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## On global bifurcation for quasilinear elliptic systems on bounded domains

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### ABSTRACT

General second order quasilinear elliptic systems with nonlinear boundary conditions on bounded domains are formulated into nonlinear mappings between Sobolev spaces. It is shown that the linearized mapping is a Fredholm operator of index zero. This and the abstract global bifurcation theorem of [Jacobó Pejsachowicz, Patrick J. Rabier, Degree theory for  $C^1$  Fredholm mappings of index 0, *J. Anal. Math.* 76 (1998) 289–319] allow us to carry out bifurcation analysis directly on these elliptic systems. At the abstract level, we establish a unilateral global bifurcation result that is needed when studying positive solutions. Finally, we supply two examples of cross-diffusion population model and chemotaxis model to demonstrate how the theory can be applied.

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## 1. Introduction

This paper is motivated by the desire to use global bifurcation theory to study quasilinear elliptic boundary value problems with a real parameter  $\lambda$ . Such a problem can be written abstractly as

$$T(\lambda, u) := (A(\lambda, u), B(\lambda, u)) = 0, \quad (1.1)$$

where  $A$  is a nonlinear elliptic operator and  $B$  is a boundary operator, and  $T$  is regarded as a mapping from a Sobolev space  $X$  to another space  $Y$  which is the product space of  $L^p$  and a trace space.

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If the principal part of  $A$  is linear, and the boundary operator  $B$  is linear (in  $u$ ) and independent of  $\lambda$ , then by first converting the problem into an “integral” equation/system in the form of  $u - K(\lambda, u) = 0$ , where  $K$  is a compact and continuous operator, we can apply Rabinowitz’s celebrated global bifurcation theory [35] to obtain the existence of a continuum of nontrivial solutions of (1.1). However, if the principal part of  $A$  is nonlinear, or if  $B$  is nonlinear and dependent of  $\lambda$ , the task of converting (1.1) into the suitable form of “a compact perturbation of the identity” is, most of the times, cumbersome.

An alternative way to treat bifurcation of quasilinear elliptic systems is to use the theory of Fredholm operators. A bounded linear mapping  $L$  from a Banach space  $X$  to another Banach space  $Y$  is said to be *Fredholm* if the dimension of its kernel  $\mathcal{N}(L)$  and the co-dimension of its range  $\mathcal{R}(L)$  are both finite. The *Fredholm index* of  $L$  is defined to be  $\text{ind}(L) = \dim \mathcal{N}(L) - \text{codim } \mathcal{R}(L)$ .  $L$  is said to be *semi-Fredholm* if  $\mathcal{R}(L)$  is closed and either  $\dim \mathcal{N}(L)$  or  $\text{codim } \mathcal{R}(L)$  is finite. A smooth mapping  $T$  from  $X$  to  $Y$  is Fredholm/semi-Fredholm, if for every  $u \in X$ ,  $D_u T(u)$  is Fredholm/semi-Fredholm.

For nonlinear Fredholm mappings with zero index, Fitzpatrick, Pejsachowicz and Rabier ([15,16,30] and references therein) have recently discovered the concepts of *parity* and a “base point degree” (one that is as useful as the Leray–Schauder degree), and established a global bifurcation result that allows us to tackle (1.1) *directly*. In particular, according to [30], with a crucial parity condition near a suspect bifurcation value of  $\lambda$ , a global bifurcation occurs if  $T$  is  $C^1$ -smooth and if the Fredholm index of  $D_u T$  (in  $u$ ) is zero. The works of Fitzpatrick, Pejsachowicz and Rabier have provided concrete ways to check the parity condition in applications of their abstract theory; however, this is not the case for the Fredholm index of the mappings stemming from nonlinear elliptic problems. Moreover, when studying reaction–diffusion equations/systems, we often desire positive solutions, and thus we need, in the new framework, a theorem about “unilateral” global bifurcation (the global property of the positive solution branch) in the spirit of Theorem 1.27 of Rabinowitz [35].

The purpose of this paper is to build a bridge between the abstract global bifurcation theory and second order quasilinear elliptic systems on bounded domains that often appear in applications as steady state reaction–diffusion systems. For such systems with general nonlinear boundary conditions, we provide some user-friendly sufficient conditions for zero Fredholm index, as well as the  $C^1$ -smoothness in the  $L^p$  setting. We also prove an abstract unilateral global bifurcation theorem in the new framework. Finally, we supply two examples of reaction–diffusion systems (one involving cross-diffusion, the other chemotaxis with nonlinear boundary conditions) to illustrate the point we are making: global bifurcation analysis can be carried out directly on the quasilinear systems with nonlinear boundary conditions. (However, these examples are not chosen to represent the full potential of Fitzpatrick–Pejsachowicz–Rabier theory and our results.)

There have been some papers that explicitly address the Fredholmness of nonlinear elliptic boundary value problems. Fitzpatrick and Pejsachowicz [16] had a part on higher order fully nonlinear single equations on bounded domains in both the  $L^2$  and Hölder settings; Rabier and Stuart [34] dealt with second order quasilinear single equations in whole  $\mathbf{R}^n$ , and subsequently Gebran and Stuart [18] studied systems of such equations with homogeneous Dirichlet boundary condition on bounded and unbounded domains. [16] contains the following idea: to show the Fredholm index of  $D_u T(u)$  is zero for every  $u$  in a connected set  $V$ , one proves (a)  $D_u T(u)$  is semi-Fredholm for all  $u \in V$ , then (b)  $D_u T(u_0)$  has zero index for a particular  $u_0 \in V$ . If both (a) and (b) hold, by the local constancy of the index for semi-Fredholm operators, the index of  $D_u T(u)$  is zero everywhere in  $V$ . On bounded spatial domains (the case we treat in this paper), by “Peetre’s Lemma,” if we have the  $L^p$  estimates for the operator  $D_u T(u)$ , then it is semi-Fredholm. On unbounded domains (the case we do not treat),  $L^p$  estimates for  $D_u T(u)$  alone are not sufficient to yield the semi-Fredholmness, and thus [34] develops a new concept of a linear bounded operator being compact modulo another one. Using this method of [34] to prove the index of  $D_u T(u)$  is zero everywhere in  $V$  requires one to prove (b) first, and then to show the compactness of  $D_u T(u) - D_u T(u_0)$  modulo  $D_u T(u)$  and  $D_u T(u_0)$ .

The papers [16,18,34] do not offer sufficient conditions for the index of  $D_u T(u_0)$  being zero, in general or in their specific problems, except Lemma 10.11 of [16] which assumes the surjectivity of the boundary operator in the  $L^2$  setting. (However, we mention a recent progress by Rabier [33] at the linear level on the Fredholm index of elliptic operators in whole  $\mathbf{R}^n$ .) Thus there is a need for a linear theory. In the  $L^p$  setting on bounded spatial domains, this linear theory should address two

issues: (a') the  $L^p$  estimates with a weak regularity assumption on the coefficients of the elliptic and boundary operators (this is to show (a) mentioned in the above paragraph); and (b') a sufficient condition for the linear operator to have zero index under a stronger regularity assumption on the coefficients (this is to show (b)). The reason for the different assumptions on the regularity of the coefficients is that  $u$  in (a) is arbitrary and  $u_0$  in (b) can be chosen to be  $C^\infty$  smooth.

The celebrated Agmon–Douglis–Nirenberg Theorem (see [3,4]) on  $L^p$  estimates assumes a stronger regularity assumption on the coefficients, especially the ones in the boundary operator. In part (i) of Theorem 2.7, we show how to modify the arguments of [3] and [4] to suit (a') mentioned in the above paragraph. Under Agmon's condition on the linear operator  $(A, B)$  (see Definition 2.4), Geymonat and Grisvard [19] (using duality method), Agranovich, Denk, Faierman and Möller [5,12] (using Agranovich–Vishik's "direct method" [6]) proved that  $(A + \sigma I, B)$  is an isomorphism (hence of index zero) and the parameter-dependent  $L^p$  estimates, where  $I$  is the embedding map from  $W^{2,p}$  to  $L^p$ , and  $\sigma$  is a complex constant along the "Agmon angle" and with large norm. Since  $I$  is compact, we have that  $(A, B)$  is Fredholm of index zero. The regularity assumptions on the coefficients in these papers are good enough for (b') (the regularity assumption in [19] is stronger than the one in [5,12]). The isomorphism and the parameter-dependent  $L^p$  estimates are also stated in Theorem 2.3 of Amann [7] under the weak regularity assumption on the coefficients (especially of the boundary operator), which is the correct assumption for (a'). However, Amann did not supply technical details in his proof. In (ii) of our Theorem 2.7, we prove the isomorphism and the parameter-dependent  $L^p$  estimates under Amann's regularity assumption, by modifying the arguments of [5,12]. Although, as mentioned before, (ii) of our Theorem 2.7 is not necessary for (b'), it is of independent interest: the result in [7] is playing an important role in applications and his fundamental Theorem 2.3 deserves an independent investigation; moreover, it gives rise, at the linear level, to criteria (see our Corollaries 2.10 and 2.11) for zero index under the weakest regularity assumption which may find applications elsewhere.

The main condition in all the isomorphism results mentioned above is Agmon's condition. Amann [7] offers several concrete and user-friendly ways to verify this condition, some of which we summarize in Remark 2.5.5.

Crandall–Rabinowitz's theorem [9] on local bifurcation from "a simple eigenvalue" is perhaps the simplest and most frequently used result to study bifurcation. As pointed out in [15], the main condition ("transversality condition," a term used in [15]) in that theorem implies the parity condition needed in the global bifurcation theorem of [30] (see Theorem 4.1 in this paper). Thus the local bifurcation is actually a global one if we merely add a Fredholm condition; see Theorem 4.3. Based on this, we establish a unilateral bifurcation result; see Theorem 4.4, which is our main contribution to the abstract theory.

## 2. The linear theory

### 2.1. Ellipticity, complementing and Agmon's conditions

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  ( $n \geq 2$ ) with  $C^2$  smooth boundary  $\partial\Omega$ .  $\partial\Omega$  may have more than one but finitely many connected components. Denote by  $\mathcal{C}$  the set of components of  $\partial\Omega$ . Let the following be  $N \times N$  real matrix-valued functions:

$$a_{ij}(x), a_i(x) \quad (i, j = 1, \dots, n), a_0(x), \quad \text{where } x \in \overline{\Omega};$$

$$b_i(x) \quad (i = 1, \dots, n), b_0(x), c(x), \quad \text{where } x \in \partial\Omega.$$

Let  $u(x)$  be a real  $N$ -dimensional column-vector-valued function, where  $x \in \Omega$ . Let

$$\delta(x) = \text{diag}(\delta_1(x), \dots, \delta_N(x)), \quad x \in \partial\Omega,$$

where each  $\delta_i(x)$  is continuous and assumes only values 0 and 1 on  $\partial\Omega$ .

Consider the second order linear operator on  $\Omega$ :

$$Au = -a_{ij}(x)\partial_i\partial_j u + a_i(x)\partial_i u + a_0(x)u, \quad x \in \Omega, \tag{2.1}$$

and the linear boundary operator

$$Bu = \delta(x)[b_i(x)\partial_i u + b_0(x)u] + (I - \delta(x))c(x)u, \quad x \in \partial\Omega, \tag{2.2}$$

where  $\partial_i = \frac{\partial}{\partial x_i}$  and the summation convention is and will be used.

**Definition 2.1.** We say that  $A$  is *elliptic* (in the sense of Petrovskii) on  $\overline{\Omega}$  if for every  $x \in \overline{\Omega}$  and every  $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n \setminus \{0\}$ ,

$$\det \mathring{A}(x, \xi) \equiv \det(a_{ij}(x)\xi_i\xi_j) \neq 0, \tag{2.3}$$

where  $\mathring{A}(x, \xi)$  is the principal symbol of the operator  $A$ :  $\mathring{A}(x, \xi) \equiv a_{ij}(x)\xi_i\xi_j$ .

**Definition 2.2.** We say that  $B$  satisfies the *complementing condition* (Lopatinskii–Shapiro) with respect to  $A$  if for every  $x \in \partial\Omega$ , every  $\xi \in \mathbf{R}^n \setminus \{0\}$  tangent to  $\partial\Omega$  at  $x$ , the only exponentially decaying solution  $v(t)$  of the following initial value problem is  $v \equiv 0$ :

$$\begin{cases} \mathring{A}\left(x, \xi + \nu(x)i\frac{d}{dt}\right)v(t) = 0, & t > 0, \\ \mathring{B}\left(x, \xi + \nu(x)i\frac{d}{dt}\right)v(0) = 0, \end{cases}$$

where  $i = \sqrt{-1}$ ,  $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$  is the unit outer normal vector to  $\partial\Omega$  at  $x$ , and

$$\mathring{B}(x, \xi) = \delta(x)(b_i(x)\xi_i) + (I - \delta(x))c(x).$$

**Definition 2.3.** We say that  $(A, B)$  is *elliptic* on  $\overline{\Omega}$  if  $A$  is elliptic on  $\overline{\Omega}$  and  $B$  satisfies the complementing condition with respect to  $A$ .

**Definition 2.4.** We say that  $(A, B)$  satisfies *Agmon’s condition* at a fixed angle  $\theta \in [-\pi, \pi)$ , if for every  $(\xi, \sigma) \in (\mathbf{R}^n \times \mathbf{C}) \setminus \{(0, 0)\}$ , where  $\sigma = 0$  or  $\arg \sigma = \theta$ , we have:

1. for every  $x \in \overline{\Omega}$ ,

$$\det(\mathring{A}(x, \xi) + \sigma I_{N \times N}) \neq 0; \tag{2.4}$$

2. for every  $x \in \partial\Omega$  and every  $\xi \in \mathbf{R}^n \setminus \{0\}$  tangent to  $\partial\Omega$  at  $x$ , the only exponentially decaying solution  $v(t)$  of the following initial value problem is  $v \equiv 0$ :

$$\begin{cases} \mathring{A}\left(x, \xi + \nu(x)i\frac{d}{dt}\right)v(t) + \sigma v(t) = 0, & t > 0, \\ \mathring{B}\left(x, \xi + \nu(x)i\frac{d}{dt}\right)v(0) = 0. \end{cases} \tag{2.5}$$

**Remark 2.5.** 1.  $\det \dot{A}(x, \xi)$  is a polynomial homogeneous in  $\xi$  with degree  $2N$ . If each  $a_{ij}(x)$  is continuous in  $\overline{\Omega}$ , then we have

$$M|\xi|^{2N} \leq |\det \dot{A}(x, \xi)|$$

for a positive constant  $M$  independent of  $x$  and  $\xi$ . For  $N = 1$ , i.e. the scalar equation case (so each  $a_{ij}$  is a scalar), this is the usual uniform ellipticity condition.

2. As summarized by Amann [7], there are other classical notions of ellipticity, each of which being stronger than (2.3). For fixed  $i, j$ , let  $a_{ij}^{rs}$  be the entry of the matrix  $a_{ij}$  located at  $r$ th row and  $s$ th column. Then the *strong ellipticity* is defined as

$$a_{ij}^{rs}(x)\xi_i\xi_j\eta_r\eta_s > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbf{R}^N \setminus \{0\}, \quad \eta \in \mathbf{R}^N \setminus \{0\};$$

and the *very strong ellipticity* is defined as

$$a_{ij}^{rs}(x)\chi_{ir}\chi_{js} > 0, \quad x \in \overline{\Omega}, \quad \chi \in \mathbf{R}^{nN} \setminus \{0\}.$$

When  $N = 1$ , ellipticity, strong ellipticity and very strong ellipticity are equivalent.

3. Sometimes the complementing condition is expressed in an algebraic way (see [4, p. 42]). That algebraic version is equivalent to the ODE version presented here, by Theorem 3.2 in [4].

4. In the scalar case, i.e.,  $N = 1$ , when  $B$  is the Dirichlet, or Neumann or Robin boundary operator,  $B$  satisfies the complementing condition with respect to any elliptic operator  $A$  (see [40, pp. 160–161]). This is not the case for systems.

5. Amann [7], whose main concern is the sectorial property of  $A$  in a parabolic equation setting, defines *normal ellipticity* of  $(A, B)$  (see [7, p. 21]), which is equivalent to Agmon’s condition (i.e. (2.4) and (2.5)) for all  $\theta \in [-\pi/2, \pi/2]$ . He proves that in the following cases,  $(A, B)$  is normally elliptic. (We emphasize that for our purpose, we only need Agmon’s condition to be satisfied at a single  $\theta \in [-\pi, \pi]$ .)

**Case 1.**  $A$  is strongly elliptic, at every  $x \in \partial\Omega$ ,  $\delta(x)$  is either the zero or identity matrix (but remember  $\delta$  is assumed to be continuous on  $\partial\Omega$ ), and  $b_i(x) = b(x)\beta_i(x)$  ( $i = 1, \dots, n$ ), where  $(\beta_1(x), \dots, \beta_n(x))$  is a nonvanishing and outward pointing vector field on  $\partial\Omega$ ,  $b(x)$  and  $c(x)$  are real  $N \times N$  nonsingular matrix-valued functions defined on  $\partial\Omega$ . Theorem 4.2 of [7] requires the continuity of  $b, c$  and  $\beta$ ; however, since Agmon’s condition (or normal ellipticity) is a property at each fixed point  $x$ , the continuity in  $x$  is not needed.

**Case 2.**  $A$  is very strongly elliptic,  $b_i(x) = a_{ij}(x)v_j(x)$ ,  $c(x) = I$ . See Proposition 4.3 of [7].

Note that in the scalar case  $N = 1$ , since ellipticity is equivalent to both strong and very strong ellipticity, we just need the ellipticity in Cases 1 and 2.

**Case 3.**  $a_{ij}(x) = a(x)\alpha_{ij}(x)$ ,  $b_i(x) = a(x)\alpha_{ij}(x)v_j(x)$ ,  $c(x) = I$ , where  $a(x)$  is a real  $N \times N$  matrix-valued function defined on  $\overline{\Omega}$ , satisfying  $(I - \delta(x))a(x)\delta(x) = 0$  for  $x \in \partial\Omega$ ,  $(\alpha_{ij}(x))$  is positive-definite for  $x \in \overline{\Omega}$ , and

$$\det(a(x) + \sigma I) \neq 0 \tag{2.6}$$

for  $x \in \overline{\Omega}$  and  $\sigma$  such that either  $\sigma = 0$  or  $\arg \sigma \in [-\pi/2, \pi/2]$ . See Theorem 4.4 of [7]—again, the smoothness assumptions and the divergence form assumed there are unnecessary. Moreover, if (2.6) holds for a fixed ray  $\arg \sigma = \theta_0$ , then Amann’s proof yields Agmon’s condition along that ray, which is enough for our purpose.

2.2.  $L^p$  estimates

We write the norm of  $L^p(\Omega)$  as  $\|u\|_{0,p,\Omega}$ . For any positive integer  $m$ , let  $W^{m,p}(\Omega)$  be the usual Sobolev space equipped with the norm

$$\|u\|_{m,p,\Omega} = \sum_{|\mu| \leq m} \|\partial^\mu u\|_{0,p,\Omega},$$

where  $\mu$  is a multi-index  $(\mu_1, \dots, \mu_n) \in (\mathbf{Z}^+)^n$  (here  $\mathbf{Z}^+ = \{0, 1, 2, \dots\}$ ),  $|\mu| = \mu_1 + \dots + \mu_n$ ,  $\partial = (\partial_1, \dots, \partial_n)$ , and  $\partial^\mu = \partial_1^{\mu_1} \dots \partial_n^{\mu_n}$ .

The trace space of  $W^{m,p}(\Omega)$  is denoted by  $W^{m-1/p,p}(\partial\Omega)$  with norm

$$\|u\|_{m-1/p,p,\partial\Omega} = \inf_{v|_{\partial\Omega}=u} \|v\|_{m,p,\Omega}.$$

The spaces defined above of course generalize to vector-valued functions. When no confusions are perceived to arise, for such (vector-valued) functions, we still use the notation  $W^{m,p}(\Omega)$ ,  $W^{m-1/p,p}(\partial\Omega)$ ,  $\dots$ , instead of  $(W^{m,p}(\Omega))^N$ ,  $(W^{m-1/p,p}(\partial\Omega))^N$ ,  $\dots$ .

For each component  $\Gamma$  of  $\partial\Omega$  (i.e.  $\Gamma \in \mathcal{C}$ ), let  $\delta_i(\Gamma) = \delta_i(x)$ ,  $x \in \Gamma$ . Define a Banach space

$$\partial W^{m,p} = \prod_{\Gamma \in \mathcal{C}} \prod_{i=1}^N W^{m-\delta_i(\Gamma)-1/p,p}(\Gamma),$$

with the norm

$$\|(u_1, \dots, u_N)\|_{\partial W^{m,p}} = \sum_{\Gamma \in \mathcal{C}} \sum_{i=1}^N \|u_i\|_{m-\delta_i(\Gamma)-1/p,p,\Gamma}.$$

We shall find the following Gagliardo–Ladyzhenskaya–Nirenberg inequalities useful (see Theorem 2.2 and Remark 2.1 in [22]). If  $q > n \geq p > 1$ , then

$$\|u\|_{0,pq/(q-p),\Omega} \leq C \|u\|_{1,p,\Omega}^{n/q} \|u\|_{0,p,\Omega}^{1-n/q} \leq \varepsilon \|\nabla u\|_{0,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}, \tag{2.7}$$

and if  $p > n$ , then

$$\|u\|_{0,\infty,\Omega} \leq C \|u\|_{1,p,\Omega}^{n/p} \|u\|_{0,p,\Omega}^{1-n/p} \leq \varepsilon \|\nabla u\|_{0,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}. \tag{2.8}$$

Moreover, if  $p > 1$ , then

$$\|\nabla u\|_{0,p,\Omega} \leq C \|u\|_{2,p,\Omega}^{1/2} \|u\|_{0,p,\Omega}^{1/2}. \tag{2.9}$$

Recall the following result of Agmon, Douglis and Nirenberg [4]:

**Theorem 2.6.** *Suppose that  $\partial\Omega$  is  $C^2$ , and  $(A, B)$  is elliptic on  $\overline{\Omega}$ .*

- (i) *If each  $a_{ij}$  is continuous on  $\overline{\Omega}$ ,  $a_i(x)$  and  $a_0 \in L^\infty(\Omega)$ ,  $b_i$  and  $b_0 \in C^1(\partial\Omega)$ , and  $c \in C^2(\partial\Omega)$ , then for any  $u \in W^{2,p}(\Omega)$ ,*

$$\|u\|_{2,p,\Omega} \leq K_1 (\|Au\|_{0,p,\Omega} + \|Bu\|_{\partial W^{2,p}} + \|u\|_{0,p,\Omega}), \tag{2.10}$$

where  $K_1 > 0$  is a constant independent of  $u$ .

(ii) Suppose that for an integer  $l \geq 0$ ,  $a_{ij}, a_i, a_0 \in C^l(\overline{\Omega})$ ,  $b_i, b_0 \in C^{l+1}(\partial\Omega)$ , and  $c \in C^{l+2}(\partial\Omega)$ . Then for any  $u \in W^{2+l,p}(\Omega)$ ,

$$\|u\|_{2+l,p,\Omega} \leq K_2(\|Au\|_{l,p,\Omega} + \|Bu\|_{\partial W^{2+l,p}} + \|u\|_{0,p,\Omega}). \tag{2.11}$$

Part (ii) is just Theorem 10.5 of [4] specialized when the order of  $A$  is 2; part (i) follows from applying part (ii) to the principal part of  $A$ , and then using the fact that the  $L^p$ -norm of the lower order terms is dominated by

$$\varepsilon \left( \sum_{|\mu|=2} \|\partial^\mu u\|_{0,p,\Omega} \right) + C(\varepsilon)\|u\|_{0,p,\Omega},$$

where the constant  $\varepsilon$  can be taken as small as we wish (Lemma 14.1 of [3]).

For our purpose, the smoothness assumptions on the coefficients of  $(A, B)$  are too stringent. We shall need, under a relaxed smoothness assumption, an injectivity and surjectivity result for the operator  $(A + \sigma I, B)$ . The following has more than what we really need, but we would like to record it here.

**Theorem 2.7.** Suppose that  $\partial\Omega$  is  $C^2$ , and each  $a_{ij}$  is continuous on  $\overline{\Omega}$ . Assume that there exists a number

$$\hat{p} \begin{cases} = p & \text{if } p > n, \\ > n & \text{if } p \leq n, \end{cases}$$

such that  $a_i, a_0 \in L^{\hat{p}}(\Omega)$ ,  $b_i, b_0 \in W^{1-1/\hat{p},\hat{p}}(\partial\Omega)$ , and  $c \in W^{2-1/\hat{p},\hat{p}}(\partial\Omega)$ .

- (i) If  $(A, B)$  is elliptic on  $\overline{\Omega}$ , then (2.10) holds.
- (ii) If Agmon's condition holds for a fixed  $\theta_0$ , the operator  $(A + \sigma I, B) : (W^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N \times \partial W^{2,p}$  is an isomorphism for every  $\sigma$  with  $\arg \sigma = \theta_0$  and large  $|\sigma|$ , and for such  $\sigma$ ,  $u \in (W^{2,p}(\Omega))^N$ ,

$$\|u\|_{2,p,\Omega} \leq K_3(\|(A + \sigma I)u\|_{0,p,\Omega} + \|Bu\|_{\partial W^{2,p}}), \tag{2.12}$$

where  $K_3$  is a constant independent of  $\sigma$  and  $u$ ,

$$\begin{aligned} \|u\|_{2,p,\Omega} &= \|u\|_{2,p,\Omega} + |\sigma| \cdot \|u\|_{0,p,\Omega}, \\ \| (v_1, \dots, v_N) \|_{\partial W^{2,p}} &= \| (v_1, \dots, v_N) \|_{\partial W^{2,p}} + \sum_{\Gamma \in \mathcal{C}} \sum_{i=1}^N |\sigma|^{(2-\delta_i(\Gamma)-1/p)/2} \|v_i\|_{0,p,\Gamma}. \end{aligned}$$

**Remark 2.8.** It is still possible to weaken the smoothness assumption. For example, if  $p < n$ , we just need to assume that  $b_i, b_0$  and  $c$  can be extended into  $\overline{\Omega}$  so that they are continuous on  $\overline{\Omega}$ ,  $b_i, b_0 \in W^{1,n}(\Omega)$ ,  $c \in W^{2,n}(\Omega)$ .

**Remark 2.9.** Part (ii) is originated from Agmon's 1962 paper [2], where he left out the proof of surjectivity. [5,12] contain (ii) under the same regularity condition as in Theorem 2.6. The regularity assumption in this theorem is the same as the one in Theorem 2.3 of [7]. (ii) and further arguments now imply the original statement of Theorem 2.3 of [7] (except the second inequality in (ii) of that theorem which is not used in the rest of [7]): (1) if Agmon's condition holds for all angles in a closed sector  $S$  on the complex plane with vertex at the origin, then there exists  $\omega > 0$  such that our (ii) holds for all  $\sigma \in S$  with  $|\sigma| > \omega$  (this is already observed in [5,12]); (2) it is well known that the set of Agmon angles is open and thus Amann's normal ellipticity implies Agmon's condition in the sector mentioned in Theorem 2.3 of [7]; (3) as will be seen in the proof below, the lower order coefficients

$a_i, a_0, b_0$  are non-essential, and so by the compactness of  $\pi(B)$  assumed by Amann, the constants  $\omega$  and  $\kappa$  are independent of the elements in  $U$  in Theorem 2.3 of [7].

**Proof of Theorem 2.7.** First we prove part (i) in two steps.

*Step 1.* We show that it is sufficient to prove (2.10) for the principal part of  $(A, B)$ . (Note that the principal part of  $B$  is obtained by dropping the  $b_0$  term.)

Consider first the case  $p \leq n$  (recall  $\hat{p} > n$ ). From Hölder inequality, (2.7) and (2.9), we find that

$$\begin{aligned} \|a_0 u\|_{0,p,\Omega} &\leq \|a_0\|_{0,\hat{p},\Omega} \|u\|_{0,\hat{p}p/(\hat{p}-p),\Omega} \\ &\leq \|a_0\|_{0,\hat{p},\Omega} (\varepsilon \|u\|_{1,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}), \end{aligned}$$

$$\begin{aligned} \|a_i \partial_i u\|_{0,p,\Omega} &\leq \|a_i\|_{0,\hat{p},\Omega} \|\partial_i u\|_{0,\hat{p}p/(\hat{p}-p),\Omega} \\ &\leq \|a_i\|_{0,\hat{p},\Omega} (\varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}), \end{aligned}$$

and

$$\begin{aligned} \|b_0 u\|_{1-1/p,p,\partial\Omega} &\leq \|b_0 u\|_{1,p,\Omega} \quad (\text{here we assume that } b_0 \text{ is extended onto } \Omega) \\ &\leq \|b_0 u\|_{0,p,\Omega} + \sum_{|\mu|=1} \|(\partial^\mu b_0)u\|_{0,p,\Omega} + \sum_{|\mu|=1} \|b_0 \partial^\mu u\|_{0,p,\Omega} \\ &\leq \|b_0\|_{1,\hat{p},\Omega} (\varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}). \end{aligned}$$

Now the contribution to the right-hand side of (2.10) from the lower order terms in  $A$  and  $B$  is controlled by

$$(\|a_0\|_{0,\hat{p},\Omega} + \|a_i\|_{0,\hat{p},\Omega} + \|b_0\|_{1,\hat{p},\Omega})(\varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}),$$

so that if (2.10) holds for the principal part of  $(A, B)$ , it does so too for the full  $(A, B)$ .

In the case of  $p > n$  (so  $\hat{p} = p$ ), from (2.8), we obtain

$$\begin{aligned} \|a_0 u\|_{0,p,\Omega} &\leq \|a_0\|_{0,p,\Omega} \|u\|_{0,\infty,\Omega} \\ &\leq \|a_0\|_{0,p,\Omega} (\varepsilon \|u\|_{1,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}), \end{aligned}$$

$$\begin{aligned} \|a_i \partial_i u\|_{0,p,\Omega} &\leq \|a_i\|_{0,p,\Omega} \|\partial_i u\|_{0,\infty,\Omega} \\ &\leq \sum_{i=1}^N \|a_i\|_{0,p,\Omega} (\varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}), \end{aligned}$$

and

$$\begin{aligned} \|b_0 u\|_{1-1/p,p,\partial\Omega} &\leq \|b_0 u\|_{1,p,\Omega} \\ &\leq \|b_0\|_{1,p,\Omega} \|u\|_{0,\infty,\Omega} + \|b_0\|_{1,\infty,\Omega} \|u\|_{1,p,\Omega} \\ &\leq \|b_0\|_{1,\hat{p},\Omega} (\varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon) \|u\|_{0,p,\Omega}). \end{aligned}$$

This completes the proof of Step 1.

From now on, assume  $(A, B)$  has only the principal part.



*Step 2.* The inequality (2.10) is proved in [4] by establishing the interior estimate (Theorem 10.3), then the boundary estimate (Theorem 10.4), and finally an argument involving partition of unity (pages 704–705 in [3] for  $N = 1$ ). In our current situation, no change needs to be made for the interior estimates because it involves only  $a_{ij}$ . In the final stage (“partition of unity stage”), some lower order terms appear, but the estimates involved are similar to the ones we have in Step 1. So we need to worry only about the boundary estimates, and in this scenario, according to pages 702–703 in [3], we only need to show that for a fixed point  $x_0 \in \partial\Omega$ ,

$$\begin{aligned} & \| (b_i(x_0) - b_i(\cdot))\partial_i u(\cdot) \|_{1-1/p,p,\partial\Omega} + \| (c(x_0) - c(\cdot))u(\cdot) \|_{2-1/p,p,\partial\Omega} \\ & \leq \varepsilon \|u\|_{2,p,\Omega} + C(\varepsilon)\|u\|_{0,p,\Omega}, \end{aligned} \tag{2.13}$$

for every  $u \in W^{2,p}(\Omega)$  whose support is contained in a small neighborhood  $U$  of  $x_0$ , where  $\varepsilon$  can be taken as small as we wish if  $U$  is small enough. Extend  $b_i$  and  $c$  so that they belong to  $W^{1,\hat{p}}(\Omega)$  and  $W^{2,\hat{p}}(\Omega)$ , respectively. Observe that the left-hand side of (2.13) is dominated by

$$\begin{aligned} & \| (b_i(x_0) - b_i(\cdot))\partial_i u(\cdot) \|_{1,p,\Omega} + \| (c(x_0) - c(\cdot))u(\cdot) \|_{2,p,\Omega} \\ & \leq \max_{x \in \bar{\Omega} \cap U} |b_i(x_0) - b_i(x)| \cdot \|u\|_{2,p,\Omega} + \sum_{|\mu|=1} \|(\partial^\mu b_i)\partial_i u\|_{0,p,\Omega \cap U} \\ & \quad + \max_{x \in \bar{\Omega} \cap U} |c(x_0) - c(x)| \cdot \|u\|_{2,p,\Omega} + \sum_{|\mu|=1} \|(\partial^\mu c)u\|_{0,p,\Omega \cap U} \\ & \quad + \sum_{|\mu_1|=2} \sum_{|\mu_2|=1, \mu_1 \geq \mu_2} \|(\partial^{\mu_2} c)\partial^{\mu_1-\mu_2} u\|_{0,p,\Omega \cap U} \\ & \equiv I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Since  $b_i$  and  $c$  are continuous on  $\bar{\Omega}$ , we have  $I_1 + I_3 \leq$  the right-hand side of (2.13) by taking  $U$  small. The other terms can be estimated as in Step 1, but in the interest of Remark 2.8, we modify the estimates as follows. Suppose  $p < n$ . Let  $r = np/(n - p)$ . Then by Hölder’s inequality and the embedding  $W^{1,p} \hookrightarrow L^r$ , we have

$$\begin{aligned} I_2 + I_4 + I_5 & \leq \sum_{|\mu|=1} \left( \int_{\Omega \cap U} |\partial^\mu b_i|^n \right)^{1/n} \left( \int_{\Omega} |\partial_i u|^r \right)^{1/r} + \sum_{|\mu|=1,2} \left( \int_{\Omega \cap U} |\partial^\mu c|^n \right)^{1/n} \left( \int_{\Omega} |u|^r \right)^{1/r} \\ & \quad + \sum_{|\mu_1|=2} \sum_{|\mu_2|=1, \mu_1 \geq \mu_2} \left( \int_{\Omega \cap U} |\partial^{\mu_2} c|^n \right)^{1/n} \left( \int_{\Omega} |\partial^{\mu_1-\mu_2} u|^r \right)^{1/r} \\ & \leq \left( \sum_{i=1}^N \|b_i\|_{1,n,\Omega \cap U} + \|c\|_{2,n,\Omega \cap U} \right) \|u\|_{2,p,\Omega}. \end{aligned}$$

Now (2.13) follows if  $U$  is taken to be small. This completes Step 2.

We turn to the proof of (ii) now, and again we divide it into two steps.

*Step 1.* We prove that if (ii) holds for the principal part  $(A_0, B_0)$  of  $(A, B)$ , then it does so for  $(A, B)$ . To this end, we use the estimates in Step 1 of the previous proof, as well as the following:

$$\rho^{1-1/p} \|u\|_{0,p,\partial\Omega} \leq K_4 (\|u\|_{1,p,\Omega} + \rho \|u\|_{0,p,\Omega}), \quad u \in W^{1,p}(\Omega), \quad \rho \geq 1, \tag{2.14}$$

where the constant  $K_4$  is independent of  $u$  and  $\rho$  (see Proposition 2.2 in [5]).

The difference between  $(A_0, B_0)$  and  $(A, B)$  can be estimated by

$$\begin{aligned}
 & \| (A - A_0)u \|_{0,p,\Omega} + \| B - B_0 \|_{\partial W^{2,p}} \\
 & \leq \| a_i \partial_i u \|_{0,p,\Omega} + \| a_0 u \|_{0,p,\Omega} + \| b_0 u \|_{1-1/p,p,\partial\Omega} \\
 & = \| a_i \partial_i u \|_{0,p,\Omega} + \| a_0 u \|_{0,p,\Omega} + \| b_0 u \|_{1-1/p,p,\partial\Omega} + |\sigma|^{(1-1/p)/2} \| b_0 u \|_{0,p,\partial\Omega} \\
 & \leq \| a_i \partial_i u \|_{0,p,\Omega} + \| a_0 u \|_{0,p,\Omega} + \| b_0 u \|_{1-1/p,p,\partial\Omega} + K_4 (\| b_0 u \|_{1,p,\Omega} + |\sigma|^{1/2} \| b_0 u \|_{0,p,\Omega}) \\
 & \leq (\| a_0 \|_{0,\hat{p},\Omega} + \| a_i \|_{0,\hat{p},\Omega} + \| b_0 \|_{1,\hat{p},\Omega}) (\varepsilon \| u \|_{2,p,\Omega} + C(\varepsilon) \| u \|_{0,p,\Omega}) \\
 & \quad + K_4 \| b_0 \|_{0,\infty,\Omega} |\sigma|^{1/2} \| u \|_{0,p,\Omega} \\
 & \leq C\varepsilon \| u \|_{2,p,\Omega}, \tag{2.15}
 \end{aligned}$$

if  $|\sigma|$  is large enough. This implies (2.12) for the full  $(A, B)$ . In particular, this implies  $(A + \sigma I, B)$  is injective. To prove the surjectivity, think of  $(A_0 + \sigma I, B_0)$  as a bounded linear operator from  $(W^{2,p}(\Omega))^N$  to  $(L^p(\Omega))^N \times \partial W^{2,p}$  with the former space equipped with the norm  $\| \cdot \|_{2,p,\Omega}$ , and the latter space equipped with the norm  $\| (u, v) \|_{L^p \times \partial W^{2,p}} = \| u \|_{L^p} + \| v \|_{\partial W^{2,p}}$ . Then  $(A_0 + \sigma I, B_0)^{-1}$  exists on  $(L^p(\Omega))^N \times \partial W^{2,p}$  with its operator norm uniformly bounded for large  $|\sigma|$ . Combining this with (2.15), we see  $(A + \sigma I, B)^{-1}$  exists on  $(L^p(\Omega))^N \times \partial W^{2,p}$  when  $|\sigma|$  is large. Step 1 is completed.

*Step 2.* From now on, we drop the lower order terms from  $A$  and  $B$ . If we follow [12] and [5], all we have to do here is to show that for any fixed  $x_0 \in \partial\Omega$ ,

$$\begin{aligned}
 & \| (b_i(x_0) - b_i(\cdot)) \partial_i u(\cdot) \|_{1-1/p,p,\partial\Omega} + |\sigma|^{(1-1/p)/2} \| (b_i(x_0) - b_i(\cdot)) \partial_i u(\cdot) \|_{0,p,\partial\Omega} \\
 & \quad + \| (c(x_0) - c(\cdot)) u(\cdot) \|_{2-1/p,p,\partial\Omega} + |\sigma|^{(2-1/p)/2} \| (c(x_0) - c(x)) u(\cdot) \|_{0,p,\partial\Omega} \\
 & \leq \varepsilon \| u \|_{2,p,\Omega},
 \end{aligned}$$

for all  $\sigma$  with large  $|\sigma|$  and  $\arg \sigma = \theta_0$ , and for all  $u \in W^{2,p}(\Omega)$  with support in a small neighborhood  $U$  of  $x_0$ , where  $\varepsilon$  is a small constant. The first and third terms above have been estimated in the proof of (i). On the other hand, the sum of the second and fourth terms is dominated by (from (2.14)), for large  $|\sigma|$ ,

$$\begin{aligned}
 & C (\| (b_i(x_0) - b_i(\cdot)) \partial_i u(\cdot) \|_{1,p,\Omega} + |\sigma|^{1/2} \| (b_i(x_0) - b_i(\cdot)) \partial_i u(\cdot) \|_{0,p,\Omega}) \\
 & \quad + C |\sigma|^{1/2} (\| (c(x_0) - c(\cdot)) u(\cdot) \|_{1,p,\Omega} + |\sigma|^{1/2} \| (c(x_0) - c(\cdot)) u(\cdot) \|_{0,p,\Omega}) \\
 & \leq \varepsilon \| u \|_{2,p,\Omega} + C |\sigma|^{1/2} \max_{x \in \bar{\Omega} \cap U} |b_i(x_0) - b_i(x)| \cdot \| u \|_{2,p,\Omega}^{1/2} \| u \|_{0,p,\Omega}^{1/2} \\
 & \quad + C |\sigma| \max_{x \in \bar{\Omega} \cap U} |c(x_0) - c(x)| \cdot \| u \|_{2,p,\Omega}^{1/2} \| u \|_{0,p,\Omega}^{1/2} + C |\sigma|^{1/2} \| \nabla c \|_{0,\infty,\Omega} \| u \|_{0,p,\Omega} \\
 & \leq 2\varepsilon \| u \|_{2,p,\Omega}.
 \end{aligned}$$

In the next to last line, we use the inequality

$$2|\sigma|^{1/2} ab \leq a^2 + |\sigma| b^2.$$

This completes the proof of Theorem 2.7.  $\square$

**Corollary 2.10.** Under all the conditions in Theorem 2.7, the operator  $(A, B) : (W^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N \times \partial W^{2,p}$  is Fredholm with zero index.

**Proof.** Write  $(A, B) = (A + \sigma I, B) - (\sigma I, 0)$ . By (ii) of Theorem 2.7,  $(A + \sigma I, B)$  is an isomorphism and hence Fredholm with index zero for all  $\sigma$  on the ray with large  $|\sigma|$ . Fix such a  $\sigma$ . Since the mapping  $u \in (W^{2,p}(\Omega))^N \mapsto (\sigma u, 0) \in (L^p(\Omega))^N \times \partial W^{2,p}$  is compact, by Theorem V.2.1 of [17],  $\text{ind}(A, B) = \text{ind}(A + \sigma I, B) = 0$ .  $\square$

$$\text{Let } W_B^{2,p}(\Omega) = \{u \in W^{2,p}(\Omega) \mid Bu = 0\}.$$

**Corollary 2.11.** Under all the conditions in Theorem 2.7, the operator  $A : (W_B^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N$  is Fredholm with zero index.

### 3. Quasilinear second order elliptic operators

Let the following be real  $N \times N$  matrix-valued functions:

$$a_{ij}(\lambda, x, z, q), \quad b_i(\lambda, x, z) \quad (i, j = 1, \dots, n),$$

where  $\lambda \in (a, b)$ ,  $z \in \mathbf{R}^N$ ,  $q \in \mathbf{R}^{nN}$ , and  $x \in \overline{\Omega}$  ( $x \in \partial\Omega$  for  $b_i$ ),

and  $N$ -dimensional real column-vector-valued functions:

$$u(x), \quad f(\lambda, x, z, q), \quad g(\lambda, x, z), \quad \text{and } h(\lambda, x, z),$$

where  $\lambda \in (a, b)$ ,  $z \in \mathbf{R}^N$ ,  $q \in \mathbf{R}^{nN}$ , and  $x \in \overline{\Omega}$  ( $x \in \partial\Omega$  for  $g$  and  $h$ ).

Consider the quasilinear second order operator in  $\Omega$  with a parameter  $\lambda$ :

$$A(\lambda, u) = -a_{ij}(\lambda, x, u, \nabla u) \partial_i \partial_j u + f(\lambda, x, u, \nabla u), \quad x \in \Omega,$$

which is associated with a quasilinear boundary operator

$$B(\lambda, u) = \delta(x)[b_i(\lambda, x, u) \partial_i u + g(\lambda, x, u)] + (I - \delta(x))h(\lambda, x, u), \quad x \in \partial\Omega.$$

We define

$$T(\lambda, u) = (A(\lambda, u), B(\lambda, u)).$$

We assume the following regularity conditions on the coefficient functions in  $A$  and  $B$ :

$$\begin{aligned} &a_{ij} \text{ and } f \in C^1((a, b) \times \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{nN}), \\ &b_i \in C^1((a, b) \times \partial\Omega \times \mathbf{R}^N), \quad \partial_{(\alpha,z)}(\partial_{(\lambda,z)} b_i) \text{ is continuous on } (a, b) \times \partial\Omega \times \mathbf{R}^N, \\ &g \in C^1((a, b) \times \partial\Omega \times \mathbf{R}^N), \quad \partial_{(\alpha,z)}(\partial_{(\lambda,z)} g) \text{ is continuous on } (a, b) \times \partial\Omega \times \mathbf{R}^N, \\ &h \in C^1((a, b) \times \partial\Omega \times \mathbf{R}^N), \quad \partial_{(\alpha,z)}^\mu(\partial_{(\lambda,z)} h), \quad |\mu| = 1, 2, \quad \text{are continuous on } (a, b) \times \partial\Omega \times \mathbf{R}^N, \end{aligned} \tag{3.1}$$

where the partial derivatives in  $x \in \partial\Omega$  are understood as the ones in the tangent space of  $\partial\Omega$ .

**Proposition 3.1.** Suppose  $p > n$ ,  $\partial\Omega \in C^3$  and that (3.1) holds. Then  $T$  is a  $C^1$  mapping from  $(a, b) \times (W^{2,p}(\Omega))^N$  to  $(L^p(\Omega))^N \times \partial W^{2,p}(\partial\Omega)$  with the partial derivatives given by

$$D_\lambda T(\lambda, u) = (D_\lambda A(\lambda, u), D_\lambda B(\lambda, u)),$$

$$D_u T(\lambda, u) = (D_u A(\lambda, u), D_u B(\lambda, u)),$$

where

$$D_\lambda A(\lambda, u) = -\frac{\partial a_{ij}}{\partial \lambda}(\lambda, x, u, \nabla u) \partial_i \partial_j u + \frac{\partial f}{\partial \lambda}(\lambda, x, u, \nabla u),$$

$$D_\lambda B(\lambda, u) = \delta(x) \left( \frac{\partial b_i}{\partial \lambda}(\lambda, x, u) \partial_i u + \frac{\partial g}{\partial \lambda}(\lambda, x, u) \right) + (I - \delta(x)) \frac{\partial h}{\partial \lambda}(\lambda, x, u),$$

$$\begin{aligned} D_u A(\lambda, u)[w] &= -a_{ij}(\lambda, x, u, \nabla u) \partial_i \partial_j w - (\nabla_q a_{ij}(\lambda, x, u, \nabla u) \nabla w) \partial_i \partial_j u \\ &\quad - (\nabla_z a_{ij}(\lambda, x, u, \nabla u) w) \partial_i \partial_j u + \nabla_q f(\lambda, x, u, \nabla u) \nabla w + \nabla_z f(\lambda, x, u, \nabla u) w, \end{aligned}$$

$$\begin{aligned} D_u B(\lambda, u)[w] &= \delta(x) [b_i(\lambda, x, u) \partial_i w + (\nabla_z b_i(\lambda, x, u) w) \partial_i u + \nabla_z g(\lambda, x, u) w] \\ &\quad + (I - \delta(x)) \nabla_z h(\lambda, x, u) w. \end{aligned}$$

Here  $\nabla_q f$  is understood as an  $N \times (nN)$  derivative matrix,  $\nabla_q a_{ij} \nabla w$  as the  $N \times N$  derivative matrix obtained by taking the dot product of  $\nabla w$  with the  $q$ -gradient of each entry of the matrix  $a_{ij}$ ,  $(\nabla_z b_i) w$  as the  $N \times N$  derivative matrix obtained by taking the dot product of  $w$  with the  $z$ -gradient of each entry of the matrix  $b_i$ .

Gebran and Stuart [18] proved that for fixed  $\lambda$ ,  $A(\lambda, \cdot) : (W^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N$  is  $C^1$  smooth under the Dirichlet boundary condition (see [18, Theorem 2.18]). Since much of the arguments for our purpose are similar to [18], we shall only show that  $B : \mathbf{R} \times (W^{2,p}(\Omega))^N \rightarrow \partial W^{2,p}(\partial\Omega)$  is  $C^1$  smooth. To that end, we need

**Lemma 3.2.** Let  $\partial\Omega \in C^3$ , and let  $f(\lambda, x, r)$  be a function defined for  $(\lambda, x, r) \in \mathbf{R}^k \times \partial\Omega \times \mathbf{R}^s$ , satisfying the regularity condition satisfied by either  $g$  or  $h$  (see (3.1)). Then there exists an extension  $\tilde{f}(\lambda, x, r)$  of  $f(\lambda, x, r)$  such that:

- (i)  $\tilde{f}$  is defined on  $\mathbf{R}^k \times \overline{\Omega} \times \mathbf{R}^s$  and  $\tilde{f}(\lambda, x, r) = f(\lambda, x, r)$  for any  $(\lambda, x, r) \in \mathbf{R}^k \times \partial\Omega \times \mathbf{R}^s$ .
- (ii)  $\tilde{f}$  satisfies the same regularity condition on  $\mathbf{R}^k \times \overline{\Omega} \times \mathbf{R}^s$  as  $f$  does on  $\mathbf{R}^k \times \partial\Omega \times \mathbf{R}^s$ .

**Proof.** Since  $\partial\Omega$  is a  $C^3$  submanifold of  $\mathbf{R}^n$ , there exists a  $C^3$  atlas  $\{(N_i, h_i)\}_{i=1}^p$  of  $\partial\Omega$  such that  $h_i(N_i)$  is the open unit ball  $B$  in  $\mathbf{R}^n$  with the center at the origin,  $h_i(N_i \cap \partial\Omega) = B \cap \{(y_1, y_2, \dots, y_n) : y_n = 0\}$ , and  $h_i(N_i \cap \Omega) = B \cap \{(y_1, y_2, \dots, y_n) : y_n > 0\}$ . Let  $\{\phi_i\}_{i=1}^p$  be a partition of unity subordinate to the covering  $\{N_i\}_{i=1}^p$ . For any  $x \in \overline{\Omega}$ , let  $I_x = \{i : x \in N_i\}$ , define

$$\tilde{f}(\lambda, x, r) = \begin{cases} \sum_{i \in I_x} (\phi_i f)(\lambda, h_i^{-1}(P(h_i(x))), r), & x \in \bigcup_{i \in I_x} N_i, \\ 0 & \text{if } I_x \text{ is empty,} \end{cases}$$

where  $P$  is the projection from  $\mathbf{R}^n$  to  $\mathbf{R}^{n-1}$ :  $P(y_1, \dots, y_n) = (y_1, \dots, y_{n-1}, 0)$ . If  $x \in \partial\Omega$ ,

$$\tilde{f}(\lambda, x, r) = \sum_{i \in I_x} (\phi_i f)(\lambda, x, r) = f(\lambda, x, r),$$

i.e., (i) holds. (ii) follows from the fact that  $P(h_i(x)) \in C^3(N_i)$  and  $\phi_i$  has compact support in  $N_i$ .  $\square$

**Proof of Proposition 3.1.** Since  $p > n$ ,  $W^{2,p}(\Omega)$  can be continuously embedded into  $C^{1,1-n/p}(\overline{\Omega})$ . So by our regularity assumptions (3.1) and Lemma 3.2, for fixed  $\lambda$ ,  $\tilde{b}_i(\lambda, x, u(x)) \in C^1(\overline{\Omega})$  (and so  $\tilde{b}_i \partial_i u \in W^{1,p}(\Omega)$ ),  $\tilde{g}(\lambda, x, u(x)) \in C^1(\overline{\Omega})$ ,  $\tilde{h}(\lambda, x, u(x)) \in W^{2,p}(\Omega)$ . Thus  $B(\lambda, u) \in \partial W^{2,p}$ .

We now show that  $B$  is differentiable at each  $(\lambda_0, u_0) \in (a, b) \times (W^{2,p}(\Omega))^N$ . It will be convenient to use the fact that  $W^{m,p}(\Omega)$  is a Banach algebra if  $mp > n$  (see Theorem 4.39 of [1]). Thus in this case,  $W^{m-1/p,p}(\partial\Omega)$  is also a Banach algebra. For the differentiability of  $B$  at  $(\lambda_0, u_0)$ , we consider

$$\begin{aligned} & B(\lambda, u) - B(\lambda_0, u_0) - D_\lambda B(\lambda_0, u_0)(\lambda - \lambda_0) - D_u B(\lambda_0, u_0)(u - u_0) \\ &= \int_0^1 \frac{d}{dt} B(t\lambda + (1-t)\lambda_0, tu + (1-t)u_0) dt - D_\lambda B(\lambda_0, u_0)(\lambda - \lambda_0) - D_u B(\lambda_0, u_0)(u - u_0) \\ &= \int_0^1 \left[ \delta(x) \left( \frac{\partial b_i}{\partial \lambda}(\lambda_*, x, u_*) \partial_i u_* - \frac{\partial b_i}{\partial \lambda}(\lambda_0, x, u_0) \partial_i u_0 + \frac{\partial g}{\partial \lambda}(\lambda_*, x, u_*) - \frac{\partial g}{\partial \lambda}(\lambda_0, x, u_0) \right) (\lambda - \lambda_0) \right. \\ &\quad + (I - \delta(x)) \left( \frac{\partial h}{\partial \lambda}(\lambda_*, x, u_*) - \frac{\partial h}{\partial \lambda}(\lambda_0, x, u_0) \right) (\lambda - \lambda_0) \\ &\quad + \delta(x) (b_i(\lambda_*, x, u_*) - b_i(\lambda_0, x, u_0)) \partial_i (u - u_0) \\ &\quad + \delta(x) ((\nabla_z b_i(\lambda_*, x, u_*)(u - u_0)) \partial_i u_* - (\nabla_z b_i(\lambda_0, x, u_0)(u - u_0)) \partial_i u_0) \\ &\quad + \delta(x) (\nabla_z g(\lambda_*, x, u_*) - \nabla_z g(\lambda_0, x, u_0))(u - u_0) \\ &\quad \left. + (I - \delta(x)) (\nabla_z h(\lambda_*, x, u_*) - \nabla_z h(\lambda_0, x, u_0))(u - u_0) \right] dt \\ &\equiv \int_0^1 [I_1 + I_2 + I_3 + I_4 + I_5 + I_6] dt. \end{aligned}$$

Here  $\lambda_* = t\lambda + (1-t)\lambda_0$  and  $u_* = tu + (1-t)u_0$ . We estimate the terms  $I_i$  one by one.

$$\begin{aligned} \|I_1\|_{1-1/p,p,\partial\Omega} &\leq |\lambda - \lambda_0| \left[ \left\| \frac{\partial b_i}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) (\partial_i u_*(\cdot) - \partial_i u_0(\cdot)) \right\|_{1-1/p,p,\partial\Omega} \right. \\ &\quad + \left\| \left( \frac{\partial b_i}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial b_i}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right) \partial_i u_0 \right\|_{1-1/p,p,\partial\Omega} \\ &\quad \left. + \left\| \frac{\partial g}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial g}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right\|_{1-1/p,p,\partial\Omega} \right] \\ &\equiv |\lambda - \lambda_0| \cdot [I_{11} + I_{12} + I_{13}]. \end{aligned}$$

As  $(\lambda, u) \rightarrow (\lambda_0, u_0)$  in  $\mathbf{R} \times (W^{2,p}(\Omega))^N$ , we have, uniformly with respect to  $t \in [0, 1]$ :

$$\begin{aligned} I_{11} &\leq C \left\| \frac{\partial b_i}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) \right\|_{1-1/p,p,\partial\Omega} \| \partial_i u_* - \partial_i u_0 \|_{1-1/p,p,\partial\Omega} \\ &\leq C \left\| \frac{\partial b_i}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) \right\|_{C^1(\partial\Omega)} \| \partial_i u_* - \partial_i u_0 \|_{1,p,\Omega} \\ &\leq C \| \partial_i u_* - \partial_i u_0 \|_{2,p,\Omega} \rightarrow 0, \end{aligned}$$

because  $\nabla_{(x,z)} \partial b_i / \partial \lambda$  is continuous and the  $C^1$  norm of  $u_*$  is bounded;

$$I_{12} \leq C \left\| \frac{\partial b_i}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial b_i}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right\|_{C^1(\partial\Omega)} \|\partial_i u_0\|_{1-1/p, p, \partial\Omega} \rightarrow 0,$$

because  $u_* \rightarrow u_0$  in  $(C^1(\partial\Omega))^N$  and that  $\nabla_{(x,z)} \partial b_i / \partial \lambda$  is continuous.  $I_{13} \rightarrow 0$  because  $u \rightarrow u_0$  in  $(C^1(\partial\Omega))^N$  and that  $\nabla_{(x,z)} \partial g / \partial \lambda$  is continuous.

Let  $\tilde{h}(\lambda, x, z)$  be the extension of  $h(\lambda, x, z)$  given in Lemma 3.2. Then

$$\begin{aligned} \|I_2\|_{2-1/p, p, \Omega} &\leq |\lambda - \lambda_0| \left\| \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right\|_{2, p, \Omega} \\ &\leq C |\lambda - \lambda_0| \left[ \left\| \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right\|_{C^1(\bar{\Omega})} \right. \\ &\quad \left. + \sum_{|\mu|=2} \left\| \partial_X^\mu \left( \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_*, \cdot, u_*(\cdot)) - \frac{\partial \tilde{h}}{\partial \lambda}(\lambda_0, \cdot, u_0(\cdot)) \right) \right\|_{0, p, \Omega} \right] \\ &= o(|\lambda - \lambda_0|) \quad \text{uniformly with respect to } t \in [0, 1], \end{aligned}$$

because  $\lambda_* \rightarrow \lambda_0, u \rightarrow u_0$  in  $(C^1(\partial\Omega))^N$ , and the smoothness of  $\partial \tilde{h} / \partial \lambda$ .

$$\begin{aligned} \|I_3\|_{1-1/p, p, \partial\Omega} &\leq \|b_i(\lambda_*, \cdot, u_*(\cdot)) - b_i(\lambda_0, \cdot, u_0(\cdot))\|_{1-1/p, p, \partial\Omega} \|\partial_i(u - u_0)\|_{1-1/p, p, \partial\Omega} \\ &= o(\|u - u_0\|_{2, p, \Omega}), \end{aligned}$$

because  $b_i \in C^1$  and  $u_* \rightarrow u_0$ .

$$\begin{aligned} \|I_4\|_{1-1/p, p, \partial\Omega} &\leq \|(\nabla_z \tilde{b}_i(\lambda_*, \cdot, u_*(\cdot))(u - u_0(\cdot)) \partial_i u_*(\cdot) - (\nabla_z \tilde{b}_i(\lambda_0, \cdot, u_0(\cdot))(u - u_0(\cdot)) \partial_i u_0(\cdot))\|_{1, p, \Omega} \\ &\leq \|(\nabla_z \tilde{b}_i(\lambda_*, \cdot, u_*(\cdot)) - \nabla_z \tilde{b}_i(\lambda_0, \cdot, u_0(\cdot)))(u - u_0) \partial_i u_*\|_{1, p, \Omega} \\ &\quad + \|(\nabla_z \tilde{b}_i(\lambda_0, \cdot, u_0(\cdot))(u - u_0(\cdot)) \partial_i(u_* - u_0))\|_{1, p, \Omega} \\ &= o(\|u - u_0\|_{2, p, \Omega}), \end{aligned}$$

$$\begin{aligned} \|I_5\|_{1-1/p, p, \partial\Omega} &\leq C \|\nabla_z g(\lambda_*, \cdot, u_*(\cdot)) - \nabla_z g(\lambda_0, \cdot, u_0(\cdot))\|_{C^1(\partial\Omega)} \|u - u_0\|_{1-1/p, p, \partial\Omega} \\ &= o(\|u - u_0\|_{2, p, \Omega}), \end{aligned}$$

$$\begin{aligned} \|I_6\|_{2-1/p, p, \partial\Omega} &\leq \|(\nabla_z \tilde{h}(\lambda_*, \cdot, u_*(\cdot)) - \nabla_z \tilde{h}(\lambda_0, \cdot, u_0(\cdot)))(u - u_0)\|_{2, p, \Omega} \\ &= o(\|u - u_0\|_{2, p, \Omega}). \end{aligned}$$

Combining these estimates, we conclude that  $B$  is differentiable at  $(\lambda_0, u_0)$ , and the arguments above actually yield the continuity of  $(D_\lambda B, D_u B)$ .  $\square$

**Theorem 3.3.** *Suppose that  $p > n$ ,  $\partial\Omega \in C^3$ , and the regularity assumption (3.1) holds. Let  $U$  be an open connected set of  $(a, b) \times (W^{2,p}(\Omega))^N$ . Assume that for each fixed  $(\lambda, u) \in U$ ,  $D_u T(\lambda, u) = (D_u A(\lambda, u), D_u B(\lambda, u))$  is elliptic on  $\overline{\Omega}$ , and that for a particular  $(\lambda_0, u_0) \in U$ ,  $D_u T(\lambda_0, u_0)$  satisfies Agmon’s condition at an angle  $\theta_0$ , then the Fredholm index of  $D_u T(\lambda, u)$  is 0 for all  $(\lambda, u) \in U$ .*

**Proof.** As can be easily checked, the coefficients of  $D_u A(\lambda, u)$  and  $D_u B(\lambda, u)$  satisfy the assumptions of Theorem 2.7. By (i) of that theorem and Peetre’s Lemma ([31, Lemma 3] and [17, Theorem VII.2.1])  $D_u T(\lambda, u)$  is semi-Fredholm, i.e., its kernel is finite-dimensional and its range is closed. Moreover, by Corollary 2.10,  $\text{ind}(D_u A(\lambda_0, u_0), D_u B(\lambda_0, u_0)) = 0$ . Finally from the continuity of the index of semi-Fredholm operators (see [17, Theorem V.1.6]), we conclude the proof.  $\square$

**Remark 3.4.** 1. In some situations, the boundary operator  $B$  is linear in  $u$ , is independent of the parameter  $\lambda$ , and the boundary condition is homogeneous:  $Bu = 0$ . In such a case, it is perhaps more convenient to think of  $A$  as an operator from  $(a, b) \times (W_B^{2,p}(\Omega))^N$  to  $(L^p(\Omega))^N$ . Then under weaker regularity conditions, we have the analogues of Proposition 3.1 and Theorem 3.3 for  $A(\lambda, u)$ . More specifically, we assume the regularity conditions weaker than (3.1):

$$\begin{aligned} a_{ij} \text{ and } f &\in C^1((a, b) \times \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{nN}), \\ b_i, b_0 &\in W^{1-\frac{1}{p}, p}(\partial\Omega), \\ c &\in W^{2-\frac{1}{p}, p}(\partial\Omega), \end{aligned}$$

and we assume  $\partial\Omega$  is  $C^2$  smooth only. Then for  $p > n$ ,  $A : (a, b) \times (W_B^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N$  is  $C^1$  smooth. In addition to the above conditions, if we also assume the ellipticity and Agmon’s condition in Theorem 3.3 with  $U$  now being an open connected set in  $(a, b) \times (W_B^{2,p}(\Omega))^N$ , then the Fredholm index of  $D_u A(\lambda, u) : (W_B^{2,p}(\Omega))^N \rightarrow (L^p(\Omega))^N$  is zero for all  $(\lambda, u) \in U$ .

2. In Proposition 3.1 and Theorem 3.3, we only need  $\partial\Omega$  to be  $C^2$  if  $B$  is linear in  $u$ .

**4. Application to global bifurcation theory**

*4.1. Abstract theory*

Let  $X$  and  $Y$  be real Banach spaces, let  $K(X, Y)$ ,  $GL(X, Y)$  and  $\Phi_0(X, Y)$  be the sets of compact linear operators, invertible linear operators and linear Fredholm operators of index 0, respectively. Let  $S(X) = S(X, X)$  for  $S = K, GL, \Phi_0$ . Let  $P \in C^0([a, b], K(X))$ , and let  $I - P(a)$ ,  $I - P(b)$  be invertible. Define the parity of  $I - P$  by

$$\sigma(I - P) = \text{deg}(I - P(a)) \cdot \text{deg}(I - P(b)),$$

where  $\text{deg}(\cdot)$  is the Leray–Schauder degree. Consider a curve  $A \in C^0([a, b], \Phi_0(X, Y))$ , and  $A(a), A(b) \in GL(X, Y)$ . Then there exists  $N \in C^0([a, b], GL(Y, X))$  such that  $N(t)A(t) = I - P(t)$  for every  $t \in [a, b]$ , where  $P \in C^0([a, b], K(X))$ . The parity of  $A$  over  $[a, b]$  is defined by

$$\sigma(A) = \sigma(I - P).$$

The existence of  $N, P$ , and the independence of  $\sigma(A)$  on  $N, P$  are reviewed in [30].

The following global bifurcation theorem is proved in [30, Theorem 6.1].

**Theorem 4.1.** *Let  $V$  be an open connected subset of  $\mathbf{R} \times X$  such that  $V \cap \{(\lambda, u_0) : \lambda \in \mathbf{R}\} = \{(\lambda, u_0) : \lambda \in (a, b)\}$ . Suppose that the mapping  $T : V \rightarrow Y$  is  $C^1$  smooth with  $T(\lambda, u_0) \equiv 0$  for  $\lambda \in (a, b)$ .*

Assume that  $D_u T(\lambda, u) : X \rightarrow Y$  is Fredholm of index 0 for every  $(\lambda, u) \in V$ , and there exist  $\lambda_- < \lambda_+$ , all in  $(a, b)$ , such that  $D_u T(\lambda_{\pm}, u_0) \in GL(X, Y)$  and  $\sigma(D_u T(\lambda, u_0)|_{[\lambda_-, \lambda_+]}) = -1$ . Let  $S = \{(\lambda, u) \in V : T(\lambda, u) = 0, u \neq u_0\}$ , and let  $S_0 = \{(\lambda, u_0) : \lambda \in [\lambda_-, \lambda_+]\}$ . Denote by  $C$  the connected component of  $\bar{S} \cup S_0$  containing  $S_0$ . Then  $C \setminus S_0 \neq \emptyset$ , either  $C$  is not compact in  $V$ , or  $C$  contains a point  $(\lambda_*, u_0)$  with  $\lambda_* \notin [\lambda_-, \lambda_+]$ .

**Remark 4.2.** 1. “ $C$  is not compact in  $V$ ” is equivalent to “ $C$  intersects  $\partial V$  or  $C$  is unbounded” if any closed and bounded subset of  $V$ , consisting of solutions  $(\lambda, u)$  of the equation  $T(\lambda, u) = 0$ , is compact. For quasilinear elliptic boundary value problems, this compactness follows from the elliptic regularity theory.

2. The parity is not easy to compute by using its definition. However, if there exists a  $\lambda_0 \in (a, b)$  such that  $D_u T(\lambda_0, u_0)$  has nontrivial null space  $\mathcal{N}(D_u T(\lambda_0, u_0))$ , and that  $D_u T(\lambda, u_0)$  is differentiable with respect to  $\lambda$  at  $\lambda_0$ , under the following “transversality condition”:

$$D_{\lambda u} T(\lambda_0, u_0)[w] \notin \mathcal{R}(D_u T(\lambda_0, u_0)), \quad \forall w \in \mathcal{N}(D_u T(\lambda_0, u_0)) \setminus \{0\}, \tag{4.1}$$

which is equivalent to

$$[D_{\lambda u} T(\lambda_0, u_0)\mathcal{N}(D_u T(\lambda_0, u_0))] \oplus \mathcal{R}(D_u T(\lambda_0, u_0)) = Y,$$

then from [15, Theorem 6.18], there exists a small  $\epsilon > 0$  such that  $D_u T(\lambda, u_0) \in GL(X, Y)$  if  $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ ,  $\lambda \neq \lambda_0$ , and  $\sigma(D_u T(\lambda, u_0)|_{[\lambda_0 - \epsilon, \lambda_0 + \epsilon]}) = (-1)^m$ , where  $m = \dim \mathcal{N}(D_u T(\lambda_0, u_0))$ . Thus when  $m$  is odd, the conclusion of Theorem 4.1 holds. We mention the well-known fact that under the transversality condition the oddness of  $m$  is also necessary for local bifurcation.

3. When the transversality condition is lacking, we may use a formula in [15] involving the “eigenvalue crossing number” to compute the parity in Theorem 4.1. Let  $A(\lambda) = D_u T(\lambda, u_0)$ . We assume that (i)  $X$  and  $Y$  are real Banach spaces with  $X \subset Y$ , and  $X$  being continuously embedded into  $Y$  by the identity map; (ii) for all  $\lambda$  near but not equal to  $\lambda_0$ ,  $A(\lambda) \in GL(X, Y)$ ; (iii) 0 is an eigenvalue of  $A(\lambda_0)$  and it is isolated in the real spectrum of the operator. From (i) and (iii), it follows that  $\dim(\bigcup_{p \geq 1} \mathcal{N}(A(\lambda_0)^p))$  is finite, which is called the *algebraic multiplicity* of the zero eigenvalue of  $A(\lambda_0)$  (see, e.g., [15, Proposition 5.11]).  $(\dim \mathcal{N}(A(\lambda_0)))$  is called the *geometric multiplicity* of the zero eigenvalue.) By Theorem 5.12 of [15], there exist  $\epsilon > 0$  and  $\beta > 0$  such that if  $0 < |\lambda - \lambda_0| \leq \epsilon$ , then  $A(\lambda)$  has only finitely many eigenvalues in  $(-\beta, 0)$ , all with finite algebraic multiplicity; let  $n(\lambda)$  be the sum of the algebraic multiplicities of all eigenvalues of  $A(\lambda)$  in  $(-\beta, 0)$ , then  $n(\lambda)$  is constant on each side of  $\lambda_0$ ; moreover  $\sigma(A(\lambda)|_{[\lambda_0 - \epsilon, \lambda_0 + \epsilon]}) = (-1)^{\chi(\lambda_0)}$ , where  $\chi(\lambda_0)$ , called *eigenvalue crossing number*, is  $n(\lambda_0 - \epsilon) - n(\lambda_0 + \epsilon)$ . Thus when  $\chi(\lambda_0)$  is odd, Theorem 4.1 applies. Under this scenario, the celebrated local bifurcation theorem of Krasnosel’skii [20] and the global bifurcation theorem of Rabinowitz [35] are recovered. The eigenvalue crossing number of  $A(\lambda)$  at  $\lambda_0$  is one of the several versions of “generalized algebraic multiplicity of  $\lambda_0$  in  $A(\lambda)$ ” (see [15] for a summary). Another user-friendly version of generalized multiplicity is the one advocated by Rabier in [32]; it can be characterized as the dimension of the null space of an operator constructed from  $A(\lambda)$  and its derivatives at  $\lambda_0$ , and thus potentially it is computable in applications.

4. It seems that the transversality condition (4.1) first appeared in [9] with  $m = 1$ , under which Crandall and Rabinowitz proved the well-known local bifurcation theorem [9, Theorem 1.7]. We now know, according to part 2 of this remark, that the local bifurcation is actually a global one, provided that all the conditions in Theorem 4.1 and part 2 of this remark are satisfied. The global version of the Crandall–Rabinowitz bifurcation theorem from a simple eigenvalue is most important in applications, so we rephrase it here:

**Theorem 4.3.** Let  $V$  be an open connected subset of  $\mathbf{R} \times X$  and  $(\lambda_0, u_0) \in V$ , and let  $\mathbf{F}$  be a continuously differentiable mapping from  $V$  into  $Y$ . Suppose that

1.  $\mathbf{F}(\lambda, u_0) = 0$  for  $(\lambda, u_0) \in V$ ,



2. the partial derivative  $D_{\lambda u} \mathbf{F}(\lambda, u)$  exists and is continuous in  $(\lambda, u)$  near  $(\lambda_0, u_0)$ ,
3.  $D_u \mathbf{F}(\lambda_0, u_0)$  is a Fredholm operator with index 0, and  $\dim \mathcal{N}(D_u \mathbf{F}(\lambda_0, u_0)) = 1$ ,
4.  $D_{\lambda u} \mathbf{F}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(D_u \mathbf{F}(\lambda_0, u_0))$ , where  $w_0 \in X$  spans  $\mathcal{N}(D_u \mathbf{F}(\lambda_0, u_0))$ .

Let  $Z$  be any complement of  $\text{span}\{w_0\}$  in  $X$ . Then there exist an open interval  $I_1 = (-\epsilon, \epsilon)$  and continuous functions  $\lambda : I_1 \rightarrow \mathbf{R}$ ,  $\psi : I_1 \rightarrow Z$ , such that  $\lambda(0) = \lambda_0$ ,  $\psi(0) = 0$ , and, if  $u(s) = u_0 + s w_0 + s \psi(s)$  for  $s \in I_1$ , then  $\mathbf{F}(\lambda(s), u(s)) = 0$ . Moreover,  $\mathbf{F}^{-1}(\{0\})$  near  $(\lambda_0, u_0)$  consists precisely of the curves  $u = u_0$  and  $\Gamma = \{(\lambda(s), u(s)) : s \in I_1\}$ . If in addition,  $D_u \mathbf{F}(\lambda, u)$  is a Fredholm operator for all  $(\lambda, u) \in V$ , then the curve  $\Gamma$  is contained in  $\mathcal{C}$ , which is a connected component of  $\bar{S}$  where  $S = \{(\lambda, u) \in V : \mathbf{F}(\lambda, u) = 0, u \neq u_0\}$ ; and either  $\mathcal{C}$  is not compact in  $V$ , or  $\mathcal{C}$  contains a point  $(\lambda_*, u_0)$  with  $\lambda_* \neq \lambda_0$ .

Another extension of the Crandall–Rabinowitz bifurcation theorem from simple eigenvalue was recently proved in Liu, Shi and Wang [23].

Near the bifurcation point  $(\lambda_0, u_0)$ , the connected component  $\mathcal{C}$  is in the form of a smooth curve. Indeed the portions of  $\mathcal{C}$  with  $s \in (0, \epsilon)$  and  $s \in (-\epsilon, 0)$  respectively could each be contained in a connected component of  $\mathcal{C} \setminus \{(\lambda_0, u_0)\}$ , and either component could be non-compact in  $V$ . Such “unilateral” global bifurcation results are very useful in studying elliptic PDEs when only positive solutions are desired. Theorems 1.27 and 1.40 of Rabinowitz [35] are the pioneering ones in this direction. (However, as pointed out by Dancer [11] and López-Gómez [24, p. 180], the proofs of these theorems contain gaps, the original statement of Theorem 1.40 of [35] is not correct, and the original statement of Theorem 1.27 of [35] is stronger than what one can actually prove so far (see Theorem 6.4.3 of [24].)) Here we prove a unilateral global bifurcation result for Fredholm operators based on López-Gómez’s interpretation of Rabinowitz’s Theorem 1.27 and our Theorem 4.3:

**Theorem 4.4.** *Suppose that all conditions in Theorem 4.3 are satisfied. Let  $\mathcal{C}$  be defined as in Theorem 4.3. We define  $\Gamma_+ = \{(\lambda(s), u(s)) : s \in (0, \epsilon)\}$  and  $\Gamma_- = \{(\lambda(s), u(s)) : s \in (-\epsilon, 0)\}$ . In addition we assume that*

1.  $\mathbf{F}_u(\lambda, u_0)$  is continuously differentiable in  $\lambda$  for  $(\lambda, u_0) \in V$ ;
2. the norm function  $u \mapsto \|u\|$  in  $X$  is continuously differentiable for any  $u \neq 0$ ;
3. for  $k \in (0, 1)$ , if  $(\lambda, u_0)$  and  $(\lambda, u)$  are both in  $V$ , then  $(1 - k)\mathbf{F}_u(\lambda, u_0) + k\mathbf{F}_u(\lambda, u)$  is a Fredholm operator.

Let  $\mathcal{C}^+$  (resp.  $\mathcal{C}^-$ ) be the connected component of  $\mathcal{C} \setminus \Gamma_-$  which contains  $\Gamma_+$  (resp. the connected component of  $\mathcal{C} \setminus \Gamma_+$  which contains  $\Gamma_-$ ). Then each of the sets  $\mathcal{C}^+$  and  $\mathcal{C}^-$  satisfies one of the following: (i) it is not compact; (ii) it contains a point  $(\lambda_*, u_0)$  with  $\lambda_* \neq \lambda_0$ ; or (iii) it contains a point  $(\lambda, u_0 + z)$ , where  $z \neq 0$  and  $z \in Z$ .

**Proof.** Recall that  $Z$  is a subspace of  $X$  which complements  $\text{span}\{w_0\}$ . From Hahn–Banach Theorem, there exists  $l \in X^*$  such that  $Z = \{u \in X : \langle l, u \rangle = 0\}$ ,  $l$  and  $w_0$  are normalized so that  $\|w_0\| = 1$  and  $\langle l, w_0 \rangle = 1$ . Then  $X = \mathbf{R} \oplus Z$  with  $u = \alpha w_0 + v$  where  $\alpha = \langle l, u \rangle$  and  $v \in Z$ .

Without loss of generality, we assume that  $u_0 = 0$ . For  $\xi > 0$  and  $\eta \in (0, 1)$ , we define

$$K_{\xi, \eta} = \{(\lambda, u) \in V : |\lambda - \lambda_0| < \xi, |\langle l, u \rangle| > \eta \|u\|\}.$$

We fix some  $\xi > 0$  and  $\eta \in (0, 1)$ . By the formula of  $u(s)$  in Theorem 4.3, the connected component  $\mathcal{C}$  satisfies that there exists  $\delta_0 > 0$ , such that for all  $0 < \delta < \delta_0$ ,

$$(\mathcal{C} \setminus \{(\lambda_0, 0)\}) \cap B_\delta((\lambda_0, 0)) \subset K_{\xi, \eta},$$

where  $B_\delta((\lambda_0, 0))$  is a ball in  $V$  centered at  $(\lambda_0, 0)$  and with radius  $\delta$ ; and for any  $(\lambda, u) \in (\mathcal{C} \setminus \{(\lambda_0, 0)\}) \cap B_\delta((\lambda_0, 0))$ ,  $u = \alpha w_0 + v$  with  $|\alpha| = |\langle l, u \rangle| > \eta \|u\|$ ,  $v \in Z$ ,  $|\lambda - \lambda_0| = o(1)$  and  $v = o(|\alpha|)$ .

We rewrite the nonlinear mapping  $\mathbf{F}(\lambda, u) = \mathbf{F}_u(\lambda, 0)u + H(\lambda, u)$ , and we define a new mapping in  $V$ :

$$\widehat{\mathbf{F}}(\lambda, u) = \begin{cases} \mathbf{F}(\lambda, u) & \text{if } \langle l, u \rangle \leq -\eta \|u\|, \\ \mathbf{F}_u(\lambda, 0)u + g\left(\frac{-\langle l, u \rangle}{\eta \|u\|}\right)H(\lambda, u) & \text{if } 0 \geq \langle l, u \rangle \geq -\eta \|u\|, \ u \neq 0, \\ -\widehat{\mathbf{F}}(\lambda, -u) & \text{if } \langle l, u \rangle \geq 0, \end{cases}$$

where  $g : \mathbf{R} \rightarrow \mathbf{R}$  is a continuously differentiable increasing function satisfying  $g(0) = 0, g(1) = 1$  and  $g'(1) = 0$ . We can easily verify that  $\widehat{\mathbf{F}}$  is continuous on  $V$ . We prove that  $\widehat{\mathbf{F}}$  is  $C^1$  for any  $(\lambda, u) \in V$  as follows. Since the definition of  $\widehat{\mathbf{F}}$  is for fixed  $\lambda$ , the partial derivative in  $\lambda$  exists and is continuous. We only need to show that the partial derivative in  $u$  exists and is continuous. This is apparently true when  $\langle l, u \rangle < -\eta \|u\|$ , and  $\widehat{\mathbf{F}}$  is  $C^1$  when  $0 > \langle l, u \rangle > -\eta \|u\|$  since  $g$  is  $C^1$  and the norm function is  $C^1$  when  $u \neq 0$ ; the derivative is continuous when  $\langle l, u \rangle = -\eta \|u\|$  and  $u \neq 0$  since  $g(1) = 1$  and  $g'(1) = 0$ ; when  $\langle l, u \rangle = 0$  the function is extended oddly with respect to the hyperplane  $Z$  thus  $\widehat{\mathbf{F}}$  is  $C^1$  when  $u \in Z$ ; finally at  $u = 0$ , since  $H(\lambda, u) = H_u(\lambda, u) = o(\|u\|)$ , we have  $\widehat{\mathbf{F}}_u(\lambda, 0) = \mathbf{F}_u(\lambda, 0)$  and the continuity of  $\widehat{\mathbf{F}}_u$  at  $(\lambda, 0)$ . This also implies that  $\widehat{\mathbf{F}}_u(\lambda, 0)$  is a Fredholm operator with index zero.

We show that  $\widehat{\mathbf{F}}_u$  is still Fredholm for all  $(\lambda, u) \in V$ . This only requires a proof when  $0 \geq \langle l, u \rangle > -\eta \|u\|$ . In this case, from direct calculation,  $\widehat{\mathbf{F}}_u(\lambda, u)[\phi] = \mathbf{F}_u(\lambda, 0)[\phi] + k_1 H_u(\lambda, u)[\phi] + k_2(\phi)H(\lambda, u)$ , where  $k_1 = g(-\langle l, u \rangle / \eta \|u\|)$  and  $k_2 : X \rightarrow \mathbf{R}$  is a bounded linear functional. Hence  $\widehat{\mathbf{F}}_u(\lambda, u)[\phi] = k_1 \mathbf{F}_u(\lambda, u)[\phi] + (1 - k_1)\mathbf{F}_u(\lambda, 0)[\phi] + k_2(\phi)H(\lambda, u)$ . By our assumption,  $k_1 \mathbf{F}_u(\lambda, u) + (1 - k_1)\mathbf{F}_u(\lambda, 0)$  is a Fredholm operator for  $k_1 \in [0, 1]$ , and so  $\widehat{\mathbf{F}}_u(\lambda, u)$  is a compact perturbation of a Fredholm operator, thus still a Fredholm operator (see for example [17]). Hence  $\widehat{\mathbf{F}}$  is Fredholm for all  $(\lambda, u) \in V$ . Finally since  $\widehat{\mathbf{F}}_{\lambda u}(\lambda, 0) = \mathbf{F}_{\lambda u}(\lambda, 0)$ , the conditions on  $D_{\lambda u} \mathbf{F}$  in Theorem 4.3 are satisfied. Now all conditions in Theorem 4.3 are satisfied for the modified mapping  $\widehat{\mathbf{F}}$ , and we can repeat the proof of Theorem 6.4.3 of [24] to obtain the desired conclusion.  $\square$

The condition that the norm function  $u \mapsto \|u\|$  is  $C^1$  for  $u \neq 0$  is not restrictive. For our primary application in this paper,  $X$  is based on  $L^p(\Omega)$  for bounded domain  $\Omega$ , and it is readily seen that the norm of  $L^p(\Omega)$  is  $C^1$  for  $u \neq 0$  and  $p \in (1, \infty)$ . In general, Restrepo [36] proved that a separable Banach space  $X$  has an equivalent norm of class  $C^1$  on  $X \setminus \{0\}$  if and only if  $X^*$  is separable. If this is the case, we work with this equivalent norm from the beginning. The second condition on the Fredholm property of  $k\mathbf{F}_u(\lambda, u) + (1 - k)\mathbf{F}_u(\lambda, u_0)$  is satisfied for elliptic operators considered in Sections 2 and 3.

#### 4.2. Example: predator–prey system with cross-diffusion

In 1979, Shigesada, Kawasaki and Teramoto [38] proposed a reaction–diffusion model with cross-diffusion and self-diffusion in addition to the passive Fickian diffusion, and this more general model incorporates the attraction/repulsion between the species. Existence/non-existence of steady state solutions for cross-diffusion systems have been investigated in [25,26], see also the survey [28,29] for more results. Here we consider a predator–prey system with cross-diffusion but not self-diffusion, and the surrounding environment of the habitat is hostile, so homogeneous Dirichlet boundary condition is imposed,

$$\begin{cases} \Delta[(1 + \alpha v)u] + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \Delta[(1 + \beta u)v] + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Here the constants  $\alpha, \beta, \lambda, b, c > 0, \mu \in \mathbf{R}$ , and  $\Omega$  is a bounded domain of  $\mathbf{R}^n$  with  $C^{2+\alpha}$  boundary. Nakashima and Yamada [27] considered the existence of positive solutions of (4.2) via fixed point index method; Kuto and Yamada [21] obtained further results for certain parameter ranges. Bifurcation theory is used in [21] in the following way: make a change of variables  $U = (1 + \alpha v)u$  and  $V = (1 + \beta u)v$ , then the equations of  $(U, V)$  are semilinear but the nonlinearity part becomes complicated. Here we deal directly with the quasilinear system (4.2).

In the following, we fix  $\alpha, \beta, \lambda, b, c$ , and let  $\mu$  be a bifurcation parameter. We now cast (4.2) into the framework discussed in the previous section. We rewrite the equations as

$$\begin{cases} (1 + \alpha v)\Delta u + \alpha u \Delta v + 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \beta v \Delta u + (1 + \beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Define  $2 \times 2$  matrix:

$$A_1(u, v) = \begin{pmatrix} 1 + \alpha v & \alpha u \\ \beta v & 1 + \beta u \end{pmatrix},$$

and for  $1 \leq i, j \leq n$ ,  $\mathbf{u} = (u, v)^T$ ,

$$a_{ij}(\mu, \mathbf{u}) = A_1 \delta_{ij} \quad \text{and} \quad f(\mu, \mathbf{u}, \nabla \mathbf{u}) = - \begin{pmatrix} 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) \\ 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) \end{pmatrix},$$

where  $\delta_{ij}$  is the Kronecker symbol. Then (4.3) is equivalent to

$$\begin{aligned} A(\mu, \mathbf{u}) &\equiv -a_{ij}(\mu, \mathbf{u}) \partial_i \partial_j \mathbf{u} + f(\mu, \mathbf{u}, \nabla \mathbf{u}) = 0, \\ \text{where } \mathbf{u} \in X &\equiv (W_B^{2,p}(\Omega))^2 = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^2. \end{aligned} \tag{4.4}$$

Here we drop the Dirichlet boundary operator  $B\mathbf{u} = \mathbf{u}$  but restrict the domain of  $A$  to a subspace of  $(W^{2,p}(\Omega))^2$  with zero boundary condition since the boundary condition is linear. We take  $p > n$ .

Because  $a_{ij}(\mu, \mathbf{u}) = A_1 \delta_{ij}$ , we can conveniently write (4.4) as

$$A(\mu, \mathbf{u}) \equiv -A_1(\mathbf{u})\Delta \mathbf{u} + f(\mu, \mathbf{u}, \nabla \mathbf{u}) = 0, \quad \mathbf{u} \in X. \tag{4.5}$$

We remark that any nonnegative solution  $\mathbf{u} \in X$  of (4.5) is a  $C^{2+\alpha}$  solution of (4.5) and hence of (4.2); we can multiply both sides of (4.5) by the inverse of  $A_1(\mathbf{u})$  and then apply the elliptic regularity theory for single equations.

The linearization of  $A(\mu, \mathbf{u})$  at  $\mathbf{u}$  is given by  $(\mathbf{w} = (w_1, w_2) \in X)$

$$D_{\mathbf{u}}A(\mu, \mathbf{u})[\mathbf{w}] = -A_1(\mathbf{u})\Delta \mathbf{w} - A_2(\mathbf{w})\Delta \mathbf{u} - A_3(\nabla \mathbf{u}) \cdot \nabla \mathbf{w} - J(\mathbf{u})\mathbf{w},$$

where

$$A_2(\mathbf{w}) = \begin{pmatrix} \alpha w_2 & \alpha w_1 \\ \beta w_2 & \beta w_1 \end{pmatrix}, \quad A_3(\nabla \mathbf{u}) = \begin{pmatrix} 2\alpha \nabla v & 2\alpha \nabla u \\ 2\beta \nabla v & 2\beta \nabla u \end{pmatrix},$$

and  $J$  is the Jacobian

$$J = \begin{pmatrix} \lambda - 2u - bv & -bu \\ cv & \mu + cu - 2v \end{pmatrix}.$$

For a small  $\varepsilon > 0$ , we define

$$X_\varepsilon = \{(u, v) \in X : u(x) > -\varepsilon, v(x) > -\varepsilon\}.$$

Then  $X_\varepsilon$  is an open connected subset of  $X$ . Clearly for  $\mathbf{u} \in X$ ,  $\text{Trace}(A_1(\mathbf{u})) > 0$  and  $\text{Det}(A_1(\mathbf{u})) > 0$ . So  $A_1(\mathbf{u})$  satisfies (2.6) and hence by Remark 2.5.5, Case 3, for any  $\mu \in \mathbf{R}$  and  $\mathbf{u} \in X_\varepsilon$ ,  $(D_{\mathbf{u}}A(\mu, \mathbf{u}), B)$  satisfies Agmon’s condition for angles  $\theta \in [-\pi/2, \pi/2]$ . By Remark 3.4.1,  $D_{\mathbf{u}}A(\mu, \mathbf{u}) : X \rightarrow Y \equiv (L^p(\Omega))^2$  is Fredholm with index 0; moreover,  $A : \mathbf{R} \times X_\varepsilon \rightarrow Y$  is  $C^1$  smooth.

Our bifurcation analysis will be based on bifurcation from semitrivial steady states of (4.5), which we now turn to. Denote by  $\lambda_1(q)$  the principal eigenvalue of

$$-\Delta\phi + q(x)\phi = \gamma\phi, \quad x \in \Omega; \quad \phi = 0, \quad x \in \partial\Omega,$$

where  $q(x)$  is a continuous function in  $\overline{\Omega}$ . And we also use the notation  $\lambda_1 = \lambda_1(0)$ . Notice that  $\lambda_1(q)$  is an increasing function in  $q$  in the sense: if  $q_1(x) \geq q_2(x)$  and  $q_1(x) \not\equiv q_2(x)$ , then  $\lambda_1(q_1) > \lambda_1(q_2)$ . It is well known that for the scalar equation

$$\Delta u + u(\lambda - u) = 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega,$$

there exists a unique positive solution  $\theta_\lambda$  if  $\lambda > \lambda_1$ . Moreover  $\{(\lambda, \theta_\lambda) : \lambda > \lambda_1\}$  is a smooth curve in  $\mathbf{R} \times W_B^{2,p}(\Omega)$ ;  $\theta_\lambda$  is stable in the sense that the linearized eigenvalue problem

$$-\Delta\phi - \lambda\phi + 2\theta_\lambda\phi = \eta\phi, \quad x \in \Omega; \quad \phi = 0, \quad x \in \partial\Omega,$$

has a positive principal eigenvalue  $\lambda_1(-\lambda + 2\theta_\lambda)$ . Thus  $-\Delta - \lambda + 2\theta_\lambda$  is invertible and  $(-\Delta - \lambda + 2\theta_\lambda)^{-1}\phi$  is positive if  $\phi$  is positive. For proofs of these facts, see for example [8,10].

We fix  $\lambda > \lambda_1$ . Thus (4.2) and (4.5) have trivial solution  $(0, 0)$  and semitrivial solution  $(\theta_\lambda, 0)$  for any  $\mu \in \mathbf{R}$ , and semitrivial solution  $(0, \theta_\mu)$  for  $\mu > \lambda_1$ . The reason for assuming  $\lambda > \lambda_1$  also comes from the fact that (4.2) has no positive solutions otherwise. This fact and more about positive solutions of (4.2) are summarized in the following

**Proposition 4.5.**

1. If  $\lambda \leq \lambda_1$ , then (4.2) has no positive solutions.
2. If  $(u, v)$  is a positive solution of (4.2), then

$$0 \leq u(x) \leq U(x) \leq M_1 \equiv \begin{cases} \lambda & \text{if } \lambda\alpha \leq b, \\ (\lambda\alpha + b)^2/4\alpha b & \text{if } \lambda\alpha > b, \end{cases}$$

$$0 \leq v(x) \leq V(x) \leq M_2 \equiv (1 + \beta M_1)(1 + cM_1),$$

where  $U(x) = (1 + \alpha v(x))u(x)$  and  $V(x) = (1 + \beta u(x))v(x)$ .

3. There exists  $\mu_0 = -cM_1$ , and  $\mu^0 > \mu_0$  such that (4.2) has no positive solution if  $\mu < \mu_0$  or  $\mu > \mu^0$ .

**Proof.** Parts 1 and 2 are proved in [27, Lemmas 1 and 2], see also [21, Lemmas 2.1 and 2.2]. For part 3, we observe that  $V$  satisfies

$$\Delta V + \frac{\mu + cu - v}{1 + \beta u}V = 0, \quad x \in \Omega; \quad V = 0, \quad x \in \partial\Omega. \tag{4.6}$$

If  $\mu < -cM_1$ , then from part 2,  $\mu + cu(x) - v(x) \leq \mu + cM_1 < 0$  for all  $x \in \overline{\Omega}$ , hence we reach a contradiction from (4.6) and the maximum principle. Thus (4.2) has no positive solutions when  $\mu < \mu_0$ .

For the upper bound of  $\mu$ , we assume (4.2) has a positive solution for all large  $\mu$ . Then by (4.6),

$$\lambda_1\left(-\frac{\mu + cu - v}{1 + \beta u}\right) = 0 = \lambda_1(-\lambda_1(0)). \tag{4.7}$$

On the other hand, by part 2, if  $\mu$  is large enough, we have on  $\Omega$

$$-\frac{\mu + cu - v}{1 + \beta u} < -\lambda_1(0).$$

This and the monotonicity of  $\lambda_1(q)$  contradict (4.7).  $\square$

We now come back to the two semitrivial solution branches:

$$\Gamma_u = \{(\theta_\lambda, 0): \mu \in \mathbf{R}\}, \quad \Gamma_v = \{(0, \theta_\mu): \mu > \lambda_1\},$$

and identify potential bifurcation points on them.

The necessary condition for bifurcation is that  $D_{\mathbf{u}}A(\mu, \mathbf{u})$  is degenerate. First we let  $\mathbf{u} = (\theta_\lambda, 0)$ . Simplifying the equations, we obtain

$$D_{\mathbf{u}}A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = - \begin{pmatrix} \Delta w_1 + (\lambda - 2\theta_\lambda)w_1 + \alpha \Delta(\theta_\lambda w_2) - b\theta_\lambda w_2 \\ \Delta[(1 + \beta\theta_\lambda)w_2] + (\mu + c\theta_\lambda)w_2 \end{pmatrix}.$$

If we set  $D_{\mathbf{u}}A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = 0$ , then the equation of  $w_2$  is equivalent to

$$\Delta W_2 + \frac{\mu + c\theta_\lambda}{1 + \beta\theta_\lambda} W_2 = 0, \quad x \in \Omega; \quad W_2 = 0, \quad x \in \partial\Omega, \tag{4.8}$$

where  $W_2(x) = (1 + \beta\theta_\lambda)w_2(x)$ . Since we look for positive solutions of (4.2), the bifurcation should take place at the principal eigenvalue so that the eigenfunction is positive. Thus the possible bifurcation point  $\mu_1$  is the one such that

$$\lambda_1\left(\frac{-\mu_1 - c\theta_\lambda}{1 + \beta\theta_\lambda}\right) = 0. \tag{4.9}$$

Similar analysis can be done on the other semitrivial branch, but  $(0, \theta_\mu)$  is not a fixed point in  $X$  so we consider the operator  $A'(\mu, \mathbf{u}) = A(\mu, \mathbf{u} + (0, \theta_\mu))$  for which  $\mathbf{u} = (0, 0)$  is always a solution of  $A'(\mu, \mathbf{u}) = 0$  for all  $\mu$ . The corresponding linearized equation is

$$D_{\mathbf{u}}A'(\mu, 0)[\mathbf{w}] = - \begin{pmatrix} \Delta[(1 + \alpha\theta_\mu)w_1] + (\lambda - b\theta_\mu)w_1 \\ \Delta w_2 + (\mu - 2\theta_\mu)w_2 + \beta \Delta(\theta_\mu w_1) + cb\theta_\mu w_1 \end{pmatrix}.$$

Thus the possible bifurcation point is  $\mu_2$  such that

$$\lambda_1\left(\frac{-\lambda + b\theta_{\mu_2}}{1 + \alpha\theta_{\mu_2}}\right) = 0. \tag{4.10}$$

Since  $\theta_\mu$  is differentiable with respect to  $\mu$ , then  $A'$  is also  $C^1$  from earlier discussions. The bifurcation analysis for  $A'$  is essentially the same as  $A$ , details can be found in, for example, [8,13].

**Lemma 4.6.** *There exists a unique  $\mu_1 \in (-\infty, \infty)$  so that (4.9) holds, and there exists a unique  $\mu_2 \in (\lambda_1, \infty)$  so that (4.10) holds. Moreover the corresponding null spaces  $\mathcal{N}(D_{\mathbf{u}}A(\mu_1, (\theta_\lambda, 0)))$  and  $\mathcal{N}(D_{\mathbf{u}}A'(\mu_2, (0, 0)))$  are one-dimensional.*

**Proof.** Define

$$f_1(\mu) = \lambda_1\left(\frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda}\right) \quad \text{and} \quad q_1(\mu) = \frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda}.$$

Then  $q_1(\mu)$  is decreasing in  $\mu$ . From the properties of  $\lambda(q)$ , we deduce that  $f_1(\mu) \rightarrow \pm\infty$  as  $\mu \rightarrow \mp\infty$  and  $f_1$  is decreasing. Hence  $\mu_1$  exists and it is unique. With  $\mu = \mu_1$ , (4.8) has a positive solution  $W_2$ . Then  $w_2 = (1 + \beta\theta_\lambda)^{-1}W_2$ , and  $w_1 = (-\Delta - \lambda + 2\theta_\lambda)^{-1}(\alpha\Delta(\theta_\lambda w_2) - b\theta_\lambda w_2)$  give rise to the unique solution of  $D_{\mathbf{u}}A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = 0$  up to a constant multiplier.

Similarly we define

$$f_2(\mu) = \lambda_1 \left( \frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right) \quad \text{and} \quad q_2(\mu) = \frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu}.$$

Since  $\theta_\mu$  is increasing in  $\mu$  (pointwisely for  $x \in \Omega$ ), then  $q_2$  and  $f_2$  are increasing in  $\mu$ . One can show that  $f_2(\mu) \rightarrow \lambda_1 + b/\alpha > 0$  as  $\mu \rightarrow \infty$ , and  $f_2(\mu) \rightarrow \lambda_1 - \lambda < 0$  as  $\mu \rightarrow \lambda_1^+$ . Hence  $\mu_2$  exists and is unique. Similarly to the above case, the null space is one-dimensional with  $w_1 > 0$ .  $\square$

Now we have the following global bifurcation theorem:

**Theorem 4.7.** *Suppose that  $\alpha, \beta, b, c > 0$  and  $\lambda > \lambda_1$ . Let  $S^+$  be the set of positive solutions to (4.2). Then there exists a connected component  $C^*$  of  $S^+$  such that the closure of  $C^*$  includes the bifurcation points  $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$  and  $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$ . In other words, bifurcations occur at both  $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$  and  $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$ , and the bifurcating continua from the two points are connected to each other.*

**Proof.** We apply Theorem 4.3 at  $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$  with  $V = \mathbf{R} \times X_\varepsilon$ . We have already observed that  $A : V \rightarrow Y$  is  $C^1$  smooth, and  $D_{\mathbf{u}}A(\lambda, u)$  is Fredholm with zero index for any  $(\lambda, u) \in V$ . We have also shown in Lemma 4.6 that  $\mathcal{N}(D_{\mathbf{u}}A(\mu_1, (\theta_\lambda, 0))) = \text{span}\{(w_1, w_2)\}$  with  $w_2 > 0$ . For the transversality condition,

$$D_{\mu\mathbf{u}}A(\mu_1, (\theta_\lambda, 0)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -w_2 \end{pmatrix} \notin \mathcal{R}(D_{\mathbf{u}}A(\mu_1, (\theta_\lambda, 0))),$$

because the equation  $\Delta[(1 + \beta\theta_\lambda)\psi] + (\mu_1 + c\theta_\lambda)\psi = w_2$  is not solvable since  $\int_\Omega (1 + \beta\theta_\lambda)w_2^2 dx \neq 0$ . Now we can apply Theorem 4.3 to obtain a connected component  $C$  of the set  $S$  of all solutions of (4.5) emanating from  $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ . Similarly we can show the existence of a connected component of  $S$  emanating from  $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$ . Moreover near the bifurcation point,  $C$  has the form  $(\mu(s), \theta_\lambda + O(s), sw_2 + o(s))$  for  $s$  small. Then the solution is positive for  $s > 0$  since  $w_2 > 0$  and  $\theta_\lambda > 0$ . Let  $P = \{(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : u > 0, v > 0 \text{ for } x \in \Omega, \partial u/\partial \nu < 0, \partial v/\partial \nu < 0 \text{ for } x \in \partial\Omega\}$ , where  $\nu$  is the unit outer normal vector field of  $\partial\Omega$ . Then  $C \cap (\mathbf{R} \times P) \neq \emptyset$ .

Let  $C^* = C \cap (\mathbf{R} \times P)$ . Let  $C^+$  and  $C^-$  be the sub-continua in Theorem 4.4 (conditions 1–3 in that theorem can be easily verified). By definition,  $C^* \subset C^+$ . By the elliptic regularity theory (see the comment below (4.5)), the first alternative in Theorem 4.4 for  $C^+$  is equivalent to “the closure of  $C^+$  intersects  $\partial V$  or is unbounded in the norm of  $\mathbf{R} \times X$ .” On the other hand, by Proposition 4.5, the positive solutions  $(u, v)$  of (4.5) are bounded in  $L^\infty$  norm, and the range of  $\mu$  for existence of such solutions is also bounded. Thus by the elliptic regularity theory again,  $C^*$  cannot be unbounded in  $\mathbf{R} \times X$  norm. Now we see that if the first alternative in Theorem 4.4 occurs, then  $C^* \cap (\mathbf{R} \times \partial P)$  contains a point  $(\mu^*, u^*, v^*)$  other than  $(\mu_1, \theta_\lambda, 0)$ . This is obviously true if the other alternatives occur.

By continuity,  $(\mu^*, u^*, v^*)$  is a solution of the equation, and  $u^* \geq 0, v^* \geq 0$ . By the maximum principle,  $u^* \equiv 0$  or  $u^* > 0$ , and the same for  $v^*$ . If  $(u^*, v^*) = (0, 0)$ , then we can show that  $D_{\mathbf{u}}A(\mu^*, 0) = -(\Delta + \lambda, \Delta + \mu^*)$  is degenerate and its null space contains a  $(w_1, w_2) \geq 0, \neq 0$ . Since  $\lambda > \lambda_1, w_1 = 0$ ; hence  $w_2 > 0$  and  $\mu^* = \lambda_1$ . Applying Theorem 4.3 to the trivial solution branch  $\{(\mu, 0, 0) : \mu \in \mathbf{R}\}$  at  $(\lambda_1, 0)$ , we have that all the nontrivial solutions of (4.5) near  $(\lambda_1, 0)$  are the semitrivial ones  $(\mu, 0, \theta_\mu)$ , contradicting the definition of  $(\mu^*, u^*, v^*)$ . Thus  $(u^*, v^*) \neq (0, 0)$ . Note that  $(\mu^*, u^*, v^*) \notin \Gamma_u$  since  $\mu = \mu_1$  is the only point on  $\Gamma_u$  where positive solutions bifurcate. We conclude  $(\mu^*, u^*, v^*) = (\mu_2, 0, \theta_{\mu_2})$ , the only possible point on  $\Gamma_v$  where positive solutions bifurcate.  $\square$

The argument about the connectedness of two components first appeared in [8], and it holds for many other predator–prey systems, see survey [14]. Our result implies the existence of positive solutions for  $\mu \in (\mu_1, \mu_2)$  or  $\mu \in (\mu_2, \mu_1)$  if  $\mu_1 \neq \mu_2$ . Indeed  $\mu \in (\mu_1, \mu_2)$  is equivalent to

$$\lambda_1 \left( \frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda} \right) < 0, \quad \lambda_1 \left( \frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right) < 0; \tag{4.11}$$

and  $\mu \in (\mu_2, \mu_1)$  is equivalent to

$$\lambda_1 \left( \frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda} \right) > 0, \quad \lambda_1 \left( \frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right) > 0. \tag{4.12}$$

It was proved in [27] under (4.11) or (4.12), that (4.2) has a positive solution. But our result implies that even when  $\mu_1 = \mu_2$ , a solution branch still connects the two bifurcation points. On the other hand, in [21] a bifurcation analysis is performed with parameter  $\lambda$  in (4.2). Under the additional conditions that  $\alpha$  is small and  $\beta$  is large, it is shown in [21] that the solution set of (4.2) possesses a component which is an unbounded or bounded curve, and the curve can be *S*-shaped. Our analysis can also be carried over to an analysis with parameter  $\lambda$ . By using the formula in [9,37] for determining the direction of the local bifurcation in Theorem 4.3, we can also determine the direction of the bifurcation curves. In fact, with  $\lambda$  as parameter, the bifurcation from  $\{(\lambda, \theta_\lambda, 0) : \lambda > \lambda_1\}$  may not even occur, and it only occurs when  $\mu > \lambda_1 > c/\beta$ , or  $c/\beta > \lambda_1 > \mu$ . In contrast, bifurcations always occur from both branches of semitrivial solutions with  $\mu$  as parameter, and the bifurcation curve is always bounded as shown in Theorem 4.7.

### 4.3. Example: Chemotactic diffusion system

Here we apply our approach to the following quasilinear elliptic system from the theory of chemotaxis, which describes the situation of a single bacterial population in a one-dimensional medium with finite length, with growth limited by a nutrient diffusing from an adjacent phase not accessible to the bacteria:

$$\begin{cases} u'' - f(u)v = 0, & x \in (0, 1), \\ \lambda v'' - \chi(v\psi'(u)u')' + (kf(u) - \theta - \beta v)v = 0, & x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = h(1 - u(1)), \\ \lambda v' - \chi v\psi'(u)u' = 0, & x = 0, 1. \end{cases} \tag{4.13}$$

Here  $f(u)$  is the consumption rate of the nutrient per cell,  $\lambda$  is a positive constant which represents the random motility of the cells,  $\chi > 0$  measures the magnitude of the chemotactic response,  $\psi(u)$  measures the sensitivity of cells to the gradient of  $u$ , the positive constants  $k$  and  $\theta$  measure the birth and the death rates of the cells,  $\beta \geq 0$  measures the overcrowding effect, and  $h > 0$  is the coefficient of the mass transfer of the substrate from the adjacent phase. From biological considerations,  $f$  and  $\psi$  satisfies  $f(0) = 0, f'(u) > 0, \psi'(u) > 0$ ; from mathematical considerations (to satisfy the regularity condition (3.1)), assume  $f \in C^1(\mathbf{R})$  and  $\psi \in C^3(\mathbf{R})$ . For more on the background of (4.13), we refer to Wang [39].

First we convert the system (4.13) to the following form:

$$\begin{cases} u'' - f(u)v = 0, & x \in (0, 1), \\ \lambda v'' - \chi(v\psi'(u)u')' + (\mu + kf(u) - kf(1) - \beta v)v = 0, & x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = h(1 - u(1)), \\ \lambda v' - \chi v\psi'(u)u' = 0, & x = 0, 1, \end{cases}$$

where  $\mu = kf(1) - \theta$ . We define  $\mathbf{F} : \mathbf{R} \times (W^{2,p}(0, 1))^2 \rightarrow (L^p(0, 1))^2 \times \mathbf{R}^4$  as follows

$$\mathbf{F}(\mu, u, v) = \begin{pmatrix} -u'' + f(u)v \\ -\lambda v'' + \chi(v\psi'(u)u')' - (\mu + kf(u) - kf(1) - \beta v)v \\ -u'(0) \\ u'(1) - h + hu(1) \\ -\lambda v'(0) + \chi v(0)\psi'(u(0))u'(0) \\ \lambda v'(1) - \chi v(1)\psi'(u(1))u'(1) \end{pmatrix}.$$

If we calculate  $D_{(u,v)}\mathbf{F}(\mu, u, v)$  and write it in form of (2.1) and (2.2), we see ( $i = 1 = j$ ):  $\delta(x) = I$ ,

$$a_{11}(x) = \begin{pmatrix} 1 & 0 \\ -v\psi'(u) & \lambda \end{pmatrix},$$

$b_1(x) = a_{11}(x)v(x)$  with  $v(1) = 1$  and  $v(0) = -1$ . The eigenvalues of  $a_{11}$  are equal to 1 and  $\lambda$ , and hence  $a_{11}$  satisfies (2.6). By Remark 2.5.5, Case 3, for any  $(\mu, u, v) \in \mathbf{R} \times (W^{2,p}(0, 1))^2$ ,  $D_{(u,v)}\mathbf{F}(\mu, u, v)$  satisfies Agmon’s condition. Now by Theorem 3.3,  $D_{(u,v)}\mathbf{F}(\mu, u, v)$  is Fredholm with index 0 and  $\mathbf{F} : \mathbf{R} \times (W^{2,p}(0, 1))^2 \rightarrow (L^p(0, 1))^2 \times \mathbf{R}^4$  is  $C^1$  smooth.

Observe that  $\mathbf{F}(\mu, 1, 0) \equiv \mathbf{0}$  from the assumptions, and that

$$D_{(u,v)}\mathbf{F}(\mu, 1, 0) \begin{bmatrix} \phi \\ \omega \end{bmatrix} = \begin{pmatrix} -\phi'' + f(1)\omega \\ -\lambda\omega'' - \mu\omega \\ -\phi'(0) \\ \phi'(1) + h\phi(1) \\ -\lambda\omega'(0) \\ \lambda\omega'(1) \end{pmatrix}.$$

At  $\mu = 0$ ,  $D_{(u,v)}\mathbf{F}(0, 1, 0)[\phi, \omega] = \mathbf{0}$  has a unique solution (up to a constant multiplier)  $(\phi_0, \omega_0) = (f(1)(hx^2 - h - 2), 2h)$ . Thus  $\dim \mathcal{N}(D_{(u,v)}\mathbf{F}(0, 1, 0)) = 1$ . On the other hand,  $D_{\mu(u,v)}\mathbf{F}(0, 1, 0)[\phi_0, \omega_0] = (0, -\omega_0, 0, 0, 0, 0)$ , so it is easy to see that  $D_{\mu(u,v)}\mathbf{F}(0, 1, 0)[\phi_0, \omega_0] \notin \mathcal{R}(D_{(u,v)}\mathbf{F}(0, 1, 0))$ . Therefore we can apply Theorem 4.3 to obtain a global branch of solutions of (4.13) bifurcating from  $(\mu, u, v) = (0, 1, 0)$ . Furthermore, Theorem 4.4, combined with the *a priori* estimates and the maximum principle as in [39], can be applied to obtain a global branch of positive solutions. We omit the details. This recovers part of Theorem 3.1 in [39], but we use a global bifurcation theorem directly on the original system instead of converting it into the form of a “compact perturbation of the identity.”

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