda} f'(u_{\lambda})e^{-i\theta} \left[ \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi_{\lambda} \right] - ih \right] = 0, \\ g_2(z) &\stackrel{\text{def}}{=} \text{Re} \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), z \right\rangle = 0, \\ g_3(z) &\stackrel{\text{def}}{=} \text{Im} \left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), z \right\rangle = 0, \end{aligned}$$

$$g_4(z, \beta, \lambda) \stackrel{\text{def}}{=} (\beta^2 - 1) \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_{\mathbb{C}}}^2 + (\lambda - \lambda_*)^2 \|z\|_{Y_{\mathbb{C}}}^2 = 0. \tag{2.8}$$

Recall that  $m_1(\xi, \alpha, \lambda)$  is defined in (2.3). We define  $G : X_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \mapsto Y_{\mathbb{C}} \times \mathbb{R}^3$  by  $G = (g_1, g_2, g_3, g_4)$  and note

$$z_{\lambda_*} = (1 - i)\xi_{\lambda_*}, \quad \beta_{\lambda_*} = 1, \quad h_{\lambda_*} = 1, \quad \theta_{\lambda_*} = \frac{\pi}{2}, \tag{2.9}$$

with  $\xi_{\lambda_*}$  defined as in Theorem 2.1. An easy calculation shows that

$$G(z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = 0.$$

Now we are in the position to give the main theorem in this section.

**Theorem 2.4.** *There exists a continuously differentiable mapping  $\lambda \mapsto (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$  from  $[\lambda_*, \lambda^*]$  to  $X_{\mathbb{C}} \times \mathbb{R}^3$  such that  $G(z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$ . Moreover, if  $\lambda \in (\lambda_*, \lambda^*)$ , and  $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda)$  solves the equation  $G = 0$  with  $h^\lambda > 0$ , and  $\theta^\lambda \in [0, 2\pi)$ , then  $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) = (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$ .*

**Proof.** Let  $T = (T_1, T_2, T_3, T_4) : X_{\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}^3$  be defined by

$$T = D_{(z, \beta, h, \theta)} G(z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*).$$

Thus, we have

$$T_1(\chi, \kappa, \epsilon, \vartheta) = (dD^2 + \lambda_*)\chi - i\epsilon \sin\left(\frac{\pi}{l}(\cdot)\right) - \lambda_* \vartheta \alpha_{\lambda_*} f'(0) \sin^2\left(\frac{\pi}{l}(\cdot)\right) + \kappa(1 - i) \sin\left(\frac{\pi}{l}(\cdot)\right) \left[ 1 + \lambda_* \alpha_{\lambda_*} f'(0) \sin\left(\frac{\pi}{l}(\cdot)\right) \right],$$

$$T_2(\chi) = \text{Re}\left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \chi \right\rangle, \quad T_3(\chi) = \text{Im}\left\langle \sin\left(\frac{\pi}{l}(\cdot)\right), \chi \right\rangle, \quad T_4(\kappa) = 2\kappa \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_{\mathbb{C}}}^2.$$

It is routine to verify that  $T$  is bijective from  $X_{\mathbb{C}} \times \mathbb{R}^3$  to  $Y_{\mathbb{C}} \times \mathbb{R}^3$ . It follows from the implicit function theorem that there exists a continuously differentiable mapping  $\lambda \mapsto (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$  from  $[\lambda_*, \lambda^*]$  (with a smaller  $\lambda^*$ ) to  $X_{\mathbb{C}} \times \mathbb{R}^3$  such that  $G(z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$ . Hence the existence is proved, and it remains to prove the uniqueness. By virtue of the uniqueness of the implicit function theorem, now we need only to show that if  $G(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda) = 0$ ,  $h^\lambda > 0$  and  $\theta^\lambda \in [0, 2\pi)$ , then

$$(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) \rightarrow (z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}) = \left( (1 - i)\xi_{\lambda_*}, 1, 1, \frac{\pi}{2} \right)$$

as  $\lambda \rightarrow \lambda_*$  in the norm of  $X_{\mathbb{C}} \times \mathbb{R}^3$ . From the definition of  $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda)$ , it is easy to see that  $\{h^\lambda\}$ ,  $\{\beta^\lambda\}$  and  $\{\theta^\lambda\}$  are bounded. From Lemma 2.2 and the first equation of Eq. (2.8) we have

$$\|z^\lambda\|_{Y_{\mathbb{C}}}^2 \leq \frac{1}{3\lambda_*} \left| \left\langle \varrho(h^\lambda, \theta^\lambda, \lambda) \left[ \beta^\lambda \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)z^\lambda \right], z^\lambda \right\rangle \right|,$$

where

$$\varrho(h^\lambda, \theta^\lambda, \lambda) = 1 + \lambda \alpha_\lambda f'(u_\lambda) e^{-i\theta} \left[ \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)\xi_\lambda \right] + \lambda m_1(\alpha_\lambda, \xi_\lambda, \lambda) - ih.$$

The boundedness of  $\{h^\lambda\}$ ,  $\{\alpha_\lambda\}$ ,  $\{\xi_\lambda\}$  and  $|f'|$  yields that there is  $M > 0$  such that  $\|Q(h^\lambda, \theta^\lambda, \lambda)\|_\infty \leq 3\lambda_*M$  for  $\lambda \in [\lambda_*, \lambda^*]$ . Thus we have

$$\|z^\lambda\|_{Y_C}^2 \leq M|\beta^\lambda| \cdot \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_C} \|z^\lambda\|_{Y_C} + M(\lambda - \lambda_*) \|z^\lambda\|_{Y_C}^2.$$

If  $M(\lambda^* - \lambda_*) < 1/2$ , then

$$\|z^\lambda\|_{Y_C} \leq 2M|\beta^\lambda| \cdot \left\| \sin\left(\frac{\pi}{l}(\cdot)\right) \right\|_{Y_C}, \quad \lambda \in [\lambda_*, \lambda^*].$$

Hence  $\{z^\lambda\}$  is bounded in  $Y_C$ . On the other hand,  $(dD^2 + \lambda_*) : X_C \cap \mathcal{R}_C(dD^2 + \lambda_*) \mapsto Y_C \cap \mathcal{R}_C(dD^2 + \lambda_*)$  has a bounded inverse, by applying  $(dD^2 + \lambda_*)^{-1}$  on  $g_1(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda) = 0$  one sees that  $\{z^\lambda\}$  is also bounded in  $X_C$ , and hence  $\{(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) : \lambda \in (\lambda_*, \lambda^*)\}$  is precompact in  $Y_C \cap \mathbb{R}^3$ . Therefore, there is a subsequence  $\{(z^n, \beta^n, h^n, \theta^n, \lambda^n)\}$  such that

$$\lambda^n \rightarrow \lambda_*, \quad (z^n, \beta^n, h^n, \theta^n) \rightarrow (z^{\lambda_*}, \beta^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}), \quad \text{as } n \rightarrow \infty,$$

by taking the limit of the equations  $G(z^n, \beta^n, h^n, \theta^n, \lambda^n) = 0$  as  $n \rightarrow \infty$ . It is not difficult to verify that  $G(z, \beta, h, \theta, \lambda_*) = 0$  has a unique solution given by  $(z, \beta, h, \theta) = (z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*)$  defined in (2.9), thus  $(z^{\lambda_*}, \beta^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}) = (z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$ . Hence,  $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) \rightarrow (z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$  as  $\lambda \rightarrow \lambda_*$  in the norm of  $Y_C \times \mathbb{R}^3$ . In addition,  $(dD^2 + \lambda_*)^{-1}$  is a continuous linear operator from  $\mathcal{R}_C(dD^2 + \lambda_*)$  into  $X_C \cap \mathcal{R}_C(dD^2 + \lambda_*)$ , we get the convergence in  $X_C \times \mathbb{R}^3$ , which follows that  $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) = (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$ .  $\square$

**Corollary 2.5.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each  $\lambda \in (\lambda_*, \lambda^*)$ , the eigenvalue problem*

$$\Delta(\lambda, iv, \tau)y = 0, \quad v \geq 0, \tau > 0, y(\neq 0) \in X_C$$

has a solution, or equivalently,  $iv \in \sigma(A_\tau(\lambda))$  if and only if

$$v = v_\lambda = (\lambda - \lambda_*)h_\lambda, \quad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{v_\lambda}, \quad n = 0, 1, 2, \dots \tag{2.10}$$

and

$$y = ry_\lambda, \quad y_\lambda = \beta_\lambda \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*)z_\lambda,$$

where  $r$  is a nonzero constant, and  $z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda$  are defined as in Theorem 2.4.

### 3. Stability of steady state solutions

In this section we study the stability of non-constant steady state solution  $u_\lambda$  of Eq. (1.1) with fixed  $\lambda$  satisfying  $0 < \lambda - \lambda_* \ll 1$ , and the time delay  $\tau$  is considered as a parameter. We recall a lemma from So and Yang [31, Lemma 4.1]:

**Lemma 3.1.** *Suppose that  $\psi$  and  $\phi$  satisfy*

$$\begin{aligned} -dD^2\psi + \lambda P(x)\psi &= 0, \quad \text{for } x \in (0, l), \\ \psi(0) &= \psi(l) = 0 \end{aligned}$$



and

$$\begin{aligned}
 -dD^2\phi + \lambda Q(x)\phi &= 0, \quad \text{for } x \in (0, l), \\
 \phi(0) = \phi(l) &= 0.
 \end{aligned}$$

Suppose  $\phi > 0$  for  $x \in (0, l)$ ,  $\lambda > 0$  and  $P(x) > Q(x)$  for  $x \in (0, l)$ . Then  $\psi \equiv 0$ .

We have the following simple facts about the spectral sets.

**Lemma 3.2.** *If  $0 < \lambda^* - \lambda_* \ll 1$  and  $\tau \geq 0$ , then 0 is not an eigenvalue of  $A_\tau(\lambda)$  for  $\lambda \in (\lambda_*, \lambda^*]$ .*

**Proof.** If  $\mu = 0$  is an eigenvalue, then  $[A(\lambda) + \lambda u_\lambda f'(u_\lambda)]y = 0$  for some  $y \neq 0$ . Since  $A(\lambda) + \lambda u_\lambda f'(u_\lambda)$  is self-adjoint, then  $y$  is real-valued. Noting that  $-[dD^2 + \lambda f(u_\lambda)]u_\lambda = 0$  and  $-f(u_\lambda) - u_\lambda f'(u_\lambda) > -f(u_\lambda)$  for  $\lambda \in (\lambda_*, \lambda^*]$ , the result follows from Lemma 3.1.  $\square$

**Lemma 3.3.** *If  $0 < \lambda^* - \lambda_* \ll 1$  and  $\tau = 0$ , then all eigenvalues of  $A_\tau(\lambda)$  have negative real parts for  $\lambda \in (\lambda_*, \lambda^*]$ .*

**Proof.** When  $\tau = 0$ ,  $A_\tau(\lambda)$  is real self-adjoint, then all eigenvalues are real-valued. Noting that  $A(\lambda)u_\lambda = 0$  and  $u_\lambda > 0$ , then 0 is the principal eigenvalue of  $-A(\lambda)$  with eigenfunction  $u_\lambda$ , it follows that the eigenvalues of  $-A(\lambda)$  are nonnegative from Lemma 3.1. In particular  $-(A(\lambda)\phi, \phi) \geq 0$  for any  $\phi \in X_{\mathbb{C}}$ . If there are  $c \geq 0$  and  $\phi \in X_{\mathbb{C}} \setminus \{0\}$  such that  $\Delta(\lambda, c, 0)\phi = 0$ , then we get

$$0 = -(A(\lambda)\phi, \phi) - \int_0^l (\lambda u_\lambda f'(u_\lambda) - c)\phi^2 dx > 0, \quad \text{for } \lambda \in (\lambda_*, \lambda^*],$$

which is a contradiction.  $\square$

We now show that  $\mu = i\nu$  is a simple eigenvalue of  $A_{\tau_n}$  for  $n = 0, 1, 2, \dots$ . For this purpose, we first give the following lemma.

**Lemma 3.4.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each fixed  $\lambda \in (\lambda_*, \lambda^*]$ ,*

$$S_n \stackrel{\text{def}}{=} \int_0^l [1 + \lambda \tau_n e^{-i\theta_\lambda} u_\lambda f'(u_\lambda)] y_\lambda^2(x) dx \neq 0, \quad n = 0, 1, 2, \dots$$

**Proof.** From the expressions of  $u_\lambda$ ,  $y_\lambda$ ,  $\tau_n$ , and the fact that  $\theta_\lambda \rightarrow \pi/2$  as  $\lambda \rightarrow \lambda_*$ , it is easy to obtain

$$S_n \rightarrow \left[ 1 + i \left( \frac{\pi}{2} + 2n\pi \right) \right] \int_0^l \sin^2 \left( \frac{\pi}{l} x \right) dx, \quad \text{as } \lambda \rightarrow \lambda_*. \tag{3.1}$$

It follows that  $S_n \neq 0$  for  $\lambda \in (\lambda_*, \lambda^*]$  and for all  $\tau_n$ ,  $n = 0, 1, 2, \dots$ .  $\square$

**Theorem 3.5.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each fixed  $\lambda \in (\lambda_*, \lambda^*]$ ,  $\mu = i\nu_\lambda$  is a simple eigenvalue of  $A_{\tau_n}$  for  $n = 0, 1, 2, \dots$*

**Proof.** From Corollary 2.5 we have  $\mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda] = \text{Span}[e^{i\nu_\lambda \cdot} y_\lambda]$ . Suppose that for some  $\phi \in \mathcal{D}(A_{\tau_n}(\lambda)) \cap \mathcal{D}([A_{\tau_n}(\lambda)]^2)$ , we have

$$[A_{\tau_n}(\lambda) - i\nu_\lambda]^2 \phi = 0.$$

This implies that

$$[A_{\tau_n}(\lambda) - i\nu_\lambda] \phi \in \mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda] = \text{Span}[e^{i\nu_\lambda \cdot} y_\lambda].$$

So there is a constant  $a$  such that

$$[A_{\tau_n}(\lambda) - i\nu_\lambda] \phi = a e^{i\nu_\lambda \cdot} y_\lambda.$$

Hence

$$\begin{aligned} \dot{\phi}(\theta) &= i\nu_\lambda \phi(\theta) + a e^{i\nu_\lambda \theta} y_\lambda, \quad \theta \in [-\tau_n, 0], \\ \dot{\phi}(0) &= A(\lambda)\phi(0) + \lambda u_\lambda f'(u_\lambda)\phi(-\tau_n). \end{aligned} \tag{3.2}$$

The first equation of Eq. (3.2) yields

$$\begin{aligned} \phi(\theta) &= \phi(0)e^{i\nu_\lambda \theta} + a\theta e^{i\nu_\lambda \theta} y_\lambda, \\ \dot{\phi}(0) &= i\nu_\lambda \phi(0) + a y_\lambda. \end{aligned} \tag{3.3}$$

From Eqs. (3.2) and (3.3) we have

$$\begin{aligned} \Delta(\lambda, i\nu, \tau_n)\phi(0) &= [A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-i\theta_\lambda} - i\nu_\lambda]\phi(0) \\ &= a(1 + \lambda \tau_n u_\lambda f'(u_\lambda)e^{-i\theta_\lambda})y_\lambda. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \int_0^l \phi(0)[\Delta(\lambda, i\nu, \tau_n)y_\lambda] dx = \int_0^l y_\lambda[\Delta(\lambda, i\nu, \tau_n)\phi(0)] dx \\ &= a \int_0^l (1 + \lambda \tau_n u_\lambda f'(u_\lambda)e^{-i\theta_\lambda})y_\lambda^2 dx. \end{aligned}$$

As a consequence of Lemma 3.4 we have  $a = 0$ , which leads to that  $\phi \in \mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda]$ . By induction we obtain

$$\mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda]^j = \mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda], \quad j = 1, 2, 3, \dots, n = 0, 1, 2, \dots$$

Therefore,  $\lambda = i\nu_\lambda$  is a simple eigenvalue of  $A_{\tau_n}$  for  $n = 0, 1, 2, \dots$ .  $\square$

Since  $\mu = i\nu$  is a simple eigenvalue of  $A_{\tau_n}$ , by using the implicit function theorem it is not difficult to show that there are a neighborhood  $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$  of  $(\tau_n, i\nu_\lambda, y_\lambda)$  and a continuously differential function  $(\mu, y) : O_n \rightarrow D_n \times H_n$  such that for each  $\tau \in O_n$ , the only eigenvalue of  $A_\tau(\lambda)$  in  $D_n$  is  $\mu(\tau)$ , and

$$\begin{aligned} \mu(\tau_n) &= i\nu_\lambda, & y(\tau_n) &= y_\lambda, \\ \Delta(\lambda, \mu(\tau), \tau) &= [A(\lambda) + \lambda u_\lambda f'(u_\lambda) e^{-\mu(\tau)\tau} - \mu(\tau)]y(\tau) = 0, & \tau &\in O_n. \end{aligned} \tag{3.4}$$

We show that  $\mu(\tau)$  move across  $\tau_n$  transversally.

**Theorem 3.6.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each  $\lambda \in (\lambda_*, \lambda^*]$ ,*

$$\operatorname{Re} \frac{d\mu(\tau_n)}{d\tau} > 0, \quad n = 0, 1, 2, \dots$$

**Proof.** Differentiating Eq. (3.4) with respect to  $\tau$  at  $\tau = \tau_n$ , we have

$$\frac{d\mu(\tau_n)}{d\tau} [-1 - \lambda u_\lambda \tau_n f'(u_\lambda) e^{-i\theta_\lambda}] y_\lambda + \Delta(\lambda, i\nu_\lambda, \tau_n) \frac{dy(\tau_n)}{d\tau} - i\nu_\lambda \lambda u_\lambda f'(u_\lambda) e^{-i\theta_\lambda} y_\lambda = 0.$$

Multiplying the equation by  $y_\lambda$  and integrating on  $[0, l]$ , we obtain

$$\begin{aligned} \frac{d\mu(\tau_n)}{d\tau} &= \frac{-i\nu_\lambda \lambda e^{-i\theta_\lambda} \int_0^l u_\lambda f'(u_\lambda) y_\lambda^2 dx}{\int_0^l [1 + \lambda \tau_n u_\lambda f'(u_\lambda) e^{-i\theta_\lambda}] y_\lambda^2 dx} \\ &= \frac{1}{|S_n|^2} \left( -i\nu_\lambda \lambda e^{-i\theta_\lambda} \int_0^l y_\lambda^2 dx \int_0^l u_\lambda f'(u_\lambda) y_\lambda^2 dx - i\nu_\lambda \lambda^2 \tau_n \left| \int_0^l u_\lambda f'(u_\lambda) y_\lambda^2 dx \right|^2 \right). \end{aligned} \tag{3.5}$$

Noting that

$$\overline{\int_0^l y_\lambda^2(x) dx} = \left| \int_0^l y_\lambda^2(x) dx \right| e^{-i\rho_\lambda},$$

where  $\rho_\lambda = \operatorname{Arg}(\int_0^l y_\lambda^2(x) dx)$ ,  $-\pi < \rho_\lambda \leq \pi$ . Then from Eq. (3.5) it follows that

$$\operatorname{Re} \frac{d\mu(\tau_n)}{d\tau} = \frac{-\nu_\lambda \lambda (\lambda - \lambda_*)}{|S_n|^2} \left| \int_0^l y_\lambda^2 dx \right| \operatorname{Re} \left\{ i e^{-i(\theta_\lambda + \rho_\lambda)} \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} dx \right\}.$$

Hence, by

$$i e^{-i(\theta_\lambda + \rho_\lambda)} \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} \rightarrow -\frac{1}{\lambda_*} \int_0^l \sin^2\left(\frac{\pi}{l}x\right) dx < 0 \quad \text{as } \lambda \rightarrow \lambda_*,$$

we have  $\operatorname{Re} \frac{d\mu(\tau_n)}{d\tau} > 0$  when  $\lambda \in (\lambda_*, \lambda^*]$ .  $\square$

From Lemma 3.2 and Theorem 3.6 we immediately have

**Theorem 3.7.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each fixed  $\lambda \in (\lambda_*, \lambda^*]$ , the infinitesimal generator  $A_\tau(\lambda)$  has exactly  $2(n + 1)$  eigenvalues with positive real part when  $\tau \in (\tau_n, \tau_{\lambda_{n+1}}]$ ,  $n = 0, 1, 2, \dots$*

The stability of the positive steady state solution can be obtained from Lemmas 3.2 and 3.3, Corollary 2.5 and Theorem 3.7.

**Theorem 3.8.** *If  $0 < \lambda^* - \lambda_* \ll 1$ , then for each fixed  $\lambda \in (\lambda_*, \lambda^*)$ , the positive steady state solution  $u_\lambda$  of Eq. (1.1) is asymptotically stable when  $\tau \in [0, \tau_0)$  and is unstable when  $\tau \in (\tau_0, \infty)$ .*

**4. Hopf bifurcation**

In this section we study the Hopf bifurcation occurring around the positive steady state solution  $u_\lambda$  with  $\tau$  as a bifurcation parameter by using the method in [3].

We first transform the steady state  $u = u_\lambda$  of Eq. (1.1) and the critical value  $\tau_n$  to the origin via the translations  $U(t) = U(\cdot, t) = u(\cdot, t) - u_\lambda(\cdot)$  and  $\alpha = \tau - \tau_n$ , then Eq. (1.1) is transformed into

$$\begin{aligned} \frac{\partial U(t)}{\partial t} &= A(\lambda)U(t) + \lambda u_\lambda f'(u_\lambda)U(t - \tau_n - \alpha) + \lambda f'(u_\lambda)U(t)U(t - \tau_n - \alpha) \\ &\quad + \lambda(U(t) + u_\lambda)[f(U(t - \tau_n - \alpha) + u_\lambda) - f(u_\lambda) - f'(u_\lambda)U(t - \tau_n - \alpha)]. \end{aligned} \tag{4.1}$$

Furthermore let  $\omega_\lambda = 2\pi/\nu_\lambda$  and  $w(t) = U(t(1 + \beta))$ , then  $U(t)$  is an  $\omega_\lambda(1 + \beta)$ -periodic solution of Eq. (4.1) if and only if  $w(t)$  is an  $\omega_\lambda$ -periodic solution of

$$\frac{dw(t)}{dt} = A(\lambda)w(t) + \lambda u_\lambda f'(u_\lambda)w(t - \tau_n) + G(\alpha, \beta, w_t), \tag{4.2}$$

where

$$\begin{aligned} G(\alpha, \beta, w_t) &= \beta A(\lambda)w(t) - \lambda u_\lambda f'(u_\lambda) \left[ w(t - \tau_n) - (1 + \beta)w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) \right] \\ &\quad + \lambda(1 + \beta)(w(t) + u_\lambda) \left[ f\left(w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) + u_\lambda\right) - f'(u_\lambda)w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) - f(u_\lambda) \right] \\ &\quad + \lambda(1 + \beta)f'(u_\lambda)w(t)w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) \\ &= \beta A(\lambda)w(t) - \lambda u_\lambda f'(u_\lambda) \left[ w(t - \tau_n) - (1 + \beta)w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) \right] \\ &\quad + \lambda(1 + \beta)(w(t) + u_\lambda) \left[ \frac{f''(u_\lambda)}{2}w^2\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) + \frac{f'''(u_\lambda)}{3!}w^3\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right) \right. \\ &\quad \left. + \mathcal{O}\left(w^4\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right)\right) \right] + \lambda(1 + \beta)f'(u_\lambda)w(t)w\left(t - \frac{\tau_n + \alpha}{1 + \beta}\right). \end{aligned}$$

We use the following notation:

- (1)  $\langle y, z \rangle^* \stackrel{\text{def}}{=} \int_0^l y(x)z(x) dx, y, z \in Y.$
- (2) From [42] or [37], for  $\phi \in \mathcal{C}_n \stackrel{\text{def}}{=} C((-\tau_n, 0]; Y), \psi \in \mathcal{C}_n^* \stackrel{\text{def}}{=} C([0, \tau_n]; Y),$

$$\langle \psi, \phi \rangle = \langle \psi(0), \phi(0) \rangle^* + \int_{-\tau_n}^0 \langle \psi(s + \tau_n), \lambda u_\lambda f'(u_\lambda)\phi(s) \rangle^* ds.$$

$$\begin{aligned}
 (3) \quad & \tilde{\Phi}(\theta) = [y_\lambda e^{i\nu_\lambda \theta}, \bar{y}_\lambda e^{-i\nu_\lambda \theta}], \quad \theta \in [-\tau_n, 0], \\
 & \tilde{\Psi}(s) = \begin{bmatrix} \frac{1}{S_n} y_\lambda e^{-i\nu_\lambda s} \\ \frac{1}{S_n} \bar{y}_\lambda e^{i\nu_\lambda s} \end{bmatrix}, \quad s \in [0, \tau_n], \\
 & \Phi(\theta) = [\Phi_1(\theta), \Phi_2(\theta)] = \tilde{\Phi}(\theta)H, \\
 & \Psi(s) = \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{bmatrix} = H^{-1} \tilde{\Psi}(s),
 \end{aligned}$$

where

$$H = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

- (4)  $\Lambda$  is the eigen-space of  $A_{\tau_n}(\lambda)$  corresponding to the eigenvalues  $\mu = \pm i\nu_\lambda$ .
- (5) Define

$$P_{\omega_\lambda} = \{f \in C(\mathbb{R}; Y) : f(t + \omega_\lambda) = f(t), t \in \mathbb{R}\}$$

with the norm  $\|\cdot\|_{P_{\omega_\lambda}}$  defined by  $\|f\|_{P_{\omega_\lambda}} = \sup_{t \in [0, \omega_\lambda]} \|f(t)\|_Y$  for  $f \in P_{\omega_\lambda}$ .

- (6)  $\mathcal{I} : P_{\omega_\lambda} \rightarrow \mathbb{R}$  is defined by  $\mathcal{I}f = \int_0^{\omega_\lambda} \langle \Psi(s), f(s) \rangle^* ds$ .

With the above notations, we can verify that  $\Phi$  is a real-valued basis of  $\Lambda$ , and  $\Psi$  is a real-valued basis of the eigen-space of the formal adjoint operator  $A_{\tau_n}^*(\lambda)$  of  $A_{\tau_n}(\lambda)$  corresponding to  $\mu = \pm i\nu_\lambda$  and  $(\Psi, \Phi) = I$ , where  $I \in \mathbb{R}^{2 \times 2}$  is the identity matrix and  $(\cdot, \cdot)$  here is the matrix multiplication. Furthermore, for  $\phi \in \mathcal{C}_n$ , the projection  $\phi^\Lambda$  of  $\phi$  onto  $\Lambda$  is  $\phi^\Lambda = \Phi(\Psi, \phi)$ .

In order to give the main result, we state the following lemma in [3]:

**Lemma 4.1.** For  $g \in P_{\omega_\lambda}$ , the equation

$$\frac{dw(t)}{dt} = A(\lambda)w(t) + \lambda u_\lambda f'(u_\lambda)w(t - \tau_n) + g \tag{4.3}$$

has an  $\omega_\lambda$ -periodic solution if and only if  $g \in \mathcal{N}(\mathcal{I})$ . Hence there is a linear operator  $\mathcal{K}$  from  $\mathcal{N}(\mathcal{I})$  to  $P_{\omega_\lambda}$  such that for each fixed  $g \in \mathcal{N}(\mathcal{I})$ ,  $\mathcal{K}g$  is the  $\omega_\lambda$ -periodic solution of Eq. (4.3) satisfying  $(\mathcal{K}g)_0^\Lambda = 0$ , where  $(\mathcal{K}g)_0 \in \mathcal{C}_n$  is defined by  $(\mathcal{K}g)_0(\theta) = (\mathcal{K}g)(\theta)$ ,  $\theta \in [-\tau_n, 0]$ .

From Lemma 4.1 and [3], up to a time translation, Eq. (4.2) has an  $\omega_\lambda$ -periodic solution  $w(t)$  if and only if there exists  $a \in \mathbb{R}$  such that

$$\begin{aligned}
 & \mathcal{I}G(\alpha, \beta, w_t) = 0, \\
 & w(t) = a\Phi_1(t) + [\mathcal{K}G(\alpha, \beta, w_t)](t), \quad t \in \mathbb{R},
 \end{aligned} \tag{4.4}$$

where  $\Phi_1(t) = \frac{1}{2}(y_\lambda e^{i\nu_\lambda t} + \bar{y}_\lambda e^{-i\nu_\lambda t}) = \text{Re}(y_\lambda e^{i\nu_\lambda t})$ ,  $t \in \mathbb{R}$ . We further introduce the change of variables by  $\alpha = a\gamma$ ,  $\beta = a\delta$ ,  $w(t) = a[\Phi_1(t) + aW(t)]$ , for  $t \in \mathbb{R}$ ,  $W \in P_{\omega_\lambda}$ , then Eq. (4.4) is equivalent to

$$\begin{aligned}
 \mathcal{I}(a, \gamma, \delta, W) & \stackrel{\text{def}}{=} \int_0^{\omega_\lambda} \langle \Psi(s), N(a, \gamma, \delta, W_s) \rangle^* ds = 0, \\
 W & = \mathcal{K}N(a, \gamma, \delta, W),
 \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
 N(a, \gamma, \delta, W_t) &= \delta A(\lambda) [\Phi_1(t) + aW(t)] \\
 &\quad - \lambda u_\lambda f'(u_\lambda) \left[ \frac{\gamma - \delta \tau_n}{1 + a\delta} \int_0^1 \dot{\Phi}_1 \left( t - \tau_n - \theta \frac{a\gamma - a\delta \tau_n}{1 + a\delta} \right) d\theta \right. \\
 &\quad \left. - \delta \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + W(t - \tau_n) - (1 + a\delta)W \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right] \\
 &\quad + \lambda(1 + a\delta) \left\{ f'(u_\lambda) [\Phi_1(t) + aW(t)] \left[ \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + aW \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right] \right. \\
 &\quad \left. + M(a, \gamma, \delta, W_t) [a(\Phi_1(t) + aW(t)) + u_\lambda] \right\} \\
 &= \delta A(\lambda) [\Phi_1(t) + aW(t)] - \lambda u_\lambda f'(u_\lambda) \left[ \frac{\gamma - \delta \tau_n}{1 + a\delta} \int_0^1 \dot{\Phi}_1 \left( t - \tau_n - \theta \frac{a\gamma - a\delta \tau_n}{1 + a\delta} \right) d\theta \right. \\
 &\quad \left. - \delta \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + W(t - \tau_n) - (1 + a\delta)W \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right] \\
 &\quad + \lambda(1 + a\delta) \left\{ f'(u_\lambda) [\Phi_1(t) + aW(t)] \left[ \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + aW \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right] \right. \\
 &\quad \left. + [a(\Phi_1(t) + aW(t)) + u_\lambda] \left[ \frac{f''(u_\lambda)}{2} \left[ \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + aW \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right]^2 \right. \right. \\
 &\quad \left. \left. + \frac{af'''(u_\lambda)}{3!} \left[ \Phi_1 \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) + aW \left( t - \frac{\tau_n + a\gamma}{1 + a\delta} \right) \right]^3 + o(a) \right] \right\},
 \end{aligned}$$

and

$$\begin{aligned}
 M(a, \gamma, \delta, W_t) &= \begin{cases} \{ f[a\Phi_1(t - \frac{\tau_n + a\gamma}{1 + a\delta}) + a^2W(t - \frac{\tau_n + a\gamma}{1 + a\delta}) + u_\lambda] - f(u_\lambda) \\ \quad - af'(u_\lambda)(\Phi_1(t - \frac{\tau_n + a\gamma}{1 + a\delta}) + aW(t - \frac{\tau_n + a\gamma}{1 + a\delta})) \} / a^2, & \text{if } a \neq 0, \\ f''(u_\lambda)\Phi_1^2(t - \tau_n)/2, & \text{if } a = 0. \end{cases}
 \end{aligned}$$

Since a periodic solution of Eq. (4.2) is a  $C_n^1$  function, where  $C_n^1 \stackrel{\text{def}}{=} C^1((-\tau_n, 0]; Y)$ , without loss of generality, we can restrict the discussion on Eq. (4.5) to  $W \in P_{\omega_\lambda}^1$ , where  $P_{\omega_\lambda}^1 = \{f \in P_{\omega_\lambda} : \dot{f} \in P_{\omega_\lambda}\}$ ,  $\|f\|_{P_{\omega_\lambda}^1} = \|f\|_{P_{\omega_\lambda}} + \|\dot{f}\|_{P_{\omega_\lambda}}$ .

**Lemma 4.2.** For any  $W \in P_{\omega_\lambda}^1$ ,  $\mathcal{S}(0, 0, 0, W) = 0$ .

**Proof.** From the definition of  $N$ ,

$$\begin{aligned}
 N(0, 0, 0, W) &= \lambda f'(u_\lambda)\Phi_1(t)\Phi_1(t - \tau_n) + \frac{\lambda}{2}u_\lambda f''(u_\lambda)\Phi_1^2(t - \tau_n) \\
 &= \frac{\lambda}{4} \left[ f'(u_\lambda)e^{-i\theta\lambda} + \frac{1}{2}u_\lambda f''(u_\lambda) \right] y_\lambda^2 e^{2i\nu_\lambda t}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\lambda}{4} \left[ f'(u_\lambda) e^{i\theta_\lambda} + \frac{1}{2} u_\lambda f''(u_\lambda) \right] \bar{y}_\lambda^2 e^{-2i\nu_\lambda t} \\
 & + \frac{\lambda}{2} \left[ f'(u_\lambda) \cos \theta_\lambda + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda \bar{y}_\lambda.
 \end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned}
 \mathcal{F}(0, 0, 0, W) &= \int_0^{\omega_\lambda} \langle \Psi(s), N(0, 0, 0, W) \rangle^* ds \\
 &= \frac{\lambda}{8} \operatorname{Re} \int_0^l \frac{1}{S_n} \left[ f'(u_\lambda) e^{-i\theta_\lambda} + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda^3 dx \int_0^{\omega_\lambda} e^{i\nu_\lambda s} ds \\
 &+ \frac{\lambda}{8} \operatorname{Re} \int_0^l \frac{1}{\bar{S}_n} \left[ f'(u_\lambda) e^{-i\theta_\lambda} + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda^2 \bar{y}_\lambda dx \int_0^{\omega_\lambda} e^{3i\nu_\lambda s} ds \\
 &+ \frac{\lambda}{4} \operatorname{Re} \int_0^l \frac{1}{\bar{S}_n} \left[ f'(u_\lambda) \cos \theta_\lambda + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda^2 \bar{y}_\lambda dx \int_0^{\omega_\lambda} e^{-i\nu_\lambda s} ds \\
 &= 0. \quad \square
 \end{aligned}$$

From the assumptions on  $f$ , one can see that  $\mathcal{F} : I_r \times \mathbb{R} \times I_r \times P_{\omega_\lambda}^1 \rightarrow \mathbb{R}$  is continuously differentiable, where  $I_r = [-r, r]$  with some  $r \in (0, 1)$ . Furthermore we have the following conclusion.

**Lemma 4.3.** For any  $W \in P_{\omega_\lambda}^1$ ,

$$\frac{\partial \mathcal{F}(0, 0, 0, W)}{\partial (\gamma, \delta)} = \omega_\lambda \begin{bmatrix} \operatorname{Re} \dot{\mu}(\tau_n) & 0 \\ -\operatorname{Im} \dot{\mu}(\tau_n) & -\nu_\lambda \end{bmatrix}.$$

**Proof.** By the definitions of  $\mathcal{F}$ ,  $\Phi$ , and  $\Psi$ , it follows that

$$\begin{aligned}
 \frac{\partial \mathcal{F}(0, 0, 0, W)}{\partial \gamma} &= \int_0^{\omega_\lambda} \left\langle \Psi(s), \frac{\partial N}{\partial \gamma}(0, 0, 0, W_s) \right\rangle^* ds \\
 &= \int_0^{\omega_\lambda} \langle \Psi(s), -\lambda u_\lambda f'(u_\lambda) \dot{\Phi}_1(s - \tau_n) \rangle^* ds = \int_0^{\omega_\lambda} H^{-1} \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix} H \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds,
 \end{aligned}$$

where the  $2 \times 2$  matrix  $(a_{ij}^1)$  is given by

$$\begin{aligned}
 a_{11}^1 &= \frac{-i\nu_\lambda e^{-i\nu_\lambda \tau_n}}{S_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda^2 dx, & a_{12}^1 &= \frac{i\nu_\lambda e^{-i\nu_\lambda (2s - \tau_n)}}{S_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda \bar{y}_\lambda dx, \\
 a_{21}^1 &= \frac{-i\nu_\lambda e^{i\nu_\lambda (2s - \tau_n)}}{\bar{S}_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda \bar{y}_\lambda dx, & a_{22}^1 &= \frac{i\nu_\lambda e^{i\nu_\lambda \tau_n}}{\bar{S}_n} \int_0^l \lambda u_\lambda f'(u_\lambda) \bar{y}_\lambda^2 dx.
 \end{aligned}$$

By using Eq. (3.5), we obtain that

$$\frac{\partial \mathcal{F}(0, 0, 0, W)}{\partial \gamma} = \omega_\lambda H^{-1} \begin{bmatrix} \dot{\mu}(\tau_n) & 0 \\ 0 & \bar{\mu}(\tau_n) \end{bmatrix} H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \omega_\lambda \begin{bmatrix} \operatorname{Re} \dot{\mu}(\tau_n) \\ -\operatorname{Im} \dot{\mu}(\tau_n) \end{bmatrix}.$$

Similarly

$$\begin{aligned} \frac{\partial \mathcal{F}(0, 0, 0, W)}{\partial \delta} &= \int_0^{\omega_\lambda} \left\langle \Psi(s), \frac{\partial N}{\partial \delta}(0, 0, 0, W_s) \right\rangle^* ds \\ &= \int_0^{\omega_\lambda} \left\langle \Psi(s), A(\lambda)\Phi_1(t) - \lambda u_\lambda f(u_\lambda)\Phi_1(t - \tau_n) + \lambda u_\lambda f'(u_\lambda)\tau_n \dot{\Phi}_1(s - \tau_n) \right\rangle^* ds \\ &= \int_0^{\omega_\lambda} H^{-1} \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{pmatrix} H \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds \\ &= \omega_\lambda H^{-1} \begin{bmatrix} i v_\lambda & 0 \\ 0 & -i v_\lambda \end{bmatrix} H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \omega_\lambda \begin{bmatrix} 0 \\ -v_\lambda \end{bmatrix}, \end{aligned}$$

where the  $2 \times 2$  matrix  $(a_{ij}^2)$  is given by

$$\begin{aligned} a_{11}^2 &= \frac{i v_\lambda}{S_n} \int_0^l y_\lambda^2 dx + \frac{i v_\lambda \tau_n e^{-i v_\lambda \tau_n}}{S_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda^2 dx, \\ a_{12}^2 &= \frac{-i v_\lambda e^{-2i v_\lambda s}}{\bar{S}_n} \int_0^l y_\lambda \bar{y}_\lambda dx - \frac{i v_\lambda \tau_n e^{-i v_\lambda (2s - \tau_n)}}{S_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda \bar{y}_\lambda dx, \\ a_{21}^2 &= \frac{i v_\lambda e^{2i v_\lambda s}}{S_n} \int_0^l y_\lambda \bar{y}_\lambda dx + \frac{i v_\lambda \tau_n e^{i v_\lambda (2s - \tau_n)}}{\bar{S}_n} \int_0^l \lambda u_\lambda f'(u_\lambda) y_\lambda \bar{y}_\lambda dx, \\ a_{22}^2 &= \frac{-i v_\lambda}{\bar{S}_n} \int_0^l \bar{y}_\lambda^2 dx - \frac{i v_\lambda \tau_n e^{i v_\lambda \tau_n}}{\bar{S}_n} \int_0^l \lambda u_\lambda f'(u_\lambda) \bar{y}_\lambda^2 dx. \quad \square \end{aligned}$$

**Lemma 4.4.** Let

$$W_\lambda(t) = \zeta_\lambda^1 e^{2i v_\lambda t} + \bar{\zeta}_\lambda^1 e^{-2i v_\lambda t} + \zeta_\lambda^2 + \Phi(t)d, \tag{4.7}$$

where

$$\zeta_\lambda^1 = \frac{-\lambda}{4} [A(\lambda) + \lambda u_\lambda f'(u_\lambda) e^{-2i\theta_\lambda} - 2i v_\lambda]^{-1} \left[ u_\lambda f'(u_\lambda) e^{-i\theta_\lambda} + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda^2, \tag{4.8}$$

$$\zeta_\lambda^2 = \frac{-\lambda}{2} [A(\lambda) + \lambda u_\lambda f'(u_\lambda)]^{-1} \left[ f'(u_\lambda) \cos \theta_\lambda + \frac{1}{2} u_\lambda f''(u_\lambda) \right] y_\lambda \bar{y}_\lambda, \tag{4.9}$$



and

$$d = -(\Psi, \zeta_\lambda^1 e^{2iv_\lambda \cdot} + \bar{\zeta}_\lambda^1 e^{-2iv_\lambda \cdot} + \zeta_\lambda^2).$$

Then

$$W_\lambda = \mathcal{K}(N(0, 0, 0, W_\lambda)).$$

**Proof.** Recall the form of  $N(0, 0, 0, W)$  given in (4.6). Then it is a direct calculation to verify that  $W_\lambda$  defined by (4.7) is an  $\omega_\lambda$ -periodic solution of the equation

$$\frac{dw(t)}{dt} = A(\lambda)w(t) + \lambda u_\lambda f'(u_\lambda)w(t - \tau_n) + N(0, 0, 0, w).$$

Furthermore, with the  $d$  defined above, we can verify that  $(\Psi(s), (W_\lambda)_0) = 0$ . Thus by the definition of  $\mathcal{K}$ , we obtain  $W_\lambda = \mathcal{K}(N(0, 0, 0, W_\lambda))$ .  $\square$

From Theorem 3.6,  $\text{Re } \dot{\mu}(\tau_n) \neq 0$ , hence Lemma 4.3 implies  $\partial \mathcal{F}(0, 0, 0, W)/\partial(\gamma, \delta)$  is nondegenerate. Now from Lemmas 4.2, 4.3 and the implicit function theorem, there exist  $a_0 > 0$ , a neighborhood  $V_0 \subseteq P^1_{\omega_\lambda}$  of  $W_\lambda$ , and unique continuously differentiable functions  $\gamma, \delta: [-a_0, a_0] \times V_0 \rightarrow [-a_0, a_0]$  such that  $\gamma(0, W_\lambda) = \delta(0, W_\lambda) = 0$  and  $\mathcal{F}(a, \gamma(a, W), \delta(a, W), W) = 0$  for  $(a, W) \in [-a_0, a_0] \times V_0$ . Next we define a mapping  $\mathcal{F}: [-a_0, a_0] \times V_0 \rightarrow P^1_{\omega_\lambda}$  by

$$\mathcal{F}(a, W) = W - \mathcal{K}(N(a, \gamma(a, W), \delta(a, W), W)).$$

Then we have  $\mathcal{F}(0, W_\lambda) = 0$  by Lemma 4.4. Moreover Lemma 4.2 implies that

$$\frac{\partial \mathcal{F}(0, 0, 0, W_\lambda)}{\partial W} = 0, \quad \left[ \frac{\partial \gamma(0, W_\lambda)}{\partial W} \right] = - \left[ \frac{\partial \mathcal{F}(0, 0, 0, W_\lambda)}{\partial(\gamma, \delta)} \right]^{-1} \frac{\partial \mathcal{F}(0, 0, 0, W_\lambda)}{\partial W} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore we are able to verify that

$$\frac{\partial \mathcal{F}(0, W_\lambda)}{\partial W} = I - \mathcal{K} \frac{\partial N(0, 0, 0, W_\lambda)}{\partial W} = I.$$

Again from the implicit function theorem, there exist a constant  $a_1 \in (0, a_0)$ , a neighborhood  $V_1 \subseteq V_0$  of  $W_\lambda$ , and a unique continuously differentiable function  $W^*: [-a_1, a_1] \rightarrow V_1$ , such that  $W^*(0) = W_\lambda$  and  $\mathcal{F}(a, W^*(a)) = 0$ , for  $a \in [-a_1, a_1]$ .

Consequently, we obtain that for any fixed  $\lambda \in (\lambda_*, \lambda^*]$ , Eq. (4.2) has an  $\omega_\lambda$ -periodic solution  $W(t)$  near zero for small  $\alpha$  and  $\beta$  if and only if  $W(t) = a\Phi_1(t) + aW^*(a)$ ,  $\alpha = a\gamma(a, W^*(a))$ , and  $\beta = a\delta(a, W^*(a))$  for  $a \in [-a_1, a_1]$ .

Hence we obtain the existence of small amplitude periodic orbits near the non-constant steady state solution  $(\lambda, u_\lambda)$  and  $\tau = \tau_n$ . But to obtain more specific information of the bifurcation, we still need to calculate the value of  $\frac{\partial \gamma(0, W^*(0))}{\partial a}$  for the direction of the bifurcation with respect to the time delay parameter  $\tau$ . Now we note that  $\gamma^*(a) = \gamma(a, W^*(a))$ ,  $\delta^*(a) = \delta(a, W^*(a))$ . Since

$$\mathcal{F}(a, \gamma^*(a), \delta^*(a), W^*(a)) = 0, \quad a \in [-a_1, a_1],$$

differentiating both sides of the above equality at  $a = 0$  yields

$$\frac{\partial \mathcal{F}(0, 0, 0, W_\lambda)}{\partial a} + \frac{\partial \mathcal{F}(0, 0, 0, W_\lambda)}{\partial(\gamma, \delta)} \begin{bmatrix} \frac{d\gamma^*(0)}{da} \\ \frac{d\delta^*(0)}{da} \end{bmatrix} = 0.$$

By Lemma 4.3 we have

$$\left[ \begin{array}{c} \frac{d\gamma^*(0)}{da} \\ \frac{d\delta^*(0)}{da} \end{array} \right] = -\frac{1}{\omega_\lambda} \left[ \begin{array}{cc} \operatorname{Re} \dot{\mu}(\tau_n) & 0 \\ -\operatorname{Im} \dot{\mu}(\tau_n) & -\nu_\lambda \end{array} \right]^{-1} \frac{\partial \mathcal{T}(0, 0, 0, W_\lambda)}{\partial a}. \tag{4.10}$$

We first prove a lemma which will be used to determine the sign of  $\frac{d\gamma^*(0)}{da}$ .

**Lemma 4.5.** *Let  $\zeta_\lambda^1$  and  $\zeta_\lambda^2$  be defined as in Lemma 4.4. Then*

$$\lim_{\lambda \searrow \lambda_*} \zeta_\lambda^1(\lambda - \lambda_*) = \frac{-(2-i)\lambda_* f'(0) \int_0^l \sin^3(\frac{\pi}{l}x) dx}{20 \int_0^l \sin^2(\frac{\pi}{l}x) dx} \sin\left(\frac{\pi}{l}x\right)$$

and

$$\lim_{\lambda \searrow \lambda_*} \zeta_\lambda^2(\lambda - \lambda_*) = 0.$$

**Proof.** For  $\lambda \in (\lambda_*, \lambda^*]$ , we decompose  $\zeta_\lambda^i$ ,  $i = 1, 2$ , as

$$\zeta_\lambda^i = \frac{1}{\lambda - \lambda_*} \left[ m_\lambda^i \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*) \zeta_\lambda^{i*} \right], \quad i = 1, 2,$$

where  $m_\lambda^i \in \mathbb{C}$ ,  $\langle \sin(\frac{\pi}{l}(\cdot)), \zeta_\lambda^{i*} \rangle = 0$ . From the definition of  $\zeta_\lambda^1$ , we can obtain

$$\begin{aligned} & (dD^2 + \lambda_*) \zeta_\lambda^{1*} + m_\lambda^1 J_\lambda^1 \sin\left(\frac{\pi}{l}(\cdot)\right) + (\lambda - \lambda_*) J_\lambda^1 \zeta_\lambda^{1*} \\ &= -\frac{1}{4} \left[ \lambda f'(u_\lambda) e^{-i\theta_\lambda} + \frac{\lambda}{2} u_\lambda f''(u_\lambda) \right] y_\lambda^2, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} J_\lambda^1(x) &= 1 - 2ih_\lambda + \lambda \alpha_\lambda [f'(u_\lambda) e^{-2i\theta_\lambda} + f'(0)] \left( \sin\left(\frac{\pi}{l}x\right) + (\lambda - \lambda_*) \xi_\lambda \right) \\ &+ \frac{\lambda}{\lambda - \lambda_*} (f(u_\lambda) - 1 - u_\lambda f'(0)) \\ &= 1 - 2ih_\lambda + \lambda \alpha_\lambda f'(0) \sin\left(\frac{\pi}{l}x\right) + \lambda \alpha_\lambda f'(u_\lambda) e^{-2i\theta_\lambda} \sin\left(\frac{\pi}{l}x\right) + o(1). \end{aligned}$$

From Eq. (4.11) we obtain that

$$\begin{aligned} m_\lambda^1 &= \frac{-\frac{1}{4} \int_0^l [\lambda f'(u_\lambda) e^{-i\theta_\lambda} + \frac{\lambda}{2} u_\lambda f''(u_\lambda)] y_\lambda^2 \sin(\frac{\pi}{l}x) dx - (\lambda - \lambda_*) \int_0^l J_\lambda^1 \zeta_\lambda^{1*} \sin(\frac{\pi}{l}x) dx}{\int_0^l J_\lambda^1 \sin^2(\frac{\pi}{l}x) dx} \\ &= \frac{-\frac{1}{4} \int_0^l \lambda f'(u_\lambda) e^{-i\theta_\lambda} y_\lambda^2 \sin(\frac{\pi}{l}x) dx - (\lambda - \lambda_*) \int_0^l J_\lambda^1 \zeta_\lambda^{1*} \sin(\frac{\pi}{l}x) dx + o(1)}{\int_0^l J_\lambda^1 \sin^2(\frac{\pi}{l}x) dx}. \end{aligned} \tag{4.12}$$

Applying (4.12) and Lemma 2.2 to Eq. (4.11), we see that  $\|r_{\lambda}^{1*}\|_Y < \infty$  for  $\lambda \in (\lambda_*, \lambda^*]$ . Meanwhile we have

$$\int_0^l J_{\lambda}^1 \sin^2\left(\frac{\pi}{l}x\right) dx \rightarrow \int_0^l (1 - 2i) \sin^2\left(\frac{\pi}{l}x\right) dx, \quad \text{as } \lambda \rightarrow \lambda_*.$$

Therefore, from (4.12) it follows that

$$\lim_{\lambda \searrow \lambda_*} m_{\lambda}^1 = \frac{-(2 - i)\lambda_* f'(0) \int_0^l \sin^3\left(\frac{\pi}{l}x\right) dx}{20 \int_0^l \sin^2\left(\frac{\pi}{l}x\right) dx}.$$

The proof of the second equality is similar.  $\square$

Now we are able to obtain the signs of  $\frac{d\gamma^*(0)}{da}$  and  $\frac{d\delta^*(0)}{da}$ .

**Lemma 4.6.** *Suppose  $0 < \lambda^* - \lambda_* \ll 1$  is satisfied. Then  $\text{Sign}\{\frac{d\gamma^*(0)}{da}\} > 0$  and  $\text{Sign}\{\frac{d\delta^*(0)}{da}\} > 0$ .*

**Proof.** Denote

$$\frac{\partial \mathcal{T}(0, 0, 0, W_{\lambda})}{\partial a} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}.$$

Then we have

$$\frac{d\gamma^*(0)}{da} = -\frac{T_1}{\text{Re } \dot{\mu}(\tau_n)\omega_{\lambda}}, \quad \frac{d\delta^*(0)}{da} = \frac{\text{Im } \dot{\mu}(\tau_n)T_1 + \text{Re } \dot{\mu}(\tau_n)T_2}{\text{Re } \dot{\mu}(\tau_n)\omega_{\lambda}\nu_{\lambda}}. \tag{4.13}$$

From the estimates of  $N(a, \gamma, \delta, W)$ , we obtain that

$$\begin{aligned} T_1 = & \int_0^{\omega_{\lambda}} \left\langle \Psi_1(s), \lambda f'(u_{\lambda}) [\Phi_1(s)W_{\lambda}(s - \tau_n) + \Phi_1(s - \tau_n)W_{\lambda}(s)] \right. \\ & + \lambda u_{\lambda} f''(u_{\lambda}) \Phi_1(s - \tau_n)W_{\lambda}(s - \tau_n) + \frac{\lambda f''(u_{\lambda})}{2} \Phi_1(s)\Phi_1^2(s - \tau_n) \\ & \left. + \frac{\lambda u_{\lambda} f'''(u_{\lambda})}{3!} \Phi_1^3(s - \tau_n) \right\rangle^* ds, \end{aligned}$$

and hence, for  $\lambda \in (\lambda_*, \lambda^*]$ ,

$$\begin{aligned} T_1 = & \frac{1}{(\lambda - \lambda_*)|S_n|^2} \int_0^{\omega_{\lambda}} \int_0^l \text{Re}\{\lambda f'(u_{\lambda}) \bar{S}_n(\lambda - \lambda_*) \zeta_{\lambda}^1 y_{\lambda} \bar{y}_{\lambda} (e^{i\theta_{\lambda}} + e^{-2i\theta_{\lambda}})\} dx ds \\ & + o((\lambda - \lambda_*)^{-2}). \end{aligned}$$

Indeed

$$\begin{aligned}
 & (\lambda - \lambda_*) \int_0^{\omega_\lambda} \int_0^l \operatorname{Re} \{ \lambda f'(u_\lambda) \bar{S}_n(\lambda - \lambda_*) \zeta_\lambda^1 y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) \} dx ds \\
 & \rightarrow \frac{\pi \lambda_*^2 [f'(0)]^2 [1 - 3(\frac{\pi}{2} + 2n\pi)] (\int_0^l \sin^3(\frac{\pi}{l}x) dx)^2}{10}, \quad \text{as } \lambda \rightarrow \lambda_*,
 \end{aligned}$$

hence  $\operatorname{Sign}\{T_1\} < 0$  for  $\lambda \in (\lambda_*, \lambda^*]$  and  $n = 0, 1, \dots$

Similarly, we can obtain for  $\lambda \in (\lambda_*, \lambda^*]$ ,

$$\begin{aligned}
 T_2 &= \frac{1}{(\lambda - \lambda_*) |S_n|^2} \int_0^{\omega_\lambda} \int_0^l \operatorname{Im} \{ \lambda f'(u_\lambda) \bar{S}_n(\lambda - \lambda_*) \zeta_\lambda^1 y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) \} dx ds \\
 &+ o((\lambda - \lambda_*)^{-2}),
 \end{aligned}$$

and

$$\begin{aligned}
 & (\lambda - \lambda_*) \int_0^{\omega_\lambda} \int_0^l \operatorname{Im} \{ \lambda f'(u_\lambda) \bar{S}_n(\lambda - \lambda_*) \zeta_\lambda^1 y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) \} dx ds \\
 & \rightarrow \frac{-\pi \lambda_*^2 [f'(0)]^2 [3 + \frac{\pi}{2} + 2n\pi] (\int_0^l \sin^3(\frac{\pi}{l}x) dx)^2}{10}, \quad \text{as } \lambda \rightarrow \lambda_*.
 \end{aligned}$$

From Eq. (3.5), it is easy to compute that

$$\begin{aligned}
 & (\lambda - \lambda_*)^{-2} |S_n|^2 \operatorname{Re} \dot{\mu}(\tau_n) \rightarrow \left( \int_0^l \sin^2\left(\frac{\pi}{l}x\right) dx \right)^2, \quad \text{as } \lambda \rightarrow \lambda_*, \\
 & (\lambda - \lambda_*)^{-2} |S_n|^2 \operatorname{Im} \dot{\mu}(\tau_n) \rightarrow -\left(\frac{\pi}{2} + 2n\pi\right) \left( \int_0^l \sin^2\left(\frac{\pi}{l}x\right) dx \right)^2, \quad \text{as } \lambda \rightarrow \lambda_*.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & (\lambda - \lambda_*)^{-1} |S_n|^4 \frac{\operatorname{Im} \dot{\mu}(\tau_n) T_1 + \operatorname{Re} \dot{\mu}(\tau_n) T_2}{\omega_\lambda} \\
 & \rightarrow \frac{\lambda_*^2 [f'(0)]^2 [3(\frac{\pi}{2} + 2n\pi)^2 - 2(\frac{\pi}{2} + 2n\pi) - 3] (\int_0^l \sin^3(\frac{\pi}{l}x) dx)^2 (\int_0^l \sin^2(\frac{\pi}{l}x) dx)^2}{20}, \\
 & \text{as } \lambda \rightarrow \lambda_*.
 \end{aligned}$$

It follows that  $\operatorname{Sign}\{\frac{d\delta^*(0)}{da}\} > 0$ . The proof is completed.  $\square$

Summarizing discussions above, we obtain the main result of the paper on local Hopf bifurcations:

**Theorem 4.7.** *Suppose that  $f(u)$  satisfies (A1) and (A2), and define  $\lambda_* = d(\pi/l)^2$ . Then there is a  $\lambda^* > \lambda_*$  with  $0 < \lambda^* - \lambda_* \ll 1$ , and for each fixed  $\lambda \in (\lambda_*, \lambda^*]$ , there exists a sequence  $\{\tau_n\}_{n=0}^\infty$  satisfying  $0 < \tau_0 < \tau_1 < \dots < \tau_n < \dots$ , such that Eq. (1.1) undergoes a Hopf bifurcation at  $(\tau, u) = (\tau_n, u_\lambda)$  for  $n = 0, 1, 2, \dots$ . More*

precisely, there is a family of periodic solutions in form of  $(\tau_n(a), u_n(x, t; a))$  with period  $T_n(a)$  for  $a \in (0, a_1)$  with  $a_1 > 0$ , such that

$$\begin{aligned} \tau_n(a) &= \frac{\theta_\lambda + 2n\pi}{\nu_\lambda} + a^2 k_n^1(\lambda) + o(a^2), & T_n(a) &= \frac{2\pi}{\nu_\lambda} (1 + a^2 k_n^2(\lambda) + o(a^2)), \\ u_n(x, t; a) &= u_\lambda(x) + \frac{a}{2} (y_\lambda(x) e^{i\nu_\lambda t} + \bar{y}_\lambda(x) e^{-i\nu_\lambda t}) + o(a), \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} k_n^1(\lambda) &= \frac{d\gamma^*(0)}{da} := k_1(n, \lambda)(\lambda - \lambda_*)^{-3} + o((\lambda - \lambda_*)^{-3}), \\ k_n^2(\lambda) &= \frac{d\delta^*(0)}{da} := k_2(n, \lambda)(\lambda - \lambda_*)^{-2} + o((\lambda - \lambda_*)^{-2}), \\ k_1(n, \lambda) &= - \frac{\operatorname{Re} \int_0^l f'(u_\lambda) \bar{S}_n m_\lambda^1 \sin(\frac{\pi}{T} x) y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) dx}{h_\lambda |\int_0^l y_\lambda^2 dx| \operatorname{Re} \{ i e^{-i(\theta_\lambda + \rho_\lambda)} \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} dx \}} \\ &= - \frac{\lambda_*^2 [f'(0)]^2 [1 - 3(\frac{\pi}{2} + 2n\pi)] (\int_0^l \sin^3(\frac{\pi}{T} x) dx)^2}{20 (\int_0^l \sin^2(\frac{\pi}{T} x) dx)^2} + o(\lambda - \lambda_*), \\ k_2(n, \lambda) &= \frac{\operatorname{Re} \int_0^l f'(u_\lambda) \bar{S}_n m_\lambda^1 \sin(\frac{\pi}{T} x) y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) dx}{h_\lambda^2 |S_n|^2 |\int_0^l y_\lambda^2 dx| \operatorname{Re} \{ i e^{-i(\theta_\lambda + \rho_\lambda)} \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} dx \}} \cdot \left( \lambda h_\lambda \left| \int_0^l y_\lambda^2 dx \right| \right. \\ &\quad \cdot \operatorname{Im} \left\{ i e^{-i(\theta_\lambda + \rho_\lambda)} \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} dx \right\} + \lambda^2 (\theta_\lambda + 2n\pi) \left| \int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda - \lambda_*} dx \right|^2 \Big) \\ &\quad + \frac{1}{h_\lambda |S_n|^2} \operatorname{Im} \int_0^l \lambda f'(u_\lambda) \bar{S}_n m_\lambda^1 y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda}) dx \\ &= \frac{\lambda_*^2 [f'(0)]^2 [3(\frac{\pi}{2} + 2n\pi)^2 - 2(\frac{\pi}{2} + 2n\pi) - 3] (\int_0^l \sin^3(\frac{\pi}{T} x) dx)^2}{20 [1 + (\frac{\pi}{2} + 2n\pi)]^2 (\int_0^l \sin^2(\frac{\pi}{T} x) dx)^2} + o(\lambda - \lambda_*), \end{aligned} \tag{4.15}$$

and  $(\theta_\lambda, \nu_\lambda, y_\lambda)$  is the associated eigen-triple in Corollary 2.5. In particular,  $k_1(n, \lambda) > 0$  and  $k_2(n, \lambda) > 0$  hence the Hopf bifurcation at  $(\tau_n, u_\lambda)$  is forward with increasing period.

**Remark 4.8.**

1. The assumption  $f(0) = 1$  of (A2) is not essential for all of our results, but it will make the computation simpler. In fact, if we denote  $\lambda_* = \frac{d\pi^2}{f^2(0)}$ , then all of our results are hold as long as  $f(0) > 0$ .
2. When we apply implicit function theorem to obtain periodic solutions, the one corresponding to  $a$  and  $-a$  are the same—that is,  $\gamma(-a) = \gamma(a)$  and  $\delta(-a) = \delta(a)$ .

**5. Bifurcation in the weak Allee effect case**

In the model (1.1), if  $f(0) > 0$  and  $f'(0) > 0$ , then the growth rate per capita is positive and it increases near  $u = 0$ . This type of growth pattern is called weak Allee effect, see [29] for details and other references. In this section, we replace the assumptions (A2) by

$$(A3) \quad f(0) = 1, \text{ and } f'(u) > 0 \text{ for } u \in [0, \delta],$$

and we consider the similar bifurcation under the conditions (A1) and (A3). Since the proofs are mostly similar to the logistic type (under (A2)), we will only indicate how statements and proofs will be modified but not give the full details. In the following, we track the changes in statements and proofs by sections.

In Section 2, the proof of Theorem 2.1 and the expression (2.2) remain valid, but now the steady state  $u_\lambda$  exists for  $\lambda \in (\lambda_{**}, \lambda_*)$  where  $\lambda_{**} < \lambda_*$  and  $|\lambda_* - \lambda_{**}| \ll 1$ . Hence we consider the Hopf bifurcation from  $(\lambda, u_\lambda)$  with  $\lambda \in (\lambda_{**}, \lambda_*)$ . Now the linear analysis in Section 2 can be carried over with the change of the base point to

$$z_{\lambda_*} = (1 + i)\xi_{\lambda_*}, \quad \beta_{\lambda_*} = 1, \quad h_{\lambda_*} = -1, \quad \theta_{\lambda_*} = \frac{3\pi}{2}, \tag{5.1}$$

and then applied the implicit function theorem to obtain the results stated in Theorem 2.4. Note that this change makes  $\nu = (\lambda - \lambda_*)h > 0$  with  $h < 0$ , and the proof still goes through. In particular, Corollary 2.5 holds with  $\nu_\lambda > 0$  and  $\tau_n > 0$  with  $n = 0, 1, 2, \dots$

In Section 3, both Lemmas 3.2 and 3.3 rely on the linear operator  $A^*(\lambda) = dD^2 + \lambda f(u_\lambda) + \lambda f'(u_\lambda)u_\lambda$ . For  $\lambda \in (\lambda_{**}, \lambda_*)$ , we claim that  $A^*(\lambda)$  has exactly one positive eigenvalue and all other eigenvalues are real-valued and negative. This is because that  $A^*(\lambda)$  is self-adjoint, then all eigenvalues are real-valued; at  $\lambda = \lambda_*$ , the principal eigenvalue of  $A^*(\lambda_*)$  is 0, and all other eigenvalues are negative; for  $\lambda \in (\lambda_{**}, \lambda_*)$ , all other eigenvalues remain negative since  $A^*(\lambda)$  is a small perturbation of  $A^*(\lambda_*)$ , but the principal eigenvalue is positive since  $f'(u) > 0$  and one can apply Lemma 3.1. This implies that the statement of Lemma 3.2 is still true, and Lemma 3.3 becomes: if  $0 < \lambda_* - \lambda_{**} \ll 1$  and  $\tau = 0$ , then there is one positive eigenvalue of  $A_\tau(\lambda)$  and all other eigenvalues of  $A_\tau(\lambda)$  have negative real parts for  $\lambda \in (\lambda_*, \lambda^*]$ .

Lemma 3.4 and Theorem 3.5 remain unchanged except that for  $n = 0, 1, 2, \dots$ ,

$$S_n \rightarrow \left[ 1 + i \left( \frac{3\pi}{2} + 2n\pi \right) \right] \int_0^l \sin^2 \left( \frac{\pi}{l} x \right) dx, \quad \text{as } \lambda \rightarrow \lambda_*. \tag{5.2}$$

Finally Theorem 3.6 still holds because of the new base point defined in (5.1). Now Theorem 3.7 becomes: if  $0 < \lambda_* - \lambda_{**} \ll 1$ , then for each fixed  $\lambda \in [\lambda_{**}, \lambda_*)$ , the infinitesimal generator  $A_\tau(\lambda)$  has exactly  $2n + 1$  eigenvalues with positive real part when  $\tau \in (\tau_n, \tau_{\lambda_{n+1}}]$ ,  $n = 1, 2, \dots$ , and  $A_\tau(\lambda)$  has exactly 1 eigenvalue with positive real part when  $\tau \in (0, \tau_1)$ . In particular, Theorem 3.8 becomes: for each fixed  $\lambda \in [\lambda_{**}, \lambda_*)$ , the positive steady state solution  $u_\lambda$  of Eq. (1.1) is unstable for  $\tau \in (0, \infty)$ .

Now consider the Hopf bifurcations in Section 4. The setup and the application of implicit function theorem go through without any alternation. The estimate in Lemma 4.5 is changed into

$$\lim_{\lambda \searrow \lambda_*} \zeta_\lambda^1(\lambda - \lambda_*) = \frac{-(2 + i)\lambda_* f'(0) \int_0^l \sin^3 \left( \frac{\pi}{l} x \right) dx}{20 \int_0^l \sin^2 \left( \frac{\pi}{l} x \right) dx} \sin \left( \frac{\pi}{l} x \right),$$

and

$$\lim_{\lambda \searrow \lambda_*} \zeta_\lambda^2(\lambda - \lambda_*) = 0.$$

Lemma 4.6 still holds. Now we are able to obtain the following Hopf bifurcation theorem for the weak Allee effect case.

**Theorem 5.1.** *Suppose that  $f(u)$  satisfies (A1) and (A3), and define  $\lambda_* = d(\pi/l)^2$ . Then there is a  $\lambda_{**} < \lambda_*$  with  $0 < \lambda_* - \lambda_{**} \ll 1$ , and for each fixed  $\lambda \in (\lambda_{**}, \lambda_*)$ , there exists a sequence  $\{\tau_n\}_{n=0}^\infty$  satisfying  $0 < \tau_0 < \tau_1 < \dots < \tau_n < \dots$ , such that Eq. (1.1) undergoes a Hopf bifurcation at  $(\tau, u) = (\tau_n, u_\lambda)$  for  $n = 0, 1, 2, \dots$ . More precisely, there is a family of periodic solutions in form of  $(\tau_n(a), u_n(x, t; a))$  with period  $T_n(a)$  for  $a \in (0, a_1)$  with  $a_1 > 0$ , such that (4.14) and (4.15) hold but with*

$$k_1(n, \lambda) = -\frac{\lambda_*^2 [f'(0)]^2 [1 + 3(\frac{3\pi}{2} + 2n\pi)] (\int_0^l \sin^3(\frac{\pi}{l}x) dx)^2}{20(\int_0^l \sin^2(\frac{\pi}{l}x) dx)^2} + o(\lambda - \lambda_*),$$

$$k_2(n, \lambda) = \frac{\lambda_*^2 [f'(0)]^2 [3(\frac{3\pi}{2} + 2n\pi)^2 + 2(\frac{3\pi}{2} + 2n\pi) - 3] (\int_0^l \sin^3(\frac{\pi}{l}x) dx)^2}{20[1 + (\frac{3\pi}{2} + 2n\pi)]^2 (\int_0^l \sin^2(\frac{\pi}{l}x) dx)^2} + o(\lambda - \lambda_*),$$

and  $(\theta_\lambda, \nu_\lambda, y_\lambda)$  is the associated eigen-triple in Corollary 2.5 (with base point (5.1)). In particular,  $k_1(n, \lambda) > 0$  and  $k_2(n, \lambda) > 0$  hence the Hopf bifurcation at  $(\tau_n, u_\lambda)$  is forward with increasing period.

### 6. Global dynamics and examples

All results in previous sections are for local bifurcations of (1.1) near a small amplitude non-constant steady state solution, hence the nonlinearity  $f(u)$  only needs to be defined in an interval  $[0, \delta]$  for small  $\delta > 0$ . Here we remark on the global dynamics of Eq. (1.1) if  $f(u)$  is extended to all  $u > 0$  appropriately.

We define the following (global) assumptions on  $f(u)$  corresponding to the logistic and weak Allee effect cases:

- (B1) There exists  $M \in (0, \infty]$  such that  $f$  is a  $C^4$  function on  $[0, M]$ ; either  $M < \infty$ ,  $f(u) > 0$  for  $u \in [0, M)$  and  $f(M) = 0$ ; or  $M = \infty$ ,  $f(u) > 0$  for  $u \in [0, \infty)$ ;
- (B2)  $f(0) = 1$  and  $f'(u) < 0$  for  $u \in [0, M)$ ; or
- (B3)  $f(0) = 1$ , there exists  $m \in (0, M)$  such that  $f'(u) > 0$  for  $u \in [0, m)$ , and  $f'(u) < 0$  for  $u \in (m, M]$ .

First we consider the logistic growth case, that is, when (B1) and (B2) are satisfied. Clearly (A1) and (A2) are satisfied for any  $\delta \in (0, M)$ . The following global existence result holds from a comparison argument:

**Theorem 6.1.** *Suppose that  $f$  satisfies (B1) and (B2). Then the problem (1.1)–(1.2) possesses a unique solution satisfying  $h(x, t)e^{\lambda t} \geq u(x, t) \geq 0$  for  $(x, t) \in [0, l] \times [-\tau, \infty)$ , and  $u(x, t) > 0$  for  $(x, t) \in (0, l) \times (0, \infty)$ , where  $h(x, t)$  is the unique solution of*

$$\frac{\partial h(x, t)}{\partial t} = d \frac{\partial^2 h(x, t)}{\partial x^2}, \quad x \in (0, l), \quad t > 0,$$

$$h(0, t) = h(l, t) = 0, \quad t \geq 0,$$

$$h(x, 0) = \eta(x, 0), \quad x \in (0, l), \tag{6.1}$$

and recall that  $\eta(x, t)$  is the initial value defined in (1.2).

This result is essentially the same as Theorem 2.1 in [7], thus we omit the proof here. For  $\lambda < \lambda_*$ , the global existence in Theorem 6.1 can be sharpened to global stability of the trivial steady state (see [3,7]):

**Theorem 6.2.** Suppose that  $f(u)$  satisfies (B1) and (B2), and  $\lambda < \lambda_*$ , then any nonnegative solution  $u(x, t)$  of the problem (1.1)–(1.2) satisfies  $\|u(\cdot, t)\|_Y \rightarrow 0$  as  $t \rightarrow \infty$  for any  $\tau \geq 0$ .

**Proof.** By taking the inner product of the first equation of Eq. (1.1) with  $u(x, t)$  and recalling the boundary conditions, it follows that

$$\frac{1}{2} \frac{d}{dt} \int_0^l u^2(x, t) dx + d \int_0^l \left| \frac{d}{dx} u(x, t) \right|^2 dx = \lambda \int_0^l u^2(x, t) f(u(x, t - \tau)) dx.$$

By the Poincaré’s inequality (see for example [9]),

$$\frac{\pi^2}{l^2} \int_0^l u^2 dx \leq \int_0^l \left| \frac{d}{dx} u \right|^2 dx, \quad \text{for } u \in H_0^1((0, l)),$$

and noting that  $f(u) \leq f(0) = 1$  for all  $u \geq 0$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_Y^2 \leq (\lambda - \lambda_*) \|u(\cdot, t)\|_Y^2.$$

The result is direct from the comparison theorem.  $\square$

For  $\lambda > \lambda_*$ , as shown in results in previous sections, various bifurcations can occur and the dynamics can be much more complicated. Here we only mention that (1.1)–(1.2) possesses a unique positive steady state  $u_\lambda$  for each  $\lambda > \lambda_*$  if  $f$  satisfies (B1) and (B2), and when  $\tau = 0$  (no delay),  $u_\lambda$  is globally asymptotically stable.

Examples for logistic growth type include the classical diffusive Hutchinson equation [23]

$$\frac{\partial v(x, t)}{\partial t} = d \frac{\partial v^2(x, t)}{\partial x^2} + rv(x, t)(k - v(x, t - \tau)), \tag{6.2}$$

which has been discussed thoroughly in [3,12,20,22,23,27,42], and we shall not discuss here. Here we illustrate Theorems 2.1, 3.8 and 4.7 by applying them to a “food-limited” population delay model with diffusion effects.

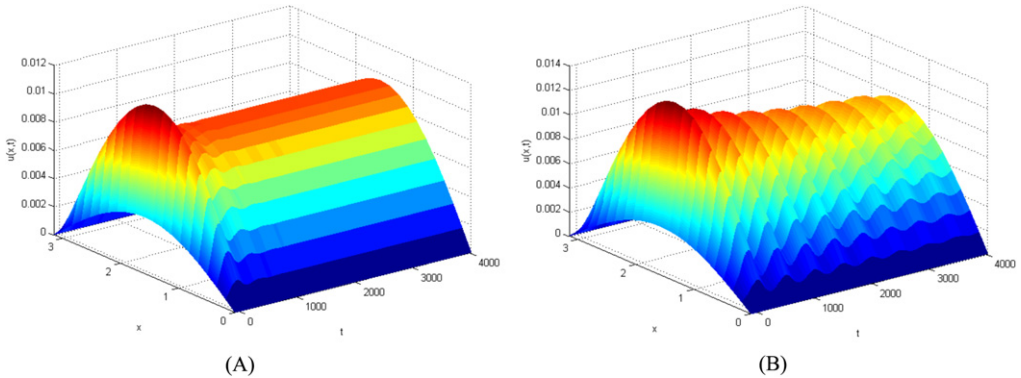
Davidson and Gourley [7] derived a generalization of “food-limited” population model in the following form

$$\frac{\partial v(x, t)}{\partial t} = d \frac{\partial v^2(x, t)}{\partial x^2} + rv(x, t) \frac{k - v(x, t - \tau)}{k + cv(x, t - \tau)}, \tag{6.3}$$

where  $r > 0, c > 0, k > 0, r/c$  is the replacement of mass in the population at saturation,  $r$  and  $k$  are the growth rate of a single-species population and the carrying capacity of the habitat, respectively. We consider the following model after rescaling Eq. (6.3), which has the form  $\tilde{t} = dt, \tilde{\lambda} = r/d, \tilde{\tau} = d\tau, u(x, t) = v(x, t)/k$  (we drop the tilde for convenience):

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial u^2(x, t)}{\partial x^2} + \lambda u(x, t) \frac{1 - u(x, t - \tau)}{1 + cu(x, t - \tau)}, \quad x \in (0, \pi), t > 0, \\ u(0, t) &= u(\pi, t) = 0, \quad t \geq 0, \end{aligned} \tag{6.4}$$





**Fig. 1.** Numerical simulation of (6.4) with  $\lambda = 1.01$  and  $c = 0.5$ . (A)  $\tau = 80$ , the solution approaches to the positive steady state. (B)  $\tau = 120$ , the solution still approaches to the positive steady state but with noticeable oscillations.

with the following initial value

$$u(x, t) = \eta(x, t), \quad x \in [0, \pi], \quad t \in [-\tau, 0], \tag{6.5}$$

where  $\eta \in C$ . Davidson and Gourley [7] have investigated the existence, uniqueness, and asymptotic stability of the nonnegative steady states of Eq. (6.3) with the zero-Dirichlet boundary condition. The existence of monotone travelling front solutions of model (6.3) is showed by Gourley [16]. There has been also extensive investigations of “food-limited” model (see e.g. [2,5,10,11,14,15,17,21,24,30,32,36, 38,40,41]).

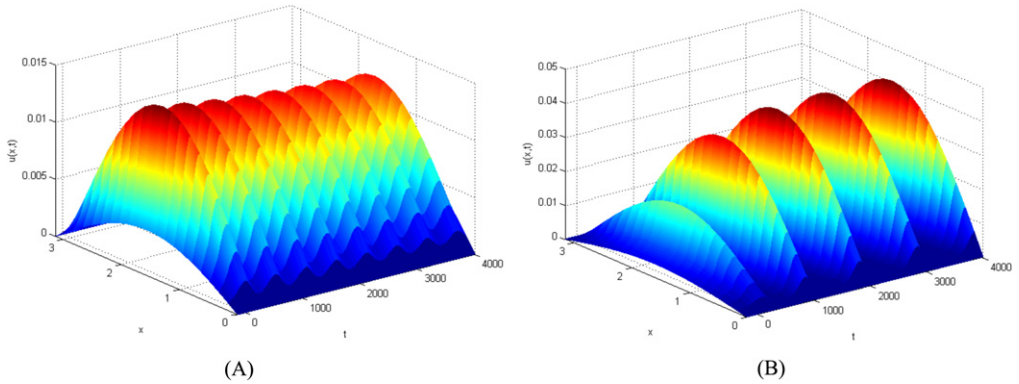
Let

$$f(u) = \frac{1 - u}{1 + cu}.$$

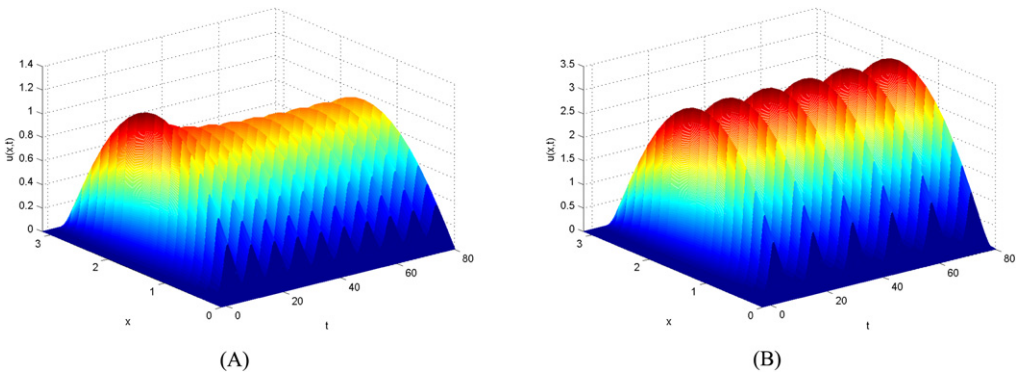
It is easy to see that  $f(0) = 1$ ,  $f(u) > 0$  and  $f'(u) < 0$  for  $u \in [0, 1]$ , and  $f(u) < 0$  for  $u > 1$ . Hence (B1) and (B2) are satisfied, and we can see that  $\lambda_* = 1$ . Applying Theorems 2.1, 3.8, 4.7 and 6.2, we have the following results:

- (i) If  $\lambda < 1$ , then 0 attracts all of positive solutions of the problem (6.4)–(6.5), for any  $\tau \geq 0$ .
- (ii) If  $0 < \lambda - 1 \ll 1$ , then Eq. (6.4) has a positive steady state solution  $u_\lambda$ .
- (iii) For each fixed  $\lambda$  with  $0 < \lambda - 1 \ll 1$ , there exists a constant  $\tau_0$  such that the positive steady state solution  $u_\lambda$  is locally asymptotically stable if  $\tau \in [0, \tau_0)$  and unstable if  $\tau \in (\tau_0, \infty)$ .
- (iv) For each fixed  $0 < \lambda - 1 \ll 1$ , there are a sequence values of delay  $\tau_n > 0$ ,  $n = 0, 1, 2, \dots$ , such that Eq. (6.4) undergoes a forward Hopf bifurcation at  $u = u_\lambda$  when  $\tau = \tau_n$ . Moreover, the period of the bifurcating periodic solutions is near  $2\pi/\nu_\lambda$ .

Here we present some numerical simulations to illustrate the above analytic results. As an example we consider (6.4)–(6.5) with  $c = 0.5$  and in the following simulation, we always use the initial condition  $\eta(x, t) = 0.008[1 + t/(2\tau)] \sin x$ . Notice that if we use  $h_\lambda \approx 1$  and  $\theta_\lambda \approx \pi/2$ , then  $\nu_\lambda \approx \lambda - \lambda_*$ ,  $\tau_0 \approx \pi(\lambda - \lambda_*)^{-1}/2$  and the period of bifurcating orbit  $T \approx 2\pi(\lambda - \lambda_*)^{-1}$  from Corollary 2.5. A steady state bifurcation occurs when  $\lambda$  crosses  $\lambda_* = 1$ , the zero solution loses its stability and the bifurcating positive steady state is asymptotically stable. For  $\lambda < 1$ ,  $u = 0$  is globally asymptotically stable from Theorem 6.2. In Figs. 1 and 2, we use  $\lambda = 1.01$ . For  $\lambda = 1.01$  and smaller  $\tau$ ,  $u_\lambda$  is stable as shown in Fig. 1(A). On the other hand, for  $\lambda = 1.01$ , the estimates for bifurcation point above show  $\tau_0 \approx 50\pi \approx 157$  and  $T \approx 200\pi \approx 628$ . Indeed in Fig. 1(B) (before bifurcation,  $\tau = 120$ ) and Fig. 2(A) (after bifurcation,  $\tau = 130$ ), one can see that  $\tau_0 \approx 125$  and  $T \approx 500$ . From Theorem 4.7, the Hopf



**Fig. 2.** Numerical simulation of (6.4) with  $\lambda = 1.01$  and  $c = 0.5$ . (A)  $\tau = 130$ , the solution converges to a time-periodic solution with small oscillations. (B)  $\tau = 200$ , the solution converges to a time-periodic solution with larger amplitude.



**Fig. 3.** Numerical simulation of (6.4) with  $\lambda = 2$  and  $c = 0.5$ . (A)  $\tau = 2$ . (B)  $\tau = 3$ .

bifurcation at  $\tau_0$  is forward. As  $\tau$  increases, both of the period and the amplitude of the stable time-periodic solution appear to be increasing with respect to the delay  $\tau$ . When  $\tau = 200$  (Fig. 2(B)), the amplitude and the period are around  $u = 0.043$  and  $T = 1000$ . Note that the amplitude of the steady state solution  $u_{1,0.1}$  is  $0.0025\pi \approx 0.0078$  from Theorem 2.1.

Our analytical results only hold for  $\lambda$  close to  $\lambda_*$ . But numerical simulations suggest that, for each  $\lambda > \lambda_*$ , when  $\tau$  is larger, the solution of (6.4)–(6.5) tends to a stable spatial nonhomogeneous time-periodic solution, see Figs. 3 and 4 for the case of  $\lambda = 2$ . In this case, the steady state is destabilized at about  $\tau = 2$ , and there is a stable periodic pattern for  $\tau > 2$ . The amplitude of the periodic solution increases significantly to  $u = 64$  as  $\tau$  increases to 6 (Fig. 4(A)), and to  $u = 2.3 \times 10^4$  as  $\tau = 12$  (Fig. 4(B)). This can be explained by the initial nearly exponential growth of the solution with large delay  $\tau$  (see [12,27]). We conjecture that as  $\tau \rightarrow \infty$ , there exists a stable periodic solution with amplitude and period both approaching  $\infty$  when  $\tau \rightarrow \infty$ . We also conjecture that a cascade of Hopf bifurcations occurs at  $(\lambda, u_\lambda)$  for a sequence of increasing delays  $\tau_n^\lambda$  for each  $\lambda > \lambda_*$ , and the values of  $\tau_n^\lambda$  and time periods of solutions decrease in  $\lambda$ .

Next we discuss the weak Allee effect case, that is when (B1) and (B3) are satisfied. Clearly (A1) and (A3) are satisfied for any  $\delta \in (0, M)$ . The global dynamics of (1.1)–(1.2) with weak Allee effect growth rate and no delay was considered in [29]. It is known that there exists a  $\lambda_K < \lambda_*$  such that there is no positive steady state solutions when  $\lambda < \lambda_K$ , but there are at least two positive steady state solutions when  $\lambda \in (\lambda_K, \lambda_*)$ . For simplicity, in the following, we only consider a special case:

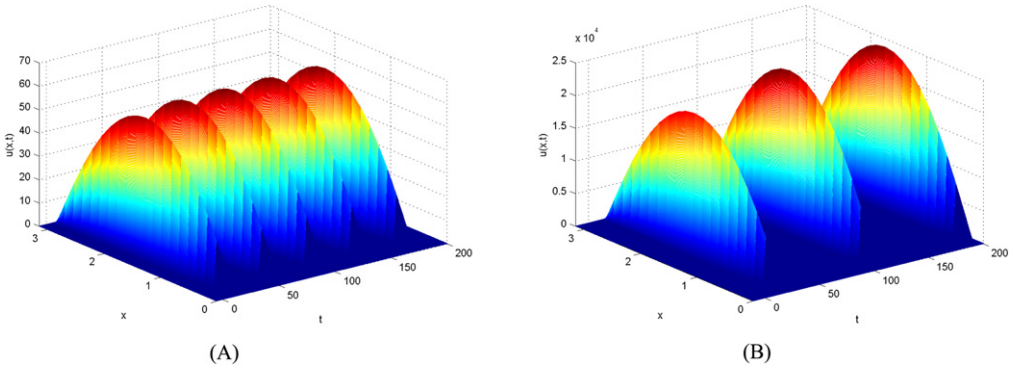


Fig. 4. Numerical simulation of (6.4) with  $\lambda = 2$  and  $c = 0.5$ . (A)  $\tau = 6$ . (B)  $\tau = 12$ .

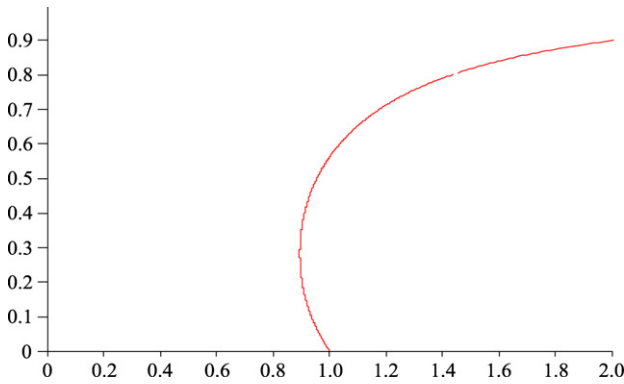


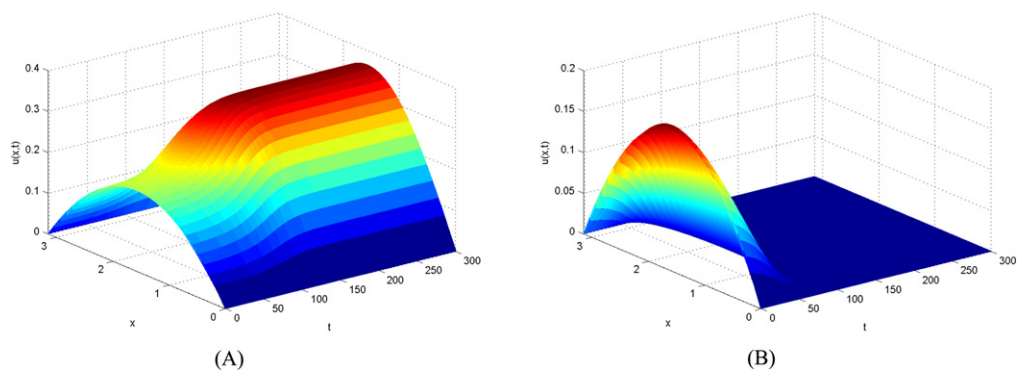
Fig. 5. The bifurcation diagram of steady state solutions of (6.6).

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial u^2(x,t)}{\partial x^2} + 2\lambda u(x,t)(1 - u(x,t - \tau)) \left( u(x,t - \tau) + \frac{1}{2} \right), \quad x \in (0, \pi), \quad t > 0, \\ u(0,t) &= u(\pi,t) = 0, \quad t \geq 0, \\ u(x,t) &= \eta(x,t), \quad x \in [0, \pi], \quad t \in [-\tau, 0]. \end{aligned} \tag{6.6}$$

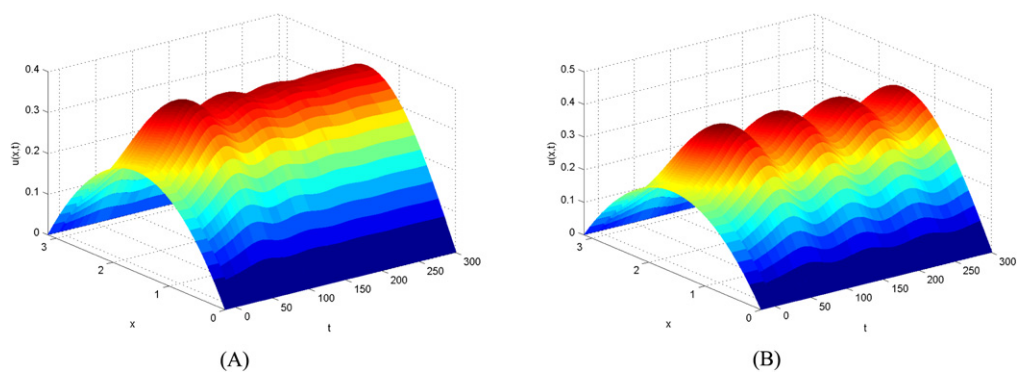
Notice that the function  $f(u) = 2(1 - u)(u + 0.5)$  clearly satisfies (B1) and (B3) with  $M = 1$ .

For (6.6), one can show that the saddle-node bifurcation point  $\lambda_K \geq 8/9 \approx 0.889$  [29]. Indeed numerical integration of the steady state equation shows that  $\lambda_K \approx 0.8928$ . Hence when  $\lambda \in (0.8928, 1)$ , (6.6) has exactly two positive steady state solutions, with the larger one locally stable and the smaller one unstable, see the bifurcation diagram in Fig. 5. For the no-delay reaction–diffusion model, there is a bistability dynamics: if the initial value is above a threshold manifold, then the solution tends to the large positive steady state; and if the initial value is below the threshold, then the solution tends to the zero steady state. Fig. 6 demonstrates that the bistability still holds for small delay: for  $\lambda = 0.9$ , with initial value  $\eta(x,t) = 0.08(\pi^2/4 - (x - \pi/2)^2)$ , the solution stabilizes at the large positive steady state (Fig. 6(A)); but for smaller initial value  $\eta(x,t) = 0.07(\pi^2/4 - (x - \pi/2)^2)$ , then extinction occurs (Fig. 6(B)).

For the reaction–diffusion model with delay (6.6), our results in Section 5 show that Hopf bifurcations occur from the smaller steady state for  $\lambda < \lambda_*$  but close to  $\lambda_*$ , but the bifurcating periodic solutions are unstable thus unlikely can be captured by numerical simulations. However a similar stable periodic pattern emerges as  $\tau$  increases for the weak Allee effect dynamics, even when  $\lambda < \lambda_*$ . In Fig. 7, with  $\lambda = 0.9 < \lambda_* = 1$ , a Hopf bifurcation appears to occur between  $\tau = 15$  and 20. We believe



**Fig. 6.** Numerical simulation of (6.6) with  $\lambda = 0.9$ . (A)  $\tau = 1$  and  $\eta(x, t) = 0.08(\pi^2/4 - (x - \pi/2)^2)$ , the solution converges to a positive steady state solution. (B)  $\tau = 1$  and  $\eta(x, t) = 0.07(\pi^2/4 - (x - \pi/2)^2)$ , the solution converges to 0.



**Fig. 7.** Numerical simulation of (6.6) with  $\lambda = 0.9$ . (A)  $\tau = 15$  and  $\eta(x, t) = 0.1(\pi^2/4 - (x - \pi/2)^2)$ , the solution still converges to a positive steady state solution. (B)  $\tau = 20$  and  $\eta(x, t) = 0.1(\pi^2/4 - (x - \pi/2)^2)$ , the solution converges to time-periodic solution with period around 135.

that this again provides evidence for Hopf bifurcations occurring along the large positive steady state solutions.

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