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Existence and nonexistence of positive solutions of semilinear elliptic equation with inhomogeneous strong Allee effect *

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Abstract In this paper, we study a semilinear elliptic equation defined on a bounded smooth domain. This type of problem arises from the study of spatial ecology model, and the growth function in the equation has a strong Allee effect and is inhomogeneous. We use variational methods to prove that the equation has at least two positive solutions for a large parameter if it satisfies some appropriate conditions. We also prove some nonexistence results.

Key words semilinear equation, Allee effect, positive solutions, existence

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Introduction

We consider the equation

$$\begin{cases} \Delta u + \lambda f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ for $n \geq 1$. The equilibrium solutions to the reaction diffusion equation (1) with the heterogenous growth pattern and zero boundary condition have been recently studied in [1-2].

In the context of population biology, the nonlinear function $f(x, u) \equiv ug(x, u)$ represents a density-dependent growth if $g(x, u)$ is a function that depends on the population density u. Traditionally, $g(x, u)$ is assumed to be declining to reflect the crowding effect of the increasing population. Allee[3] (see also [4]) suggested that physiological and demographic precesses often possess an optimal density with the decreasing response as either higher or lower density. Such a growth pattern is called an Allee effect. If the growth rate per capita is negative when u is

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small, we call it a strong Allee effect; if the growth rate per capita is small than the maximum but still positive for a small u, we call it a weak Allee effect.

The solution set of (1) and bifurcation diagrams were considered in [2] when $f(x,u)$ is of the weak Allee effect type, and we also refer to [2] for more discussions on biological applications. For the strong Allee effect case, in [5], a typical case $f(x,u)\equiv u(u-b)(c-u)$ with $0<2b<c$ and similar cases were considered when Ω is a ball of any dimension. Here, a critical value λ_* exists. There exist two positive solutions when $\lambda > \lambda_*$, and there are no solutions when $\lambda < \lambda_*$. The condition $0 < 2b < c$ here is to ensure the existence of positive solutions to (1), and for a more general $f(u)$, the condition becomes[6-7]

$$\int_0^c f(u)du > 0, \tag{2}$$

if $f(u)$ satisfies $f(0) = f(b) = f(c) = 0$ for $0 < b < c$, $f(u) < 0$ in $(0,b)$, and $f(u) > 0$ in (b,c). Besides, when (2) is not satisfied, (1) has no positive solutions. The nonlinear $f(u)$ with a strong Allee effect is also called the bistable type as $u = 0$ and $u = c$ are both stable solutions to the ordinary differential equation (ODE) $u' = f(u)$. The condition (2) can be interpreted as $u = c$ being a more stable equilibrium than $u = 0$.

The main goal of this paper is to generalize the existence/nonexistence and multiplicity results from a homogeneous $f(u)$ to an inhomogeneous $f(x,u)$ as in (1). Our main result is that, if a condition like (2) holds for x belonging to an open subset of Ω, then the existence of two positive solutions can be established as the homogeneous case. We also prove a nonexistence result when a converse condition to (2) holds, which generalizes the main result of [7].

In this paper, we consider (1) with a strong Allee effect growth. More specifically, we assume that $f(x,u)$ satisfies (see Fig. 1)

(f1) For any $u \geq 0$, $f(\cdot,u) \in C^{1,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$, and for any $x \in \bar{\Omega}$, $f(x,\cdot) \in C^1(\mathbb{R}^+)$.

(f2) For any $x \in \bar{\Omega}$, there exist $b(x), c(x) \in C^{1,\alpha}(\Omega)$ $(0 < \alpha < 1)$, where $0 < b(x) < c(x) \leq M$ and $M > 0$ is a constant, such that $f(x,0) = f(x,b(x)) = f(x,c(x)) = 0$.

(f3) For almost all $x \in \bar{\Omega}$, $f(x,s) < 0$ for any $s \in (0,b(x)) \bigcup (c(x),\infty)$, and $f(x,s) > 0$ for any $s \in (b(x),c(x))$.

(f4) For almost all $x \in \bar{\Omega}$, there exists $N > 0$ such that $f(x,s)/s \leq N$, $s \geq 0$.

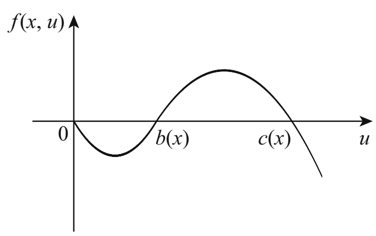

Fig. 1 Sketch of $f(x,u)$ satisfing (f1)–(f4) for a fixed x

1 Main results

To consider the existence of solutions of (1) that satisfies (f1)–(f4), we observe that

Lemma 1.1 *Suppose that $f(x,u)$ satisfies* (f1)–(f3). *If $u(x)$ is an integrable function in Ω, and there is a measurable subset Ω_0 of Ω with a positive measure such that*

$$\int_0^{c(x)} f(x,s)ds > 0 \ \textit{in } \Omega_0 \quad \textit{and} \quad \int_0^{c(x)} f(x,s)ds \leq 0 \ \textit{in } \Omega\backslash\Omega_0,$$

then

$$\int_0^{u(x)} f(x,s)ds \le \int_0^{c(x)} f(x,s)ds \ \textit{in}\ \Omega_0 \quad \textit{and} \quad \int_0^{u(x)} f(x,s)ds \le 0 \ \textit{in}\ \Omega\backslash\Omega_0.$$

The proof is clear from the conditions and integration, so we omit it. We also recall the proof in [8].

Lemma 1.2 *For any $v \in X \equiv H_0^1(\Omega)$, define $\langle Tu, v\rangle_X = \int_\Omega f(x,u)vdx$, where $f(x,u)$ satisfies* (f1)–(f4)*. Then T is a compact operator on X.*

Our main result of the existence is the following:

Theorem 1.3 *If $f(x,u)$ satisfies* (f1)–(f4)*, and B_1 is an open subset of Ω such that*

$$\int_0^{c(x)} f(x,s)ds > 0, \quad x \in B_1, \tag{3}$$

then for the sufficiently large λ, (1) *has at least two positive solutions, and for the samll λ,* (1) *has no solutions.*

Proof We redefine $f(x,u)$ such that $f(x,u) \equiv 0$ when $u \in (-\infty, 0)\bigcup(c(x),\infty)$, but it does not change the solution set of (1) by the maximum principle since all the solutions of (1) satisfy $0 \le u(x) \le c(x)$. Define

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(x,u)dx, \\ F(x,u) &= \int_0^{u(x)} f(x,s)ds. \end{aligned}$$

Here, $I_\lambda(u)$ is a C^2 function in X (see [6,8]). From the regularity assumptions on $f(x,u)$, any critical point u of $I_\lambda(\cdot)$ is a classical solution of (1), and from the maximum principle and the definition of the modified $f(x,u)$ above, u either is zero or satisfies $0 < u(x) < c(x)$ for any $x \in \Omega$.

Step 1: First, we prove $I_\lambda(u)$ satisfies the Palais-Smale condition.

Assume that $\{u_m\}$ is a sequence in X, and for any $m \in \mathbb{N}$, $|I_\lambda(u_m)| \le C$ and $\|I'_\lambda(u_m)\| \to 0$ as $m \to \infty$. Since $\int_0^{c(x)} f(x,s)ds > 0$ in B_1, there exists a measurable set $\Omega_0 \subset \Omega$ with a positive measure such that $\int_0^{c(x)} f(x,s)ds > 0$ in Ω_0 and $\int_0^{c(x)} f(x,s)ds \le 0$ in $\Omega\backslash\Omega_0$. Then

$$\begin{aligned} C \ge I_\lambda(u_m) &= \frac{1}{2}\int_\Omega |\nabla u_m(x)|^2 dx - \lambda \int_\Omega F(x,u_m(x))dx \\ &= \frac{1}{2}\int_\Omega |\nabla u_m(x)|^2 dx - \lambda \int_\Omega \Big(\int_0^{u_m(x)} f(x,s)ds\Big)dx \\ &\ge \frac{1}{2}\int_\Omega |\nabla u_m(x)|^2 dx - \lambda \int_{\Omega_0} \Big(\int_0^{u_m(x)} f(x,s)ds\Big)dx \\ &\quad - \lambda \int_{\Omega\backslash\Omega_0} \Big(\int_0^{u_m(x)} f(x,s)ds\Big)dx. \end{aligned}$$

From Lemma 1.1, $\int_0^{u(x)} f(x,s)ds \le \int_0^{c(x)} f(x,s)ds$ in Ω_0 and $\int_0^{u(x)} f(x,s)ds \le 0$ in $\Omega\backslash\Omega_0$. Then

$$\begin{aligned} C \ge I_\lambda(u_m) &\ge \frac{1}{2}\int_\Omega |\nabla u_m(x)|^2 dx - \lambda \int_{\Omega_0} \Big(\int_0^{c(x)} f(x,s)ds\Big) dx \\ &\ge \frac{1}{2}\|u_m(x)\|^2 - \lambda \int_{\Omega_0} A dx \\ &= \frac{1}{2}\|u_m(x)\|^2 - \lambda A|\Omega_0|, \end{aligned}$$

i.e., $C + \lambda A|\Omega_0| \ge \frac{1}{2}\|u_m(x)\|^2$, where $A = \max\limits_{0\le u(x)\le c(x)} |F(x,u)|$. So $\{u_m\}$ is bounded in X.

We denote the duality mapping of X by $D : X = H_0^1(\Omega) \to X^* = H^{-1}(\Omega)$. Then, for any $u, \varphi \in X$,

$$Du[\varphi] = \int_\Omega \nabla u \cdot \nabla \varphi dx,$$

and

$$\begin{aligned} I'_\lambda(u)[\varphi] &= \int_\Omega \nabla u \cdot \nabla\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx \\ &= Du[\varphi] - J'_\lambda(u)[\varphi]. \end{aligned}$$

Here, $J_\lambda(u) = \lambda \int_\Omega F(x,u)dx$. Then

$$I'_\lambda(u) = Du - J'_\lambda(u).$$

Hence,

$$\begin{aligned} u &= D^{-1}I'_\lambda(u) + D^{-1}J'_\lambda(u), \\ u_m &= D^{-1}I'_\lambda(u_m) + D^{-1}J'_\lambda(u_m). \end{aligned}$$

Because $\{u_m\}$ is bounded in X, we find $\{u_m\}$ has a subsequence $\{u_{m_j}\}$ in X and $\{u_{m_j}\}$ converges to u weakly. By Lemma 1.2, $J'_\lambda(\cdot)$ is a compact operator, and then

$$J'_\lambda(u_{m_j}) \to J'_\lambda(u), \quad j \to \infty$$

in X. However, $\|I'_\lambda(u_m)\| \to 0$ when $m \to \infty$, thus

$$u_{m_j} = D^{-1}I'_\lambda(u_{m_j}) + D^{-1}J'_\lambda(u_{m_j}) \to 0 + D^{-1}J'_\lambda(u) = u, \quad j \to \infty.$$

Hence, the Palais-Smale condition holds.

Step 2: We prove the existence of a positive solution.

From the proof above, $I_\lambda(u) \ge -\lambda A|\Omega_0|$. Here, $\inf I_\lambda(u)$ is a critical value since I_λ satisfies the Palais-Smale condition and it is bounded from below.

However, because $u = 0$ is also a solution of (1), in order to attain a positive solution of (1), we must exclude the possibility of $\inf I_\lambda(u)$ equal to 0. In fact, we only need to verify that when λ is large, there exists a $u_0 \in H_0^1(\Omega)$ such that $I_\lambda(u_0) < 0 = I_\lambda(0)$. We define $u_0(x) = 0$ in $\partial\Omega \bigcup (\Omega\backslash B_2)$ and $u_0(x) = c(x)$ in B_1, and define u_0 properly in $B_2\backslash B_1$, where B_2 is a ball having the same center with B_1 and $\Omega \supset B_2 \supset B_1$. Then

$$\begin{aligned} I_\lambda(u_0) &= \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda \int_\Omega F(x,u_0)dx \\ &= \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda \int_{B_1} F(x,u_0)dx - \lambda \int_{\Omega\setminus B_1} F(x,u_0)dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda \int_{B_1} F(x,c(x))dx - \lambda \int_{B_2\setminus B_1} (-A)dx \\ &= \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda \int_{B_1} F(x,c(x))dx - \lambda(-A(|B_2|-|B_1|)). \end{aligned}$$

Since $\int_0^{c(x)} f(x,s)ds > 0$ when $x \in B_1$ and $\int_0^{c(x)} f(x,s)ds$ is continuous, then there must exist an open subset B_0 with $\bar{B}_0 \subset B_1$ and $\delta > 0$ such that $|B_0| > 0$ and $\int_0^{c(x)} f(x,s)ds \ge \delta$ for $x \in B_0$. Choose a small enough B_2 such that $\delta|B_0| + A(|B_1| - |B_2|) > 0$. Then

$$\begin{aligned} I_\lambda(u_0) &\le \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda\delta|B_0 \cap B_2| - \lambda(-A(|B_2|-|B_1|)) \\ &= \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \lambda(\delta|B_0| + A(|B_1|-|B_2|)). \end{aligned}$$

Therefore, when λ is large enough, $I_\lambda(u_0) < 0$. Consequently, when λ is large enough, (1) has a positive solution $u_1(x)$ satisfying $I_\lambda(u_1) = \inf I_\lambda(u) < 0$.

Step 3: We use the mountain pass theorem to prove that (1) has another positive solution u_2.

Fix $\lambda > 0$ such that $I_\lambda(u_0) < 0$. Because $I_\lambda \in C^2(X,\mathbb{R})$, $I_\lambda(0) = 0$, and $I'_\lambda(0) = 0$, for $\forall \varepsilon > 0$, there exists $\delta' > 0$ such that for $\|w\| \le \delta'$,

$$|I_\lambda(w) - I''_\lambda(0)[w,w]| \le \varepsilon \|w\|^2.$$

Because $I_\lambda(0) = 0$ and $I'_\lambda(0) = 0$,

$$I''_\lambda(0)[w,w] = \int_\Omega |\nabla w|^2 dx - \lambda \int_\Omega f_u(x,0) w^2 dx.$$

Since $f_u(x,0) < 0$,

$$I''_\lambda(0)[w,w] \ge \int_\Omega |\nabla w|^2 dx = \|w\|^2.$$

If we choose $\varepsilon = \dfrac{1}{2}$, when $\|w\| = \delta'$,

$$I_\lambda(w) \ge \frac{1}{2}\|w\|^2 = \frac{1}{2}\delta'^2.$$

Let $\rho = \delta'$ and $\alpha = \dfrac{1}{2}\delta'^2$. If $\|u\| = \rho$, $I_\lambda(u) \ge \alpha > 0$. Besides, since $I_\lambda(0) = 0$, from the proof of Step 2, there exists $u_0 \in X$ such that $I_\lambda(u_0) < 0$ and $\|u_0\| > \rho$. So, from the mountain pass theorem, I_λ has another critical point u_2 such that

$$I_\lambda(u_2) \ge \alpha > 0 > I_\lambda(u_1).$$

From the discussions before Step 1, u_2 is another positive solution of (1).

Step 4: Finally, we show that (1) has no positive solutions when λ is small.

Let $(\Lambda_1, \phi_1(x))$ be the principal eigen-pair of the problem

$$\begin{cases} \Delta\phi + \Lambda\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

such that $\phi_1(x) > 0$ in Ω. We multiply (4) by u and multiply (1) by ϕ_1, respectively. After the subtraction, the integration in Ω yields

$$\int_\Omega (\Lambda_1 u\phi_1 - \lambda\phi_1 f(x,u))dx = \int_\Omega u\phi_1 \left(\Lambda_1 - \lambda\frac{f(x,u)}{u}\right) dx = 0.$$

If $\lambda < \dfrac{\Lambda_1}{N}$, by (f4),

$$\Lambda_1 - \lambda\frac{f(x,u)}{u} \geq \Lambda_1 - \lambda N > 0.$$

That is a contradiction. So, for the small λ, (1) has no positive solutions.

Remark 1.4 As an example of Theorem 1.3, we consider the following inhomogeneous cubic nonlinearity case:

$$\begin{cases} \Delta u + \lambda u(u - b(x))(c(x) - u) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $b(x)$, $c(x)$ are C^1 functions such that $0 < b(x) < c(x)$ for any $x \in \bar{\Omega}$.

It is standard to verify that $f(x,u) = u(u - b(x))(c(x) - u)$ satisfies (f1)–(f4). Then

$$\begin{aligned} \int_0^{c(x)} f(x,s)ds &= \int_0^{c(x)} s(s - b(x))(c(x) - s)ds \\ &= \int_0^{c(x)} [-s^3 + (b(x) + c(x))s^2 - b(x)c(x)s]ds \\ &= -\frac{1}{4}s^4\Big|_0^{c(x)} + \frac{b(x)+c(x)}{3}s^3\Big|_0^{c(x)} - \frac{b(x)c(x)}{2}s^2\Big|_0^{c(x)} \\ &= \frac{1}{12}(c(x))^3(c(x) - 2b(x)). \end{aligned}$$

Then, by Theorem 1.3, if there exists an open subset $B_1 \subset \Omega$ such that $c(x) > 2b(x)$ in B_1, then (5) has at least two positive solutions for the large λ.

From the proof of Theorem 1.3, (5) has no positive solutions when $\lambda < \Lambda/N$. A better bound for the nonexistence range of λ can be obtained for this special case as we show below. Recall that $(\Lambda_1, \phi_1(x))$ is the principal eigen-pair of the problem (4). From the assumptions, there exist $b_1, b_2, c_1, c_2 > 0$ such that $b_2 \geq b(x) \geq b_1 > 0$ and $c_2 \geq c(x) \geq c_1 > 0$. Similar to Step 4 of the proof of Theorem 1.3, we have

$$\begin{aligned} 0 &= \int_\Omega (u\Delta\phi_1 - \phi_1\Delta u)dx \\ &= \int_\Omega u\phi_1[\lambda(u - b(x))(c(x) - u) - \Lambda_1]dx \\ &= \int_\Omega u\phi_1[-\lambda b(x)c(x) + \lambda(b(x) + c(x))u - \lambda u^2 - \Lambda_1]dx. \end{aligned}$$

So,

$$\lambda\int_\Omega (b(x) + c(x))u^2\phi_1 dx = \lambda\int_\Omega b(x)c(x)u\phi_1 dx + \lambda\int_\Omega u^3\phi_1 dx + \int_\Omega u\phi_1 dx.$$

Since

$$\lambda\int_\Omega (b(x) + c(x))u^2\phi_1 dx \leq \lambda(b_2 + c_2)\int_\Omega u^2\phi_1 dx$$

and

$$\lambda \int_\Omega b(x)c(x)u\phi_1 dx \geq \lambda b_1 c_1 \int_\Omega u\phi_1 dx,$$

we have

$$\lambda(b_2+c_2)\int_\Omega u^2\phi_1 dx \geq (\lambda b_1 c_1 + \Lambda_1)\int_\Omega u\phi_1 dx + \lambda \int_\Omega u^3\phi_1 dx$$
$$\geq 2\sqrt{\lambda(\lambda b_1 c_1 + \Lambda_1)}\int_\Omega u^2\phi_1 dx,$$

where, in the last inequality, we use the inequality $a^2+b^2 \geq 2ab$. Hence, $\lambda(b_2+c_2) \geq 2\sqrt{\lambda(\lambda b_1 c_1 + \Lambda_1)}$, i.e.,

$$\lambda \geq \frac{4\Lambda_1}{(b_2+c_2)^2 - 4b_1c_1}$$

is a necessary condition for the existence of positive solutions of (5).

Now, we turn to the nonexistence of the positive solutions of (1) when (3) does not hold for any $x \in \Omega$. We define $\overline{c} = \max_{\overline{\Omega}} c(x)$ and $\overline{b} = \min_{\overline{\Omega}} b(x)$. Our main result of the nonexistence is the following:

Theorem 1.5 *Define* $\overline{f}(u) = \max_{x\in\overline{\Omega}} f(x,u) \geq f(x,u)$. *If* $\int_0^{\overline{c}} \overline{f}(u) \leq 0$, *then* (1) *has no positive solutions for any* $\lambda > 0$.

For the proof of Theorem 1.5, we first recall a theorem in [6]. If f is a C^1 function and there exists $0 \leq s_0 \leq s_1 \leq s_2$ such that

$$\begin{cases} f(s_i) = 0\ (i=1,2), \quad f(s_0) \leq 0, \\ f(s) < 0 \text{ for } s_0 < s < s_1, \\ f(s) > 0 \text{ for } s_1 < s < s_2 \end{cases} \tag{6}$$

and

$$\int_{s_0}^{s_2} f(s)ds \leq 0, \tag{7}$$

we have

Lemma 1.6 *Assume that* $f(u)$ *satisfies* (6) *and* (7). *Let* Ω *be a bounded domain with a smooth boundary. If* (1) *has a positive solution* u, *then* u *cannot satisfy*

$$\begin{cases} u_{\max} = \max_{x\in\Omega} u(x) \in (s_1, s_2), \\ u(x) > 0. \end{cases}$$

Proof of Theorem 1.5 If there exists a positive solution (λ, u_*) for (1), since $\Delta u_* + \lambda\overline{f}(u_*) \geq \Delta u_* + f(x,u_*) = 0$, then u_* is a subsolution of

$$\begin{cases} \Delta u + \lambda \overline{f}(u) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

$\overline{c}$ is the supersolution of (8). Therefore, by the standard comparison arguments, (8) has a positive solution $\overline{u}$ such that $u_* \leq \overline{u} \leq \overline{c}$. However, if we let $s_0 = 0$, $s_1 = \overline{b}$, $s_2 = \overline{c}$, and $\overline{f}$ satisfy (6) and (7), then by Lemma 1.6, (8) has no positive solutions. This is a contradiction. So (1) has no positive solutions if $\int_0^{\overline{c}} \overline{f}(u)du \leq 0$.

Remark 1.7 We notice that $\int_0^{\overline{c}} \overline{f}(u)du \leq 0$ implies (3) does not hold for any $x \in \bar{\Omega}$, but not vice versa. We conjecture that the nonexistence holds with a weaker condition:

$$\int_0^{c(x)} f(x,s)ds \leq 0 \text{ for any } x \in \bar{\Omega}.$$

A direct consequence of Theorem 1.5 is

Corollary 1.8 *If* $c(x) \equiv 1$ *for all* $x \in \bar{\Omega}$, *and* $\max_{x \in \bar{\Omega}} b(x) \geq \frac{1}{2}$, *then* (5) *has no positive solutions for any* $\lambda > 0$.

In [9], Dancer and Yan proved that, when $c(x) \equiv 1$ and $\{x \in \Omega : b(x) < 1/2\}$ is of positive measure, (5) may have many positive solutions of local minimum type. Corollary 1.8 shows that the condition on $b(x)$ is necessary for the existence of non-trivial solutions.

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