DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **11**, Number **3**, May **2009**

doi:10.3934/dcdsb.2009.11.717

pp. **717–740**

THE ROLE OF HIGHER VORTICITY MOMENTS IN A VARIATIONAL FORMULATION OF BAROTROPIC FLOWS ON A ROTATING SPHERE

Chjan C. Lim

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, New York 12180, USA

Junping Shi

Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23187, USA
and
Department of Mathematics
Harbin Normal University
Harbin, Heilongjiang 150025, China

(Communicated by Shouhong Wang)

Abstract. The effects of a higher vorticity moment on a variational problem for barotropic vorticity on a rotating sphere is examined rigorously in the framework of the Direct Method. This variational model differs from previous work on the Barotropic Vorticity Equation (BVE) in relaxing the angular momentum constraint, which then allows us to state and prove theorems that give necessary and sufficient conditions for the existence and stability of constrained energy extremals in the form of super and sub-rotating solid-body steady flows. Relaxation of angular momentum is a necessary step in the modeling of the important tilt instability where the rotational axis of the barotropic atmosphere tilts away from the fixed north-south axis of planetary spin. These conditions on a minimal set of parameters consisting of the planetary spin, relative enstrophy and the fourth vorticity moment, extend the results of previous work and clarify the role of the higher vorticity moments in models of geophysical flows.

1. **Introduction.** The question of how many of the infinite number of Casimir invariants should be retained in the formulation of a tractable and physically relevant equilibrium statistical mechanics theory for ideal 2D flows, is at the center of many recent works in the field. Chorin [8], Majda and Holen [25] and others have concluded that it is enough to keep the energy, total circulation and the quadratic vorticity moment known as enstrophy. See Miller [26] and Robert [29] for another

2000 *Mathematics Subject Classification.* Primary: 76M30, Secondary: 76E20, 45K05.

Key words and phrases. barotropic vorticity model, higher vorticity moment, variational method.

C. Lim is partially supported by ARO grant W911NF-05-1-0001 and DOE grant DE-FG02-04ER25616; and J. Shi is partially supported by NSF grant DMS-0314736, DMS-0703532, NSFC grant 10671049, Longjiang Scholarship of Heilongjiang Province, and Summer Research Grant of College of William and Mary.

point of view. There is little doubt that the enstrophy constraints capture most of the physics in this problem. However, this question remains a matter of discussion, and informs the investigation presented here.

Apart from the above issue of the number of vorticity invariants to retain in statistical mechanics, there is the related question of how many invariants to use in dynamical crude closure models of 2D turbulence, and also in the variational formulation of the dynamics of ideal 2D flows.

In this paper, a variational formulation for barotropic flows is presented, with the specific aim of modeling super-rotation and the tilt instability in planetary atmospheres. We seek a minimal inviscid model for rotating barotropic flows that has features related to these phenomena. The Barotropic Vorticity Equation (BVE) is the simplest geophysically relevant PDE which is the basis of our model. Since the problem we are treating here involves full spherical geometry approximating the very small amplitude random topography and non-uniform Coriolis parameter, key invariants of the BVE are the reduced energy -equal to the total mechanical energy minus the relative angular momentum- and the vorticity moments.

Due to the small random topography, a cumulative mountain torque can nonetheless change the three components of the fluid's angular momentum, conveniently taken to be the z component aligned with the north pole of planetary spin, and the x and y components. Thus, to first order accuracy in the modelling process, we will use spherical geometry in the analysis below except for not fixing the angular momentum, to address the relaxation of barotropic vorticity on a rotating planet.

There is another good reason for not fixing the angular momentum, besides our stated aim of providing a minimal model of super-rotation - from a mathematical point of view [28], adding the angular momentum constraints to the energy - relative enstrophy theory in [21] over-constrain or over-determine the variational formulation, for the simple reason that in that earlier theory as in the current formulation, the objective functional is total mechanical energy (kinetic energy measured from the rest frame minus some constants) which contains a second term that is proportional to the angular momentum aligned with planetary spin. Moreover, it was shown there that the energy extremals (constrained only by relative enstrophy) contain maximum amount of z-component of the angular momentum.

Hence, the reader should bear in mind that the statistical mechanics and stability properties arising from the variational analysis presented here, are for barotropic flows on a rotating sphere, that exchange energy and angular momentum inviscidly with their respective reservoirs residing in both the small scales of the flow and the massive rotating planet. This generalized variational model captures vital properties of the barotropic component of a vertically averaged, multi-layered, rotating atmosphere as it relaxes, without the complications of a full-blown theory based on the general circulation model (GCM)(cf. [9]). Such a theory that is based on the GCM, will include a detailed mechanism for the exchange of angular momentum between the fluid and the bottom topography (due to spin-ups and spin-downs) but will likely be too complex to be analyzed by classical mathematical techniques.

To begin the construction of a minimalist variational model, we have to choose how many and which invariants of the BVE to retain. The generalized BVE models retain the effects of spherical geometry on differential rotation in the sense that the β parameter varies with latitude. They are non-divergent models in the sense that the Rossby Radius of deformation is taken to be infinite. Thus, these ideal flows are strictly 2D and the surface of the fluid is a rigid lid. They differ substantially from more realistic models of the atmosphere which start usually from a damped driven multi-layer flow, that is insolated differentially and interacts with the bottom topography through a very complex torque mechanism. Nonetheless, Cho and Polvani [6] and Yoden and Yamada [39], amongst others, have studied the long time asymptotics of the barotropic flows on a sphere numerically (and therefore with some unavoidable artificial viscosity), and found geophysically interesting end-states.

A relationship between the inviscid variational theory in this paper and more realistic damped quasi-2D flows can be found in terms of the Principle of Selective Decay [40] which says that the asymptotic states of the damped system must have minimum enstrophy to energy ratio; the constrained energy extremals of our variational models can be viewed as minimum enstrophy states for given energy in a dual variational principle formulated originally by Leith [18]. More specifically, with damping, both the energy and enstrophy decay to zero as a result of viscosity, and Selective Decay states that a positive measure set of initial data satisfies the asymptotic property that the enstrophy decays at a faster rate than the kinetic energy. Now by Poincare's inequality there is a lower bound for the Dirichlet quotient in a periodic domain or the surface of a sphere, which then yields the useful constraint that the asymptotic quasi-steady states form a one dimensional subspace, that is, they must be decaying multiples of the ground state of the Laplace-Beltrami operator for the flow domain. Because of this, the constrained extremals that we find by fixing the enstrophy and maximizing the energy or equivalently, fixing the energy and minimizing the enstrophy captures the important long-time quasi-steady properties of the solutions of a damped 2D Navier-Stokes system. Therefore, the variational analysis in this paper is relevant not just for short times but is indeed important in the asymptotic sense as well because of the multiple time-scales extant in inverse cascades of nearly 2D flows. In one unit of the slow dissipation time, relative enstrophy can therefore be assumed fix while relaxation of energy to large scales proceed at a much faster time-scale.

The first author [21] gave a complete analysis of an earlier generalized BVE model in terms of a constrained variational formulation where the fixed-frame kinetic energy or Lagrangian of the BVE is the objective functional and the relative enstrophy and total circulation are constraints. Precisely because angular momentum is not fixed, he was able to state and prove a rigorous result on symmetry-breaking between the pro and retrograde rigidly rotating steady states in terms of the relevant physical parameters in the theory, namely, the values of the kinetic energy with respect to the relative enstrophy and planetary spin. We focus here on the effects of constraining the higher relative vorticity invariants in the generalized BVE models without applying any constraint on the angular momentum. This is not to say that variational and statistical mechanics models which invoke the conservation of angular momentum are without merit. In fact, previous work have mainly focussed on such models [12], [35], [34] and produced interesting results on stability and the energy and enstrophy spectra of rotating barotropic flows, confirming what Kraichnan [14] and others have found numerically.

This is a good point to discuss the types of constraints that one could apply to the remaining active or robust invariants of the inviscid barotropic models. When the bottom topography is non-trivial, it can be shown that the enstrophy is no longer conserved by the dynamics (cf. Salmon et al [30], Shepherd [32]). However, provided the bottom topography is small (nearly a perfect sphere), the dynamics of inviscid barotropic models can be assumed to fix the vorticity moments but not the angular momentum. At the level of statistical equilibrium models, one usually distinguish between applying the relative enstrophy constraints microcanonically (fixing it as in Ding and Lim [11]), canonically (let it relax to equilibrium in contact with a reservoir of infinite capacity) or in a well-defined mean field way as in Lim [22]. There is an important difference between the first two Gibbs probability measures: the former is equivalent to the Kac's spherical model (cf. Kac and Berlin [3] and Lim [22]) for ferromagnetism and the latter is a Gaussian model. The former is well-defined at all positive and negative temperatures. The latter is not well-defined at low temperatures but is exactly solvable where it is defined. The spherical model for barotropic flows on a rotating sphere that do not fix the angular momentum has been recently solved in closed form [23]. It is a better formulation for the calculations of phase transitions than the classical energy-enstrophy theory because it is well-defined at all temperatures (cf. Lim [22]). Ding and Lim [11] gives the results of extensive Monte-Carlo simulations of this spherical model for the equilibrium statistics of barotropic flows, and reports that at least three different equilibrium vorticity states can be distinguished as planetary spin, relative enstrophy and temperature (or expected value of the energy) are changed. They confirmed the mean field calculations of critical temperatures for the generalized barotropic flows in Lim [22] and in Lim and Singh [24].

At the level of variational models for generalized barotropic flows, the microcanonical constraint corresponds to fixing the relative enstrophy while optimizing usually the kinetic energy of the barotropic flows. On the other hand, the canonical constraint is associated with a Lagrange multiplier which arise in necessary conditions for constrained energy extremals. We show that these two levels of description are equivalent for variational models of generalized barotropic flows.

Next, we explain how the angular momentum invariants (and constraints) are related to the relative, total enstrophy and kinetic energy in barotropic flows. The kinetic energy (more correctly the fluid's total mechanical energy relative to a fixed-frame and thus, the non-conserved Lagrangian of the BVE) of barotropic flows on a sphere,

$$\begin{aligned} H[q] &= -\frac{1}{2}\int_{S^2} dx\ \psi q = -\frac{1}{2}\int dx\ \psi(x)\left[w(x) + 2\Omega\cos\theta\right] \\ &= -\frac{1}{2}\int dx\ \psi(x)w(x) - \Omega\int dx\ \psi(x)\cos\theta \end{aligned} \tag{1}$$

where Ω denotes planetary spin, ψ denotes the stream function, w is the relative vorticity, and q is the total vorticity, can be written in the form of an ellipsoid in the Hilbert space of square-integrable vorticity distributions with zero total circulation, that is shifted to the left along the axis for aligned rigid rotation by an amount proportional to the planetary spin Ω, according to

$$H = -\frac{1}{2}\sum_{l\geq 1,m}\frac{\alpha_{lm}^2}{\lambda_{lm}} + \frac{1}{2}\Omega C\alpha_{10}. \tag{2}$$

Here α_{lm} and λ_{lm} are respectively the Fourier coefficients and Laplacian eigenvalues - discussed in mathematical settings below - for the lm-th spherical harmonics ψ_{lm} in the Fourier expansion of the relative vorticity. The z component of the angular momentum, that is aligned with the north pole due to planetary spin, is associated with the spherical harmonic ψ_{10}; the x and y components are associated respectively with the harmonics $\psi_{1,1}$ and $\psi_{1,-1}$. Adding small amounts of x and y angular momenta to a solid-body steady-state with a large z angular momentum is kinematically equivalent to tilting the axis of atmospheric rotations away from the planetary spin axis (or north-south axis), hence the name tilt instability that is usually associated with these two spherical harmonics. If we fix the z angular momentum, then as Frederiksen and Sawford observed in [12], we are essentially working with a reduced kinetic energy expression in which the symmetry-breaking term is missing, and the energy is then represented by an ellipsoid that is no longer shifted. Similarly, if we also fix the x and/or y angular momenta, the resulting energy ellipsoid and relative enstrophy sphere are concentric in a reduced Hilbert space spanned by only the spherical harmonics ψ_{lm}, with the total wave-number l greater than 1. This effectively reduces the statistical mechanics and variational theory of the BVE on a rotating sphere to that of the non-rotating sphere, where the pro and retrograde rigid rotation states are now perfectly symmetrical. Thus, we now see how fixing angular momentum changes the other key invariants in the problem, namely relative enstrophy and energy.

We now explain how total enstrophy is related to relative enstrophy and angular momentum. According to the expansion of the total enstrophy,

$$\begin{aligned}\Gamma[q] &= \int_{S^2} dx\ q^2 = \int_{S^2} dx\ \ [w + 2\Omega\cos\theta]^2 \\ &= \int_{S^2} dx\ w^2 + 4\Omega \int_{S^2} dx\ w\cos\theta + 4\Omega^2 C^2,\end{aligned} \tag{3}$$

where C is a universal constant, the total enstrophy is essentially (modulo a constant) the sum of the relative enstrophy and the z angular momentum. Thus, fixing both the relative enstrophy and the z angular momentum is equivalent to fixing total enstrophy in the statistical mechanics and variational theories of barotropic flows. Conversely, fixing the total enstrophy is clearly not identical to any (equality type) constraints on the relative enstrophy and z angular momentum, which instead, are now constrained by implicit or inequality bounds. Fixing the relative enstrophy without fixing any of the x, y, or z angular momenta is clearly equivalent to implicit inequality constraints on these components of angular momenta, because the parts of the relative enstrophy associated with ψ_{lm}, $m = -1, 0, +1$ are now allowed to fluctuate provided the square-norm of relative vorticity is fixed. Precisely in this way, the first generalized barotropic variational model by Lim and our current work allow angular momentum to fluctuate, in order to achieve our stated aim of a (minimalist-blackbox) variational model of the complex torque mechanisms for the exchange of angular momentum between atmosphere and planet in relaxing barotropic flows on a perfect rotating sphere. Only such models are simultaneously relevant to super-rotation and the tilt instability and at the same time, amenable to a treatment based on classical applied mathematics.

In this paper we focus on the application and analysis of the fourth vorticity moment in the BVE model, without imposing any constraint on the angular momentum. The principal aim is to see how the phenomenon of symmetry-breaking observed in the first BVE-variational theory under the constraint of only the relative enstrophy, depend on the higher vorticity moments. In Section 2, a summary of the BVE with respect to its geophysical fluid dynamical properties will be given. The methods of variational theory such as the Direct Method of the Calculus of

Variations and general functional analysis are used extensively in Sections 3 – 5 of this paper to extract the physically relevant rigorous results that the higher vorticity moments make a difference in the variational theory of the BVE.

2. Barotropic vorticity equation.

This equation can be written in the form

$$\frac{\partial}{\partial t}\omega + J(\psi, \omega + 2\Omega\cos\theta) = 0$$

where ω is the relative vorticity,

$$\omega = \Delta\psi,$$

ψ is the stream function, and the absolute or total vorticity is given by the sum,

$$q = \Delta\psi + 2\Omega\cos\theta,$$

of relative and planetary vorticity, and Ω is the rate of spin of the coordinate system in which the solid planet (taken to be the unit sphere S^2) is at rest. The operator Δ is the Laplace-Beltrami operator for the sphere S^2, J is the Jacobian operator, and θ is co-latitude which is 0 at the north pole N.

A more realistic model is the divergent shallow-water equations (SWE) on a rotating sphere (cf. Cho and Polvani [6] and Ding and Lim [10]) which we will discuss next in order to state explicitly the approximations in the BVE. For this purpose let us denote by U, L and H, the velocity, length and depth scales respectively. Then two important dimensionless numbers are the Rossby and Froude numbers respectively,

$$R = \frac{U}{2\Omega L},\ F = \frac{U}{\sqrt{gH}},$$

where g is the gravitational constant. Within the SWE model, the relative importance of convective / local flow to rotational effects is measured by the Rossby number R. The Froude number F measures the relative importance within the SWE model of the convective / local flow effects to gravity-depth effects. In a definite sense, a small Rossby number $R \ll 1$ signals the importance of rotational effects: Ω has to be relatively large or the scale L of the flow has to be relatively large in order for rotation of the planet to be important. On the other hand, a large Froude number $F \gg 1$ implies the importance of gravity effects, since in this case, the gravity waves are relatively slow, and cannot be time-averaged out of the problem.

The Rossby Radius of deformation,

$$L_R = \frac{\sqrt{gH}}{2\Omega} = \frac{RL}{F},$$

measures the relative importance of gravity-depth effects to rotational effects. When it is of $O(1)$, both gravity and rotational effects are relevant to the problem, and only when $L_R \gg 1$ that rotational and convective or inertial effects dominate. It is convenient to label the square of L_R/L by the Burgers number

$$B = \frac{R^2}{F^2} = \frac{L_R^2}{L^2}.$$

Small values of B signals the importance of gravity-depth effects over rotational effects; it inludes the case when the Rossby number R itself is relatively small, that is, when rotational effects dominate convective or inertial effects, as well as the case when R is $O(1)$, that is, when rotational effects are comparable to convective or inertial effects.

The BVE can be characterized as the limit of the SWE when L_R tends to ∞ or equivalently when the depth scale H tends to ∞ with Ω and g fixed. The flow (u, v) in the BVE model is strictly 2D, that is, ω is a scalar field and the top and bottom boundary conditions are idealized away by taking, in effect, the depth scale H to ∞. Thus, in a definite sense, the BVE models a rotating fluid of infinite depth. This fact partly accounts for its tractability relative to the more complex SWE model where boundary conditions at the top and bottom of the fluid are retained. In this definite sense, the BVE model is non-divergent, i.e., $div\ (u, v) = 0$, while the SWE is a divergent model.

Finally when the spin $\Omega = 0$, the BVE reduces to the non-rotating non-divergent model on a sphere, which has less interesting physical properties. One of the significant properties of the generalized BVE model ($\Omega > 0$) is the existence of symmetry-breaking between its pro and counter-rotating solid-body rotation steady states, which was discovered by the first author in [21], using only the relative enstrophy constrain. One of the aims of this paper is to see whether this asymmetry is preserved when higher order vorticity moments are included in the variational theory of the BVE model.

3. **Mathematical settings.** In the following, S^2 is the unit sphere $\{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$; and the spherical coordinate representation is $S^2 = \{(\phi, \theta) : 0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi\}$, where ϕ is the longitude, and θ is the colatitude; $L^p(S^2)$ is the space of real-valued measurable functions $f : S^2 \to \mathbf{R}$ such that $\int_{S^2} |f|^p dx < \infty$ for $p > 0$, and the norm of the space is

$$||f||_p = \left(\int_{S^2} |f(x)|^p dx\right)^{1/p}. \tag{4}$$

In this paper we will consider the space

$$V_0 = \left\{w \in L^4(S^2) : \int_{S^2} w(x)dx = 0\right\}. \tag{5}$$

The norm of V_0 is $||\cdot||_4$ from

$$\int_{S^2} w^2 dx \leq \left(\int_{S^2} w^4 dx\right)^{1/2} \cdot \left(\int_{S^2} 1dx\right)^{1/2} = \sqrt{4\pi}\left(\int_{S^2} w^4 dx\right)^{1/2}, \tag{6}$$

where 4π is the surface area of S^2.

The key to our analysis is the Laplace-Beltrami operator Δ on S^2 and its inverse. The solution of

$$\Delta u = f(x), \quad x \in S^2, \tag{7}$$

is given by

$$u(x) = -\frac{1}{2\pi}\int_{S^2} f(y) \ln \frac{1}{|x-y|} dy, \tag{8}$$

where $|x - y|$ is the Euclidean distance in $\mathbf{R}^3$. For any $f \in L^2(S^2)$, we define $G(f)$ to be the solution u of (7) defined as in (8), and we also define

$$V_1 = \left\{w \in L^2(S^2) : \int_{S^2} w(x)dx = 0\right\}. \tag{9}$$

An orthonormal basis of V_1 is given by the spherical harmonics

$$Y_l^m(\theta, \phi) = c_l^m e^{im\phi} P_l^{|m|}(\cos\theta), \tag{10}$$

where $l \geq 1$, $-l \leq m \leq l$, c_l^m is a real normalizing constant defined by

$$c_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \tag{11}$$

so that $\int_{S^2} Y_l^m Y_{l'}^{m'} dx = \delta_{mm'}\delta_{ll'}$, and $P_l^m(\cdot)$ is the Legendre polynomials. Here we define a real valued orthonormal basis of V_1:

$$\begin{aligned} \psi_{lm}(\theta,\phi) &= \sqrt{2}c_l^m \cos(m\phi)P_l^{|m|}(\cos\theta), \;\; 0 < m \leq l, \\ \psi_{lm}(\theta,\phi) &= \sqrt{2}c_l^m \sin(m\phi)P_l^{|m|}(\cos\theta), \;\; -l \leq m < 0, \\ \psi_{l0}(\theta,\phi) &= c_l^0 P_l^0(\cos\theta). \end{aligned} \tag{12}$$

We recall that

$$\psi_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \;\; ||\psi_{10}||_2 = 1, \quad \text{and } ||\psi_{10}||_4 = \left(\frac{9}{20\pi}\right)^{1/4}. \tag{13}$$

The operator G satisfies the following basic properties:

Lemma 3.1. *Let G be defined as above.*

1. *The set of the eigenvalues of G is $\{-\frac{1}{l(l+1)} : l \geq 1\}$, and the corresponding eigenfunctions are ψ_{lm}, $-l \leq m \leq l$, the spherical harmonic functions on S^2; the set $\{\psi_{lm} : l \geq 1, -l \leq m \leq l\}$ is an orthonormal basis of V_1;*
2. *$G : V_1 \to V_1$ is self-adjoint, i.e. for any $f, g \in V_1$, $\langle G(f), g\rangle = \langle f, G(g)\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner-product in $L^2(S^2)$;*
3. $$0 \leq -\int_{S^2} G(w)w dx \leq \frac{1}{2}\int_{S^2} w^2 dx, \quad \text{for any } w \in V_1. \tag{14}$$

Proof. Parts 1-2 are standard. Part 3 and the equation below follows directly from results in [35] and [34]. □

Note that the inequality (14) can also be written as in [35] and [34]

$$\Lambda_1 = 2 = \min_{\substack{\varphi \in H^2(S^2) \\ \varphi \not\equiv 0, \int \varphi = 0}} \frac{\int_{S^2} (\Delta\varphi)^2 dx}{\int_{S^2} |\nabla\varphi|^2 dx}, \tag{15}$$

where $\varphi = G(w)$. The Rayleigh quotient in (15) defines the first buckling eigenvalue of the surface S^2.

The total kinetic energy (modulo a constant) of the fluid motion relative to a fixed-frame is given by (see [21])

$$H[w] = -\frac{1}{2}\int_{S^2} G(w)[w + 2\Omega\cos\theta]dx, \tag{16}$$

and by using Lemma 3.1 and (13), we obtain

$$H[w] = -\frac{1}{2}\int_{S^2} G(w)wdx + \frac{1}{2}\Omega||\cos\theta||_2 \int_{S^2} \psi_{10} wdx. \tag{17}$$

In this paper we study the energy extremals of H with conserved relative enstrophy

$$\Gamma_2[w] \equiv \int_{S^2} w^2 dx = M_2 > 0, \tag{18}$$

and also the first non-zero higher vorticity moment

$$\Gamma_4[w] \equiv \int_{S^2} w^4 dx = M_4^2 > 0. \tag{19}$$

From (6), M_2 and M_4 must satisfy

$$M_2 \leq \sqrt{4\pi} M_4. \tag{20}$$

Notice that from Stokes's Theorem, we also have zero total circulation

$$\Gamma_1[w] \equiv \int_{S^2} w dx = 0, \tag{21}$$

thus $w \in V_1$. In summary, we will

$$\begin{aligned} &\text{extremize } H[w] \text{ on} \\ V = \Big\{ & w \in L^4(S^2) : \Gamma_2[w] \equiv \int_{S^2} w^2 dx = M_2 > 0, \\ & \Gamma_4[w] \equiv \int_{S^2} w^4 dx = M_4^2 > 0, \int_{S^2} w dx = 0 \Big\}. \end{aligned} \tag{22}$$

To use the Lagrange principle for constrained variational problem, we consider the augmented energy functional:

$$\begin{aligned} &E(w, \lambda, \mu) = H[w] + \lambda \Gamma_2[w] + \mu \left(\Gamma_4[w]\right)^{1/2} \\ &= -\frac{1}{2}\int_{S^2} G(w) w dx + \frac{1}{2}\Omega || \cos\theta ||_2 \int_{S^2} \psi_{10} w dx + \lambda \int_{S^2} w^2 dx + \mu \left(\int_{S^2} w^4 dx \right)^{1/2}, \end{aligned} \tag{23}$$

for $w \in V_0 = \{ w \in L^4(S^2) : \int_{S^2} w dx = 0 \}$. Note that we use $||w||_4^2$ instead of $||w||_4^4$ to get the correct dimension of the energy. The Euler-Lagrange equation of (23) is

$$G(w) - 2\lambda w - \frac{2\mu}{||w||_4^2} w^3 = \frac{1}{2}\Omega || \cos\theta ||_2 \psi_{10}. \tag{24}$$

Lemma 3.2. *Suppose that for some M_2, M_4, $w \in V \subset V_0$ is an extremal of (23) (or a solution of (24)). Then w is an extremal of the problem (22) for these values.*

Proof. The tangent space of V at w is defined as

$$T = \left\{ \xi \in L^4(S^2) : \int_{S^2} \xi dx = \int_{S^2} \xi w dx = \int_{S^2} \xi w^3 dx = 0 \right\}. \tag{25}$$

For any $\xi \in T$, the variation of $H(\cdot)$ along the direction ξ is

$$\begin{aligned} \delta H(w)[\xi] &= -\int_{S^2} G(w) \xi dx + \frac{1}{2}\Omega || \cos\theta ||_2 \int_{S^2} \psi_{10} \xi dx \\ &= -2\lambda \int_{S^2} w \xi dx - \frac{2\mu}{||w||_4^2} \int_{S^2} w^3 \xi dx = 0, \end{aligned} \tag{26}$$

from the Euler-Lagrange equation (24) and (25). Thus w is an extremal for (22). □

Multiplying (24) by w and integrating over S^2, we obtain an energy identity:

$$-\frac{1}{2}\int_{S^2} G(w) w dx + \lambda \int_{S^2} w^2 dx + \mu \left(\int_{S^2} w^4 dx \right)^{1/2} + \frac{1}{4}\Omega || \cos\theta ||_2 \int_{S^2} \psi_{10} w dx = 0. \tag{27}$$

By comparing (23) and (27), we find that

$$E(w,\lambda,\mu) = \frac{1}{4}\Omega||\cos\theta||_2 \int_{S^2} \psi_{10} w dx. \tag{28}$$

Thus the augmented energy functional always equals to a multiple of the angular momentum if w is an extremal. Although the identity (28) is more mathematical than physical (in the augmented energy functional (23), all terms are of quadratic nature except the angular momentum), it suggests that the inclusion of an angular momentum constraint will lead to an over-constrained problem. In the sequel to this paper where the angular momentum is included, it will be shown that the constraints structure of the new variational formulation is equivalent to imposing a total enstrophy constraint on the BVE. Both (27) and (28) will be useful in our later discussions.

The constrained variational problem (23) which only conserves the relative enstrophy is considered in [21]. In that case the Euler-Lagrange equation is linear:

$$G(w) - 2\lambda w = \frac{1}{2}\Omega||\cos\theta||_2\psi_{10}, \tag{29}$$

and for non-resonant $\lambda \neq -[l(l+1)]^{-1}$, $l \geq 1$, (29) has a unique solution:

$$W_\lambda = -\frac{\Omega||\cos\theta||_2}{1+4\lambda}\psi_{10}, \tag{30}$$

and when $\lambda_l = -[l(l+1)]^{-1}$, $l \geq 2$, the solution is the one in (30) plus some higher spherical harmonics at the eigenvalue λ_l. The solution W_λ can also be expressed in term of the enstrophy and energy $H(\cdot)$, which we will recall in Section 5, when compared to the model conserving both enstrophy and fourth order vorticity moment.

The stability of an extremal w of (23) with respect to the augmented energy E is determined by the second variation:

$$\delta^2 E(w,\lambda,\mu)[h,k]$$
$$= -\int_{S^2} G(h)kdx + 2\lambda\int_{S^2} hkdx + \frac{6\mu}{||w||_4^2}\int_{S^2} w^2hkdx - \frac{4\mu}{||w||_4^6}\int_{S^2} w^3hdx\int_{S^2} w^3kdx, \tag{31}$$

for $h,k \in V_0$, or equivalently, the linear eigenvalue problem:

$$L(w)[h] \equiv G(h) - 2\lambda h - \frac{6\mu}{||w||_4^2}w^2h + \frac{4\mu w^3}{||w||_4^6}\int_{S^2} w^3hdx = -\gamma h. \tag{32}$$

The eigenvalues of L can be characterized by the minimax values of a Rayleigh quotient:

$$R[w,h] \equiv \frac{\delta^2E(w,\lambda,\mu)[h,h]}{\int_{S^2} h^2dx}. \tag{33}$$

We call w a stable minimizer if all eigenvalues $\gamma_i > 0$, and we call w a stable maximizer if all eigenvalues $\gamma_i < 0$. And an extremal w is non-degenerate if $L(w)$ is invertible, *i.e.* zero is not an eigenvalue.

Proposition 1. *Let w be an extremal of (23).*

1. *If $\mu \geq 0$,$\lambda \geq 0$ and $(\lambda,\mu) \neq (0,0)$, then w is a stable minimizer;*
2. *If $\mu \leq 0$, $\lambda \leq -1/4$ and $(\lambda,\mu) \neq (-1/4,0)$, then w is a stable maximizer.*

Proof. From Cauchy-Schwarz's inequality, we have

$$\int_{S^2} w^2h^2dx \int_{S^2} w^4dx \geq \left(\int_{S^2} w^3hdx\right)^2. \tag{34}$$

Thus when $\mu > 0$ and $\lambda > 0$, $R[w,h] > 0$ for any $h \neq 0$; and when $\mu < 0$ and $\lambda < -1/4$, $R[w,h] < 0$ for any $h \neq 0$ from (14). □

Remark. When $\mu = 0$, the stability of W_λ is defined by (33) with $\mu = 0$, and it is easy to see that the solution W_λ of (29) is a stable minimizer when $\lambda \geq 0$, and it is a stable maximizer when $\lambda < -1/4$. Moreover when $-1/4 < \lambda < 0$, any solution of (29) is a saddle point with both positive and negative spectral points.

4. **Minimizers and maximizers.** In this section, we apply the direct method of calculus of variations to obtain the energy minimizers and maximizers for suitable parameters (λ, μ). We use the following standard theorem (see for example, [37] Theorem 1.2):

Theorem 4.1. *Suppose that V is a reflexive Banach space with norm $||\cdot||$, and let $M \subset V$ be a weakly closed subset of V. Suppose that $E : M \to \mathbf{R} \cup \{+\infty\}$ is coercive and weakly lower semi-continuous on M with respect to V, that is suppose the following conditions are fulfilled:*

1. *$E(u) \to \infty$ as $||u|| \to \infty$, $u \in M$;*
2. *For any $u \in M$, any sequence (u_n) in M such that $u_m \rightharpoonup u$ weakly in V there holds: $E(u) \leq \liminf_{n\to\infty} E(u_n)$.*

Then E is bounded from below on M and attains its infimum in M.

Our main existence theorem is

Theorem 4.2. *Let $E(\cdot, \lambda, \mu)$ be defined as in (23).*

1. *For $\lambda \in \mathbf{R}$, and $\mu > \max\{0, -\sqrt{4\pi}\lambda\}$, $E(\cdot, \lambda, \mu)$ has a global minimizer $\Psi^m_{\lambda,\mu} \in V_0$ such that $E(\Psi^m_{\lambda,\mu}, \lambda, \mu) = \inf_{w\in V_0} E(w, \lambda, \mu)$;*
2. *For $\lambda \in \mathbf{R}$, and $\mu < \min\{0, -\sqrt{4\pi}\lambda - \dfrac{\sqrt{\pi}}{2}\}$, $E(\cdot, \lambda, \mu)$ has a global maximizer $\Psi^M_{\lambda,\mu} \in V_0$ such that $E(\Psi^M_{\lambda,\mu}, \lambda, \mu) = \sup_{w\in V_0} E(w, \lambda, \mu)$;*
3. *When $\mu = 0$, and $\lambda > 0$, $E(\cdot, \lambda, 0)$ has a global minimizer $\Psi^m_{\lambda,\mu} \in V_0$ such that $E(\Psi^m_{\lambda,0}, \lambda, 0) = \inf_{w\in V_0} E(w, \lambda, 0)$;*
4. *When $\mu = 0$, and $\lambda < -1/4$, $E(\cdot, \lambda, 0)$ has a global maximizer $\Psi^M_{\lambda,0} \in V_0$ such that $E(\Psi^M_{\lambda,\mu}, \lambda, 0) = \sup_{w\in V_0} E(w, \lambda, 0)$.*

Proof. We note that $L^4(S^2)$ is a reflexive Banach space, and V_0 is a closed subspace of $L^4(S^2)$. The weakly lower semi-continuity is obvious as the functional E is continuous with respect to L^4 norm from (14) and (6). When $\mu > 0$ and $\lambda > 0$, for any $w \in V_0$,

$$\begin{aligned} E(w, \lambda, \mu) \geq& \mu||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2||w||_2 \\ \geq& \mu||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2(4\pi)^{1/4}||w||_4 \to \infty, \end{aligned} \tag{35}$$

as $||w||_4 \to \infty$; and when $\mu > 0$ and $\mu > -\sqrt{4\pi}\lambda > 0$,

$$\begin{aligned} E(w,\lambda,\mu) \geq &(\mu + \lambda\sqrt{4\pi})||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2||w||_2 \\ \geq &(\mu + \lambda\sqrt{4\pi})||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2(4\pi)^{1/4}||w||_4 \to \infty, \end{aligned} \tag{36}$$

as $||w||_4 \to \infty$. Thus in both cases, E is coercive, and the global minimizer of E exists by Theorem 4.1.

When $\mu < 0$ and $\lambda \leq -1/4$,

$$\begin{aligned} -E(w,\lambda,\mu) \geq &|\mu|||w||_4^2 + \frac{1}{2}\int_{S^2} G(w)w dx + \frac{1}{4}||w||_2^2 - \frac{1}{2}\Omega||\cos\theta||_2||w||_2 \\ \geq &|\mu|||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2(4\pi)^{1/4}||w||_4 \to \infty, \end{aligned} \tag{37}$$

as $||w||_4 \to \infty$; and when $\mu < 0$, $\lambda > -1/4$ and $\mu + \sqrt{4\pi}\lambda + \dfrac{\sqrt{\pi}}{2} < 0$,

$$\begin{aligned} -E(w,\lambda,\mu) \geq &-\mu||w||_4^2 + \frac{1}{2}\int_{S^2} G(w)w dx - \lambda||w||_2^2 - \frac{1}{2}\Omega||\cos\theta||_2||w||_2 \\ \geq &-\mu||w||_4^2 - \left(\frac{1}{4} + \lambda\right)||w||_2^2 - \frac{1}{2}\Omega||\cos\theta||_2(4\pi)^{1/4}||w||_4 \\ \geq &\left[-\mu - \sqrt{4\pi}\left(\frac{1}{4} + \lambda\right)\right]||w||_4^2 - \frac{1}{2}\Omega||\cos\theta||_2(4\pi)^{1/4}||w||_4 \to \infty, \end{aligned} \tag{38}$$

as $||w||_4 \to \infty$. Thus in both cases, $-E$ is coercive, and the global maximizer of E exists by Theorem 4.1. The cases of $\mu = 0$ can be similarly proved, or use the previous results in [21] and the remark at the end of Section 2. □

In more restrictive parameter regions than the ones in Theorem 4.2, we can show that the extremal of (23) is unique, thus the global minimizer or maximizer obtained in Theorem 4.2 is the unique extremal of (23). The uniqueness result is based on the stability lemma Proposition 1 and the uniqueness for $\mu = 0$ which is proved in [21].

Theorem 4.3. *Let $\Psi^M_{\lambda,\mu}$ and $\Psi^m_{\lambda,\mu}$ be as in Theorem 4.2. Then*

1. *If $\mu \geq 0$ and $\lambda > 0$, then (24) has a unique solution $\Psi^m_{\lambda,\mu}$, and $\Psi^m_{\lambda,\mu}$ is zonal, i.e. $\Psi^m_{\lambda,\mu}(\phi,\theta) \equiv \Psi^m_{\lambda,\mu}(\theta)$;*
2. *If $\mu \leq 0$ and $\lambda < -1/4$, then (24) has a unique solution $\Psi^M_{\lambda,\mu}$, and $\Psi^M_{\lambda,\mu}$ is zonal.*

Proof. When $\mu > 0$ and $\lambda > 0$, we prove that the functional $E(w,\lambda,\mu)$ is strictly convex in w, *i.e.* $E(kw_1 + (1-k)w_2) < kE(w_1) + (1-k)E(w_2)$ for $w_1, w_2 \in V_0$ and $k \in (0,1)$. It is sufficient to show the convexity of each term of E and the strict convexity of at least one term of E. The functional $w \mapsto \lambda\int_{S^2} w^2 dx = \lambda||w||_2^2$ and $w \mapsto \mu(\int_{S^2} w^4 dx)^{1/2} = \mu||w||_4^2$ are both strictly convex from Hölder's inequality and Cauchy-Schwarz's inequality, and $\mu > 0$ and $\lambda > 0$; the functional $w \mapsto \int_{S^2}\psi_{10} w dx$ is linear thus convex; finally $w \mapsto -\int_{S^2} G(w)w$ is convex by using the decomposition $w = \sum_{l\geq 1}\alpha_{lm}\psi_{lm}$ and Cauchy-Schwarz's inequality:

$$-\int_{S^2} G(w_1)w_2 dx \leq -\frac{1}{2}\int_{S^2} G(w_1)w_1 dx - \frac{1}{2}\int_{S^2} G(w_2)w_2 dx.$$

Hence $E(w, \lambda, \mu)$ is strictly convex in w, which implies the minimizer is unique. On the other hand, from Proposition 1, any critical point of E when $\mu \geq 0$ and $\lambda > 0$ is a minimizer. Therefore $\Psi^m_{\lambda,\mu}$ obtained in Theorem 4.2 is the unique critical point of E. The proof of uniqueness for $\mu < 0$ and $\lambda < -1/4$ is similar and we can show that $-E$ is strictly convex in that case. The case of $\mu = 0$ is proved in [21].

Finally let V_z be the subspace of V_0 generated by zonal spherical harmonics, and we restrict the definition of $E(w, \lambda, \mu)$ above to $V_z \times \mathbf{R} \times \mathbf{R}$. Then all of the arguments in the last paragraph and the proof of Theorem 4.2 can be applied mutatis mutandis. Again we obtain a unique minimizer in the space $V_z \subset V_0$. Thus the minimizer in V_z and the one in V_0 must coincide, and $\Psi^p_{\lambda,\mu}$ is zonal for $p = m, M$. □

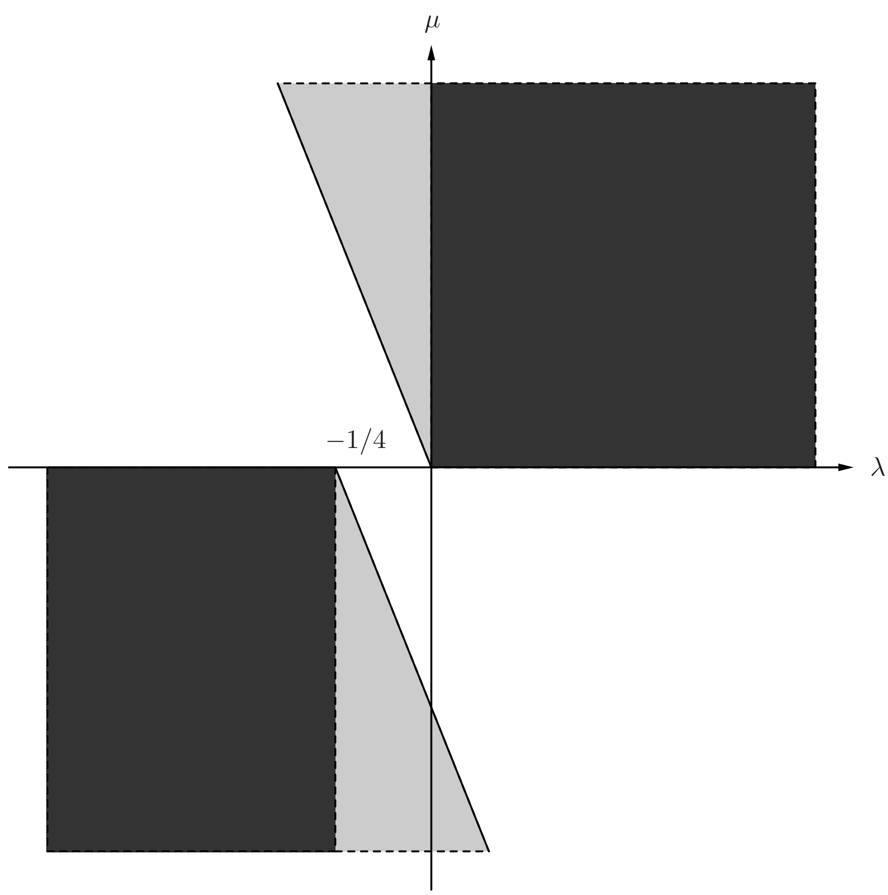

Figure 1. Parameter regions of existence and uniqueness of minimizer/maximizers. Shaded regions are where the existence holds, and darker shaded regions are where the uniqueness holds. In the next section, we define the upper right dark region to be Q_1, and the lower left dark region to be Q_2.

The parameter regions of the existence and uniqueness of extremals of $E(\lambda, \mu)$ are depicted on Figure 1. The upper right shaded region is the parameter region where a energy minimizer exists, and the darker part is where the uniqueness also holds. All solutions we find for the upper right shaded region are stable minimizers. Similarly the lower left shaded region is the parameter region where an energy maximizer exists, and the darker part is where the uniqueness also holds. The blank region on Figure 1 (or a subset of that region) is where we expect to have saddle type extremals, which will be discussed in a forthcoming paper. Note that when $\mu = 0$ and $\lambda \in (-1/4, 0)$, the existence of saddle solutions has been shown in [21] by the first author.

Next we show that the unique minimizer(maximizer) is continuously differentiable in certain function spaces with respect to parameters λ and μ, which will be useful for further estimates and properties of solutions. We start with a lemma about the regularity of the solutions on the spatial variable x:

Lemma 4.4. *Suppose that $w \in V_0$ is a solution of (24), and $\lambda\mu > 0$. Then $w \in C^{2,\alpha}(S^2)$ for any $\alpha \in [0, 1)$.*

Proof. Note that (24) can be written as $G(w) = h$ where $h = 2\mu||w||_4^{-2}w^3 + 2\lambda w + (\Omega/2)||\cos\theta||_2\psi_{10}$. Since G is the inverse of Laplace-Beltrami operator and $w \in L^2(S^2)$, then $h \in H^{2,2}(S^2) \hookrightarrow C^{0,\alpha}(S^2)$ for any $\alpha \in [0, 1)$ from elliptic estimates and Sobolev embedding theorem. This implies that $h_1(x) = 2\mu||w||_4^{-2}w^3(x) + 2\lambda w(x)$ is a function in the class of $C^{0,\alpha}(S^2)$. When $\lambda\mu > 0$, the function $h_2(y) = 2\mu||w||_4^{-2}y^3 + 2\lambda y$ is a monotone function thus invertible. Hence $w(x) = h_2^{-1} \circ h_1(x)$ is a function in the class of $C^{0,\alpha}(S^2)$ as well. From the Hölder estimates of elliptic equations, $G(w) \in C^{2,\alpha}(S^2)$, thus $h_1 \in C^{2,\alpha}(S^2)$. Also h_2 and its inverse are smooth, hence $w = h_2^{-1} \circ h_1 \in C^{2,\alpha}(S^2)$. □

We notice that repeating the arguments in the proof of Lemma 4.4 can produce higher regularity of the solution, but it is not needed for our purpose. Because the regularity of the solution, we can view the solution of (24) as the zero point of the mapping $F : \mathbf{R}^2 \times (C^{0,\alpha}(S^2)\backslash\{0\}) \to C^{0,\alpha}(S^2)$ defined by

$$F(\lambda, \mu, w) \equiv G(w) - 2\lambda w - \frac{2\mu}{||w||_4^2}w^3 - \frac{1}{2}\Omega||\cos\theta||_2\psi_{10}. \tag{39}$$

Now we can prove the differentiability of the minimizer/maximumizer with respect to the parameters:

Theorem 4.5. *Let $\Psi^M_{\lambda,\mu}$ and $\Psi^m_{\lambda,\mu}$ be as in Theorem 4.2. Then*

1. *The map $(\lambda, \mu) \mapsto \Psi^M_{\lambda,\mu}$ is continuously differentiable from $(-\infty, -1/4) \times (-\infty, 0)$ to $C^{0,\alpha}(S^2)$;*
2. *The map $(\lambda, \mu) \mapsto \Psi^m_{\lambda,\mu}$ is continuously differentiable from $(0,\infty) \times (0,\infty)$ to $C^{0,\alpha}(S^2)$.*

Proof. We prove the second case and the proof for the first case is similar. We fix $(\lambda_0, \mu_0) \in (0,\infty) \times (0,\infty)$. The linearized operator D_wF at $w = \Psi^m_{\lambda,\mu}$ is $L(w)$ defined in (32) for $h \in C^{0,\alpha}(S^2)$. We claim that $L(w)$ is a Fredholm operator $C^{0,\alpha}(S^2) \to C^{0,\alpha}(S^2)$ with index zero. Indeed from Lemma 4.4, $G : C^{0,\alpha}(S^2) \to C^{2,\alpha}(S^2)$ is continuous, and the embedding from $C^{2,\alpha}(S^2)$ to $C^{0,\alpha}(S^2)$ is compact, thus $G : C^{0,\alpha}(S^2) \to C^{0,\alpha}(S^2)$ is compact. It is well-known that Riesz-Schauder theory holds for the operator $G - 2\lambda I$, and $K = G - 2\lambda I$ is a Fredholm operator with index zero. Moreover $L(w)$ is a K-compact perturbation of K, then from the perturbation theory of Fredholm operators (see [13] Theorem 5.26), $L(w)$ is also a Fredholm operator with index zero.

The kernel of $L(w)$ at $w = \Psi^m_{\lambda,\mu}$ $N(L(\Psi^m_{\lambda,\mu}))$ is $\{0\}$ from Proposition 1. Since $L(w)$ is Fredholm with index zero, then the range $R(L(\Psi^m_{\lambda,\mu}))$ is of co-dimension zero, thus $(L(\Psi^m_{\lambda,\mu})) = C^{0,\alpha}(S^2)$. From Banach open mapping theorem, $L(\Psi^m_{\lambda,\mu})$ is invertible with a bounded inverse. We can easily check that the mapping F defined in (39) is continuously differentiable, thus we can apply implicit function theorem to $F(\lambda, \mu, w) = 0$ at $(\lambda_0, \mu_0, \Psi^m_{\lambda,\mu})$ so that the solutions of $F(\lambda, \mu, w) = 0$ near it are in form of $(\lambda, \mu, w(\lambda, \mu))$ for a C^1 function $w(\lambda, \mu)$. But the solution of $F(\lambda, \mu, w) = 0$ is unique in the parameter range $\lambda > 0$, $\mu > 0$ according to Theorem 4.3, and this implies the result claimed in the theorem. □

Now we show that for the minimizers/maximizers, both the enstrophy and the fourth vorticity moment approach zero as $|\lambda|, |\mu| \to \infty$ in the regions where a stable minimizer or a stable maximizer exists. We need the following estimates:

Proposition 2. *Let $\Psi^M_{\lambda,\mu}$ and $\Psi^m_{\lambda,\mu}$ be as in Theorem 4.2.*

1. *If $\mu > 0$ and $\lambda > 0$, then*

$$||\Psi^m_{\lambda,\mu}||_2 \le \frac{\sqrt{\pi}\Omega||\cos\theta||_2}{2(\mu + \lambda\sqrt{4\pi})}, \quad \textit{and } ||\Psi^m_{\lambda,\mu}||_4^2 \le \frac{\sqrt{\pi}\Omega^2||\cos\theta||_2^2}{8\mu(\mu + \lambda\sqrt{4\pi})}. \tag{40}$$

2. *If $\mu < 0$ and $\lambda < -1/4$, then*

$$||\Psi^M_{\lambda,\mu}||_2 \le \frac{-\sqrt{\pi}\Omega||\cos\theta||_2}{2(\mu + \lambda\sqrt{4\pi} + \sqrt{\pi}/2)}, \quad \textit{and } ||\Psi^M_{\lambda,\mu}||_4^2 \le \frac{-\sqrt{\pi}\Omega^2||\cos\theta||_2^2(\mu + 2\sqrt{\pi}\lambda)}{8\mu(\mu + \lambda\sqrt{4\pi} + \sqrt{\pi}/2)^2}. \tag{41}$$

Proof. First we assume $\mu > 0$ and $\lambda > 0$. To prove the L^2 estimate in (40), we multiply (24) by w and integrate it on S^2, then

$$2\lambda||w||_2^2 + 2\mu||w||_4^2 = \int_{S^2} G(w)wdx - \frac{1}{2}\Omega||\cos\theta||_2 \int_{S^2} \psi_{10} w dx. \tag{42}$$

From (20),

$$2\lambda||w||_2^2 + 2\mu||w||_4^2 \ge \frac{1}{\sqrt{\pi}}(\mu + \lambda\sqrt{4\pi})||w||_2^2. \tag{43}$$

On the other hand,

$$\int_{S^2} G(w)wdx + \frac{1}{2}\Omega||\cos\theta||_2 \int_{S^2} \psi_{10} w dx \le \frac{1}{2}\Omega||\cos\theta||_2||w||_2. \tag{44}$$

Hence the L^2 estimate in (40) can be obtained by combining (42), (43) and (44). The L^4 estimate can be obtained from the L^2 estimate and

$$2\mu||w||_4^2 \le \frac{1}{2}\Omega||\cos\theta||_2||w||_2. \tag{45}$$

The proof for (41) is similar. □

Next we prove the following monotonicity results regarding the angular momentum, enstrophy and higher order moment of the minimizers (maximizers), which will be crucial in analysis in the next section. A solution w of (24) is called pro-rotating if the angular momentum $\Gamma_a(w) = \int_{S^2} w\psi_{10}dx$ is positive, and it is counter-rotating if the angular momentum is negative.

Proposition 3. *Let $\Psi^M_{\lambda,\mu}$ and $\Psi^m_{\lambda,\mu}$ be as in Theorem 4.2.*

1. *Let $\Gamma_a(w) = \int_{S^2} w\psi_{10}dx$ be the angular momentum of the $w \in V_0$. Then for either $\mu \geq 0$ and $\lambda \geq 0$, or $\mu \leq 0$ and $\lambda \leq -1/4$,*
$$\frac{\partial \Gamma_a(\Psi)}{\partial p} > 0, \tag{46}$$
where $p = \lambda$ or μ, and $\Psi = \Psi^M_{\lambda,\mu}$ or $\Psi^m_{\lambda,\mu}$ depending on the value of (λ,μ). Moreover the minimizer $\Psi^m_{\lambda,\mu}$ is always counter-rotating, and the maximizer $\Psi^M_{\lambda,\mu}$ is always pro-rotating.
2. *If $\mu \geq 0$ and $\lambda \geq 0$, then*
$$\frac{\partial \Gamma_2(\Psi^m_{\lambda,\mu})}{\partial \lambda} < 0, \quad \text{and} \quad \frac{\partial \Gamma_4(\Psi^m_{\lambda,\mu})}{\partial \mu} < 0. \tag{47}$$
3. *If $\mu \leq 0$ and $\lambda \leq -1/4$, then*
$$\frac{\partial \Gamma_2(\Psi^M_{\lambda,\mu})}{\partial \lambda} > 0, \quad \text{and} \quad \frac{\partial \Gamma_4(\Psi^M_{\lambda,\mu})}{\partial \mu} > 0. \tag{48}$$

Proof. From Theorem 4.5, the extremals are differentiable with respect to the parameters. By differentiating the energy identity (27) with respect to λ and μ, we obtain
$$\begin{aligned} &-\int_{S^2} G(\Psi)\frac{\partial \Psi}{\partial \lambda}dx + 2\lambda\int_{S^2}\Psi\frac{\partial \Psi}{\partial \lambda}dx + \frac{2\mu}{||\Psi||_4^2}\int_{S^2}\Psi^3\frac{\partial \Psi}{\partial \lambda}dx \\ &+\frac{1}{4}\Omega||\cos\theta||_2\int_{S^2}\psi_{10}\frac{\partial \Psi}{\partial \lambda}dx + \int_{S^2}\Psi^2dx = 0, \end{aligned} \tag{49}$$
and
$$\begin{aligned} &-\int_{S^2} G(\Psi)\frac{\partial \Psi}{\partial \mu}dx + 2\lambda\int_{S^2}\Psi\frac{\partial \Psi}{\partial \mu}dx + \frac{2\mu}{||\Psi||_4^2}\int_{S^2}\Psi^3\frac{\partial \Psi}{\partial \mu}dx \\ &+\frac{1}{4}\Omega||\cos\theta||_2\int_{S^2}\psi_{10}\frac{\partial \Psi}{\partial \mu}dx + \left(\int_{S^2}\Psi^4dx\right)^{1/2} = 0. \end{aligned} \tag{50}$$
On the other hand, by multiplying (24) by $\partial\Psi/\partial p$ ($p = \lambda$ or μ) and integrating on S^2, we obtain
$$\begin{aligned} &-\int_{S^2} G(\Psi)\frac{\partial \Psi}{\partial p}dx + 2\lambda\int_{S^2}\Psi\frac{\partial \Psi}{\partial p}dx + \frac{2\mu}{||\Psi||_4^2}\int_{S^2}\Psi^3\frac{\partial \Psi}{\partial p}dx \\ &+\frac{1}{2}\Omega||\cos\theta||_2\int_{S^2}\psi_{10}\frac{\partial \Psi}{\partial p}dx = 0. \end{aligned} \tag{51}$$
Comparing (49) or (50) with (51), we obtain
$$\frac{1}{4}\Omega||\cos\theta||_2\int_{S^2}\psi_{10}\frac{\partial \Psi}{\partial \lambda}dx = \int_{S^2}\Psi^2dx, \tag{52}$$
and
$$\frac{1}{4}\Omega||\cos\theta||_2\int_{S^2}\psi_{10}\frac{\partial \Psi}{\partial \mu}dx = \left(\int_{S^2}\Psi^4dx\right)^{1/2}. \tag{53}$$
Notice that (52) and (53) implies (46). From (27), $\Gamma_a(\Psi^m_{\lambda,\mu}) < 0$ when $\lambda \geq 0$, $\mu \geq 0$, and similarly $\Gamma_a(\Psi^m_{\lambda,\mu}) > 0$ when $\lambda \leq -1/4$, $\mu \leq 0$.

To prove the monotonicity of enstrophy and higher order momentum, we differentiate the Euler-Lagrange equation (24) with respect to λ and μ, and we obtain the variational equations:

$$L(\Psi)\left[\frac{\partial\Psi}{\partial\lambda}\right] = 2\Psi,\ \ L(\Psi)\left[\frac{\partial\Psi}{\partial\mu}\right] = \frac{2}{||\Psi||_4^2}\Psi^3, \tag{54}$$

where $L(W)$ is defined in (32), $\Psi = \Psi^m_{\lambda,\mu}$ or $\Psi^M_{\lambda,\mu}$, and $\partial\Psi/\partial p$ is the partial derivatives with respect to $p = \lambda$ or μ. We first consider the minimizer case. From (54) and Proposition 1,

$$\frac{\partial\Gamma_2(\Psi^m_{\lambda,\mu})}{\partial\lambda} = 2\int_{S^2}\Psi^m_{\lambda,\mu}\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\lambda}dx = \int_{S^2}L(\Psi^m_{\lambda,\mu})\left[\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\lambda}\right]\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\lambda}dx < 0, \tag{55}$$

and

$$\begin{aligned}\frac{\partial(\Gamma_4(\Psi^m_{\lambda,\mu}))^{1/2}}{\partial\mu} &= \frac{2}{||\Psi^m_{\lambda,\mu}||_4^2}\int_{S^2}(\Psi^m_{\lambda,\mu})^3\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\mu}dx \\ = \int_{S^2}L(\Psi^m_{\lambda,\mu})\left[\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\mu}\right]&\frac{\partial\Psi^m_{\lambda,\mu}}{\partial\mu}dx < 0.\end{aligned} \tag{56}$$

The case of maximizers is similar by using (55) and (56) and Proposition 1. ◻

To conclude this section, we investigate the asymptotic profiles of the minimizers and maximizers when parameters λ and/or μ approach $\pm\infty$. From Proposition 3, it is important to understand the asymptotic behavior of these solutions when $\lambda = 0$ or $\mu = 0$. When $\mu = 0$, the constraint on the fourth order moment is ignored, and from [21], we have

$$\Psi^p_{\lambda,0} = -\frac{\Omega||\cos\theta||_2}{1+4\lambda}\psi_{10}, \tag{57}$$

where $p = m$ or M depending on λ, and $\lambda \geq 0$ or $\lambda < -1/4$. The solutions $\Psi^p_{0,\mu}$ cannot be explicitly solved because of the nonlinearity in the equation:

$$G(\Psi) - \frac{2\mu}{||\Psi||_4^2}\Psi^3 = \frac{1}{2}\Omega||\cos\theta||_2\psi_{10}. \tag{58}$$

We first assume $\mu > 0$. From Proposition 2, $||\Psi_{0,\mu}||_4 \to 0$ as $\mu \to \infty$. Let $N_\mu = ||\Psi_{0,\mu}||_4$ and $\phi_\mu = N_\mu^{-1}\Psi_{0,\mu}$. Then ϕ_μ satisfies

$$\mu^{-1}G(\phi_\mu) - 2\phi_\mu^3 = k\mu^{-1}N_\mu^{-1}\psi_{10}, \tag{59}$$

where $k = \Omega||\cos\theta||_2/2$. From (59),

$$\begin{aligned}k\mu^{-1}N_\mu^{-1} = k\mu^{-1}N_\mu^{-1}\int_{S^2}\psi_{10}^2dx &\leq 2\int_{S^2}|\phi_\mu|^3\cdot|\psi_{10}|dx + \mu^{-1}\int_{S^2}|G(\phi_\mu)\psi_{10}|dx \\ &\leq 2||\phi_\mu||_4\cdot||\psi_{10}||_4 + \frac{1}{2\mu}||\phi_\mu||_2\cdot||\psi_{10}||_2.\end{aligned} \tag{60}$$

Thus $\mu^{-1}N_\mu^{-1}$ is bounded as $\mu \to \infty$. From (59) and $G = \Delta^{-1}$, then ϕ_μ^3 is bounded in $H^2(S^2)$. Hence subsequences of $\{\phi_\mu^3\}$ and $\{\mu^{-1}N_\mu^{-1}\}$ converge simultaneously, and the limits $(\phi^3_\infty, h_\infty)$ satisfy

$$\Delta(\phi^3_\infty) = kh_\infty\psi_{10}. \tag{61}$$

However (61) has a solution if and only if $\phi_\infty^3 = a\psi_{10}$, the eigenvalue corresponds to ψ_{10}. On the other hand, $||\phi_\infty||_4 = 1$ thus $a^{4/3}\int_{S^2}\psi_{10}^{4/3}dx = 1$. Hence

$$a = -\frac{1}{||\psi||_{4/3}}, \quad h_\infty = \frac{4}{\Omega||\psi_{10}||_{4/3}||\cos\theta||_2}. \tag{62}$$

Therefore the asymptotic limit of $\Psi_{0,\mu}$ as $\mu \to \infty$ is

$$\Psi_{0,\mu} = -\mu^{-1}\frac{\Omega||\psi_{10}||_{4/3}^{2/3}||\cos\theta||_2}{4}\psi_{10}^{1/3}. \tag{63}$$

The case of $\mu < 0$ can be handled similarly, and indeed (63) also holds in that case. The formula (63) can be interpreted as following: if the enstrophy Γ_2 is not preserved but the fourth vorticity moment Γ_4 is preserved, then at least for small Γ_4, the profile of the extremal is of form $k\psi_{10}^{1/3}$. In fact the same argument also applies to any $2n$-th vorticity moment, and the profile will be of form $k\psi_{10}^{1/(2n-1)}$. When n is larger, $\psi_{10}^{1/(2n-1)}$ tends to 1 on the northern hemisphere, and tends to -1 on the southern hemisphere. This suggests that as higher even order vorticity moments are individually constrained, the extremals of the above variational problem tend to the step function vorticity distribution. One can argue that since sharp barotropic vorticity transitions have not been found in either numerical simulations of the GCM or observational data of planetary atmospheres, the physical relevance of the individual higher even order vorticity moments decreases with n. However, this is not to say that the first few even order vorticity moments are not important collectively. Indeed, the rigorous results in this paper are mathematical statements of their physical significance in a natural variational formulation of the BVE.

5. **Stability in constrained variational problem.** According to the results in the last section, we define parameter regions:

$$Q_1 = \{(\lambda,\mu) : \lambda \geq 0, \mu \geq 0\}, \quad Q_2 = \{(\lambda,\mu) : \lambda \leq -1/4, \mu \leq 0\}. \tag{64}$$

We have proved in Section 4, that for each give $(\lambda,\mu) \in Q_i$ $(i = 1,2)$, there is a unique critical point of the augmented energy functional $E(w,\lambda,\mu)$. We shall show that the enstrophy and fourth order moment of this critical point is uniquely determined.

Proposition 4. *Let $\Psi_{\lambda,\mu}^M$ and $\Psi_{\lambda,\mu}^m$ be as in Theorem 4.2. Define the mapping:*

$$G_2(\lambda,\mu) = \left[\int_{S^2}\left(\Psi_{\lambda,\mu}^p(x)\right)^2 dx\right]^2, \quad G_4(\lambda,\mu) = 4\pi\int_{S^2}\left(\Psi_{\lambda,\mu}^p(x)\right)^4 dx, \tag{65}$$

and $G(\lambda,\mu) = (G_2(\lambda,\mu), G_4(\lambda,\mu))$, where $p = m$ or M. Then $G : Q_i \to \mathbf{R}_+^2$ is a continuous one-to-one mapping, and $G(Q_i)$ is a closed subset of $\mathbf{R}_+^2 = \{(x,y) : x > 0, y > 0\}$, for $i = 1,2$.

Proof. It is equivalent to consider the mappings

$$\widetilde{G_2}(\lambda,\mu) = \int_{S^2}\left(\Psi_{\lambda,\mu}^p(x)\right)^2 dx, \quad \widetilde{G_4}(\lambda,\mu) = \left[\int_{S^2}\left(\Psi_{\lambda,\mu}^p(x)\right)^4 dx\right]^{1/2}, \tag{66}$$

and $\widetilde{G}(\lambda,\mu) = (\widetilde{G_2}(\lambda,\mu), \widetilde{G_4}(\lambda,\mu))$. From the calculations in the proof of Proposition 3, the Jacobian of the $\widetilde{G}$ is

$$D\widetilde{G} = \begin{pmatrix} \langle L(\Psi)\partial_\lambda\Psi, \partial_\lambda\Psi\rangle & \langle L(\Psi)\partial_\mu\Psi, \partial_\lambda\Psi\rangle \\ \langle L(\Psi)\partial_\lambda\Psi, \partial_\mu\Psi\rangle & \langle L(\Psi)\partial_\mu\Psi, \partial_\mu\Psi\rangle \end{pmatrix}, \tag{67}$$

where $\langle \cdot, \cdot \rangle$ is the inner-product of $L^2(S^2)$, $\Psi = \Psi^p_{\lambda,\mu}$, $p = m$ or M, and

$$\partial_\mu \Psi = \frac{\partial \Psi}{\partial \mu}, \quad \partial_\lambda \Psi = \frac{\partial \Psi}{\partial \lambda}. \tag{68}$$

We claim that the Jacobian matrix $D\widetilde{G}$ in (67) is invertible for any $(\lambda, \mu) \in Q_i$ $(i = 1, 2)$, *i.e.* the determinant of $D\widetilde{G}$ is nonzero. Indeed,

$$det(D\widetilde{G}) = \langle L(\Psi)\partial_\lambda \Psi, \partial_\lambda \Psi \rangle \cdot \langle L(\Psi)\partial_\mu \Psi, \partial_\mu \Psi \rangle - \langle L(\Psi)\partial_\mu \Psi, \partial_\lambda \Psi \rangle^2, \tag{69}$$

since $L(\Psi)$ is self-adjoint. We have shown in Proposition 1 that L is a negative definite operator when $(\lambda, \mu) \in Q_1$ and L is a positive definite operator when $(\lambda, \mu) \in Q_2$. From the standard results in functional analysis, if L is a positive definite self-adjoint operator, then

$$\langle Lu, u \rangle \cdot \langle Lv, v \rangle = \langle L^{1/2}u, L^{1/2}u \rangle \cdot \langle L^{1/2}v, L^{1/2}v \rangle \geq \left(\langle L^{1/2}u, L^{1/2}v \rangle \right)^2 = (\langle Lu, v \rangle)^2, \tag{70}$$

from the Cauchy-Schwarz's inequality. The equality holds only when $u = kv$ for some constant k.

Applying the above arguments to $L = L(\Psi)$ or $-L(\Psi)$, we find $det(D\widetilde{G}) \geq 0$, and the equality holds only when $\partial_\lambda \Psi = k\partial_\mu \Psi$. From (54), $\partial_\lambda \Psi = k\partial_\mu \Psi$ implies $||\Psi||_4^2 \Psi = \Psi^3$ for almost everywhere $x \in S^2$, which can only happen when Ψ is a constant function. But $\int_{S^2} \Psi(x)dx = 0$, thus the only possible constant solution is $\Psi(x) = 0$, which is not possible when $(\lambda, \mu) \in Q_i$. Therefore $det(D\widetilde{G}) > 0$ for any $(\lambda, \mu) \in Q_i$, and from standard result in multi-variable calculus, $\widetilde{G}$ is one-to-one mapping from Q_i to $\mathbf{R}^2_+$. The continuity is from the continuous dependence of solutions on the parameters. □

Combining with Proposition 3, we have the following corollary regarding the angular momentum of the minimizers/maximizers:

Corollary 1. *Let $\Psi^M_{\lambda,\mu}$ and $\Psi^m_{\lambda,\mu}$ be as in Theorem 4.2, and let Γ_2 and Γ_4 be defined as in (18) and (19) respectively.*

1. *$\Psi^p_{\lambda,\mu}$ can be parameterized by the enstrophy Γ_2 and the fourth order momentum Γ_4;*
2. *For the minimizer $\Psi^m_{\lambda,\mu}$, we have*

$$\frac{\partial \Gamma_a(\Psi^m_{\lambda,\mu})}{\partial \Gamma_2} > 0, \quad \frac{\partial \Gamma_a(\Psi^m_{\lambda,\mu})}{\partial \Gamma_4} > 0; \tag{71}$$

and for the maximizer $\Psi^M_{\lambda,\mu}$, we have

$$\frac{\partial \Gamma_a(\Psi^M_{\lambda,\mu})}{\partial \Gamma_2} < 0, \quad \frac{\partial \Gamma_a(\Psi^M_{\lambda,\mu})}{\partial \Gamma_4} < 0. \tag{72}$$

Proof. Part (1) is from Proposition 4, since $(\lambda, \mu) \to (\Gamma_2, \Gamma_4)$ is an invertible change of variables. Part 2 can be obtained since

$$\begin{pmatrix} \dfrac{\partial \Gamma_a(\Psi)}{\partial \Gamma_2} \\ \dfrac{\partial \Gamma_a(\Psi)}{\partial \Gamma_4} \end{pmatrix} = [D\widetilde{G}]^{-1} \begin{pmatrix} \dfrac{\partial \Gamma_a(\Psi)}{\lambda} \\ \dfrac{\partial \Gamma_a(\Psi)}{\mu} \end{pmatrix}, \tag{73}$$

then the result follows from the positive (negative) definiteness of the matrix $[D\widetilde{G}]^{-1}$ and (46). □

Now we determine the images of the mapping G, which will determine the range of the enstrophy and fourth order momentum of the minimizers/maximizers found in Theorem 4.2. From the natural restriction (6), we have

$$G_2 \leq G_4, \text{ or } \frac{G_2}{G_4} \leq 1. \tag{74}$$

On the portion $\mu = 0$ of the boundary of Q_i ($\lambda \geq 0$ or $\lambda < -1/4$), $\Psi = C\psi_{10}$, thus the ratio of G_2 and G_4 on that part is

$$\frac{G_2}{G_4} = \frac{\left(\int_{S^2} (\cos\theta)^2 dx\right)^2}{4\pi \int_{S^2} (\cos\theta)^4 dx} = \frac{5}{9} \approx 0.555556. \tag{75}$$

On the other hand, along the boundary $\lambda = 0$, the asymptotic profile of Ψ is given by (63), thus when $\mu \to \infty$,

$$\frac{G_2}{G_4} \to \frac{\left(\int_{S^2} (\cos\theta)^{2/3} dx\right)^2}{4\pi \int_{S^2} (\cos\theta)^{4/3} dx} = \frac{21}{25} = 0.84. \tag{76}$$

From Proposition 2, $G_2 \to 0$ and $G_4 \to 0$ as $\lambda \to \infty$ or $\mu \to \infty$, so $(G_2, G_4) = (0, 0)$ is a limit point of $G(Q_i)$ for both $i = 1$ and 2. Notice that $G(0, 0) = (a, 9a/5)$, where $a = \Omega^4 ||\cos\theta||_2^4$, is a vertex of the region $G(Q_1)$; and $G(-1/4, 0) \to (\infty, \infty)$ along the line $G_4 = (9/5)G_2$. Summarizing the discussion, we have

Proposition 5. *Let $\mathcal{Q}_i = G(Q_i)$ be the image of Q_i under the mapping G, ($i = 1, 2$).*

1. *$\mathcal{Q}_1$ is an unbounded closed region in the wedge*
$$\{(G_2, G_4) : G_2 > 0, G_4 > 0, 0 \leq G_2 \leq G_4\}, \tag{77}$$
$\mathcal{Q}_1 \supset \{G_4 = 9G_2/5, \Omega^4 ||\cos\theta||_2^4 \geq G_2 > 0\}$, and the slope of the tangent line of $\partial\mathcal{Q}_1$ at $(0, 0)$ is 0.84.
2. *$\mathcal{Q}_2$ is an unbounded closed region in the wedge*
$$\left\{(G_2, G_4) : G_2 > 0, G_4 > 0, \frac{5}{9}G_4 \leq G_2 \leq G_4\right\}, \tag{78}$$
$\mathcal{Q}_2 \supset \{G_4 = 9G_2/5, G_2 > 0\}$, and the slope of the tangent line of $\partial\mathcal{Q}_2$ at $(0, 0)$ is 0.84.

Note that even though one of the boundary curves of $\mathcal{Q}_1$ is $G_4 = 9G_2/5$, we do not know whether $G_2 \geq 5G_4/9$ for all possible minimizers. But we believe that $\mathcal{Q}_1$ is also in the wedge defined in (78).

Finally we return to our original extremal problem (22):

Theorem 5.1. *Let $\mathcal{Q}_i$ be as defined in Proposition 5.*

1. *Suppose that $(M_2^2, 4\pi M_4^2) \in \mathcal{Q}_1$. Then (22) has a global minimizer Ψ^m.*
2. *Suppose that $(M_2^2, 4\pi M_4^2) \in \mathcal{Q}_2$. Then (22) has a global maximizer Ψ^M.*

Proof. We assume that $(M_2^2, 4\pi M_4^2) \in \mathcal{Q}_1$. Then there exists a unique $(\lambda_*, \mu_*) = G^{-1}(M_2^2, 4\pi M_4^2)$ such that $\lambda_* > 0$, $\mu_* > 0$. From Theorem 4.2, $E(\lambda_*, \mu_*, w)$ has a unique critical point $\Psi^m_{\lambda_*, \mu_*}$, which is the global minimizer of $E(\lambda_*, \mu_*, w)$ in

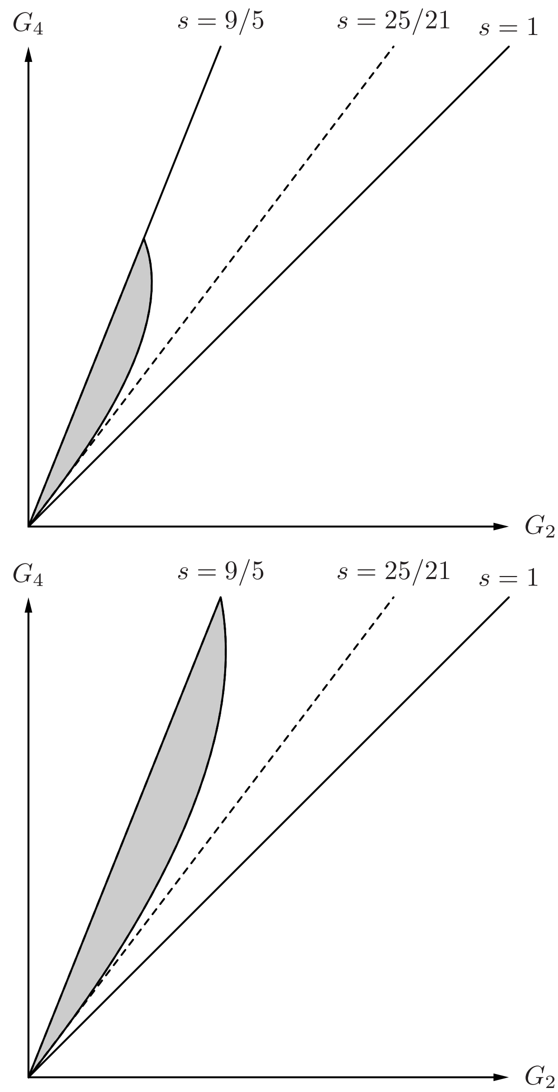

FIGURE 2. Admissible (enstrophy, fourth vorticity moment) regions for minimizers (top) and maximizers (bottom). Graphs are for illustration only, both shaded regions are unbounded, but the one for maximizer is larger than the one for minimizer.

V_0. Then $\Psi^m_{\lambda_*,\mu_*}$ satisfies $\int_{S^2}[\Psi^m_{\lambda_*,\mu_*}]^2 dx = M_1$ and $\int_{S^2}[\Psi^m_{\lambda_*,\mu_*}]^4 dx = M_2^2$. From Lemma 3.2, $\Psi^m_{\lambda_*,\mu_*}$ is also an extremal of (22).

We claim that $\Psi^m_{\lambda_*,\mu_*}$ is the global minimizer of $H(w)$ (defined in (17)). Suppose not, then there exists $\Phi \in V$ such that $H(\Phi) < H(\Psi^m_{\lambda_*,\mu_*})$. Since $\Gamma_i(\Phi) = \Gamma_i(\Psi^m_{\lambda_*,\mu_*})$ for $i = 2, 4$, then $E(\lambda_*, \mu_*, \Phi) < E(\lambda_*, \mu_*, \Psi^m_{\lambda_*,\mu_*})$, which is a contradiction. The proof of the other part is similar. $\square$

6. **Conclusions.**

In [21], the first author found that for each fixed enstrophy level $M_2 > 0$, there are two extremals for the energy function $H(w)$. When M_2 is small, the two extremals are, a global energy minimizer which is counter-rotating, and a global energy maximizer which is pro-rotating. But when M_2 is large, while the pro-rotating state is still the global energy maximizer, the counter-rotating state becomes an energy saddle point.

In the following, we use $G_2 = M_2^2$ and $G_4 = 4\pi M_4^2$ for discussion. From the analytical and qualitative results in Sections 4 and 5, if we fix enstrophy level G_2 to be small, and decrease the fourth order moment quantity G_4, then

1. When G_4 is large (greater than $(9/5)G_2$), there is no energy maximizers and we conjecture that there is no energy minimizer as well;
2. When $G_4 = (9/5)G_2$, we recover the minimizer and maximizer found in [21];
3. When G_4 is in a range which is less than $(9/5)G_2$, we still have both the minimizer and maximizer. But notice that when there is no restriction on G_4, the minimizer/maximizer we find is on $G_4 = (9/5)G_2$. Thus when G_4 moves away from the line $G_4 = (9/5)G_2$, the energy of the the minimizer increases, and the energy of the maximizer decreases;
4. When G_4 is further smaller (but still larger than the cutoff level $G_4 = G_2$ to make V_0 nonempty), again it appears that there is no energy extremals as suggested by the tangent line of $\partial\mathcal{Q}_i$ at $(0, 0)$ is 0.84.

Thus our new results confirm the previous ones in [21], and also provide new information to the original problem with infinite Casimir conserved quantities. It is natural that when more constraints are added, then the energy of the the minimizer increases, and the energy of the maximizer decreases, which happens when $(G_2, G_4) \in \mathcal{Q}_i$. When $G_4 < (9/5)G_2$, the nonexistence of the extremal suggests that minimizing/maximizing sequence exists, and the energy function will approach that of the absolute minimizer/maximizer achieved at $G_4 = (9/5)G_2$, but these minimizing/maximizing sequences are non-convergent on that enstrophy-fourth moment surface. We predict that when more higher order moment constraints are added to the setting of the variational problem, the wedge $\{(G_2, G_4) : G_2 > 0, G_4 > 0, \frac{5}{9}G_4 \leq G_2 \leq G_4\}$ will be eventually filled by the values of maximizers, as suggested by the calculations of G_2/G_4 in Section 5. Indeed, when more higher order moment constraints are added, the maximizer/minimizer with asymptotic form $\psi^{1/p}$ for some large odd number p is possible for certain choices of moment values. Then similar to (76),

$$\frac{G_2}{G_4} \to \frac{\left(\int_{S^2} (\cos\theta)^{2/p} dx\right)^2}{4\pi \int_{S^2} (\cos\theta)^{4/p} dx} = \frac{p^2 + 4p}{p^2 + 4p + 4} \to 1, \quad p \to \infty. \tag{79}$$

Such extremals will occupy the (G_2, G_4) values close to the line $G_2 = G_4$. Therefore, this implies, when all Casimir constraints are imposed, for any (G_2, G_4) in $\{(G_2, G_4) : G_2 > 0, G_4 > 0, (5/9)G_4 \leq G_2 \leq G_4\}$, an energy maximizer exists, and a minimizer only exists for small G_2 and G_4.

Finally we give an explanation of the existence of energy maximizer but not energy minimizer on any fixed enstrophy surface $S = \{w \in L^2(S^2) : ||w||_2^2 = M_2\}$ for large M_2, from the view of functional analysis (which has been proved in [21] by using different proof.) Let $\{w^n\}$ be an energy minimizing sequence on S. Then $\{w^n\} \subset B = \{w \in L^2(S^2) : ||w||_2^2 \leq M_2\}$. Since $L^2(S^2)$ is reflexive, from [15] Chapter 10 Theorem 7 (page 104), w^n has a subsequence weakly convergent to w^∞, and $w^\infty \in B$ from Mazur Lemma ([15] Chapter 10 Theorem 6, page 103). Since G is a compact operator, the weak convergence implies $H(w^n) \to H(w^\infty)$ as $n \to \infty$.

Thus the maximum is achieved at $w^\infty \in B$, and from previous argument we always have $\int_{S^2} \psi_{10} w^\infty dx > 0$. Thus the maximum of H on B must be achieved on S since $H(kw^\infty)$ is increasing in $k > 0$. This shows that the maximizer always exists on S for any $M_2 > 0$, but it is clear that this argument does not work for minimizers. Notice that same proof applies to fixed p-th moment since $L^p(S^2)$ a is reflexive Banach space for $\infty > p > 1$.

Acknowledgements. The authors thank anonymous reviewers for helpful comments and suggestions.

REFERENCES

[1] V. I. Arnold, *Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid*, Sov. Math Dokl., **6** (1965), 773–777.

[2] V. I. Arnold, *On an a priori estimate in the theory of hydrodynamical stability*, AMS Translations, Ser.2, **79** (1969), 267–269.

[3] T. H. Berlin and M. Kac, *The spherical model of a ferromagnet*, Phys. Rev., **86** (1952), 821–835.

[4] E. Caglioti, P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for 2D Euler equations: a statistical mechanics description*, Comm. Math. Phys., **143** (1992), 501–525.

[5] S. J. Chern, "Math Theory Of The Barotropic Model in GFD," PhD thesis, Cornell University, 1991.

[6] J. Cho and L. Polvani, *The emergence of jets and vortices in freely evolving, shallow-water turbulence on a sphere*, Phys Fluids, **8** (1995), 1531–1552.

[7] J. Cho and L. Polvani, *The robustness of self-organized zonal jets in unforced, turbulent vorticity fields*, in "Vortex flows and related numerical methods, II (Montreal, PQ, 1995)," 447–462, ESAIM Proc., 1, Soc. Math. Appl. Indust., Paris, 1996.

[8] A. Chorin, "Vorticity and Turbulence," Applied Mathematical Sciences, 103, Springer-Verlag, New York, 1994.

[9] A. Del Genio, W. Zhuo and T. Eichler, *Equatorial superrotation in a slowly rotating GCM: implications for Titan and Venus*, Icarus, **101** (1993), 1–17.

[10] X. Ding and C. C. Lim, *First order phase transition and high energy coherent spots on a rapidly-rotating sphere*, preprint, 2008.

[11] X. Ding and C. C. Lim, *Phase transitions to super-rotation in a coupled Barotropic fluid - rotating sphere system*, Phys. A, **374** (2007), 152–164.

[12] J. S. Frederiksen and B. L. Sawford, *Statistical dynamics of 2D inviscid flows on a sphere*, J. Atmos Sci, **31** (1980), 717–732.

[13] T. Kato, "Perturbation Theory for Linear Operators," Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[14] R. H. Kraichnan, *Statistical dynamics of two-dimensional flows*, J. Fluid Mech., **67** (1975), 155–175.

[15] P. D. Lax, "Functional analysis," Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2002.

[16] T. D. Lee, *On some statistical properties of hydrodynamical and magnetohydrodynamical fields*, Quart. Appl. Math., **10** (1952), 69–74.

[17] C. Leith, *Diffusion approximation for 2D turbulence*, Phys. Fluids, **11** (1968), 671–673.

[18] C. Leith, *Minimum enstrophy vortices*, Phys. Fluids, **27** (1984), 1388–1395.

[19] C. C. Lim, *A long range spherical model and exact solutions of a energy-enstrophy theory for 2-D turbulence*, Phys. Fluids, **13** (2001), 1961–1973.

[20] C. C. Lim, *Energy maximizers and robust symmetry breaking in vortex dynamics on a non-rotating sphere*, SIAM J. Applied Math, **65** (2005), 2093–2106.

[21] C. C. Lim, *Energy extremals and nonlinear stability in an Energy-relative enstrophy theory of the coupled barotropic fluid—rotating sphere system*, J. Math Phys., **48** (2007), 1–21.

[22] C. C. Lim, *Free energy extremals in a simple mean field theory for the coupled barotropic fluid-rotating sphere system*, Disc. Cont. Dyna. Sys. A, **19** (2007), 361–386.

[23] C. C. Lim, *Energy-Enstrophy theory for coupled fluid–rotating sphere system–exact solutions for super-rotations*, in "Proceedings of the International Workshop Collective Phenomena in Macroscopic Systems,"(G. Bertin, R. Pozzoli, M. Rome' & K.R. Sreenivasan Eds), World Scientific, Singapore, (2007), 96–106.

[24] C. C. Lim and R. Singh Mavi, *Phase transitions for barotropic flows on a sphere by the Bragg method*, Physica A, **380** (2007), 43–60.

[25] A. J. Majda and M. Holen, *Dissipation, topography, and statistical theories for large-scale coherent structure*, Comm. Pure Appl. Math., **50** (1997), 1183–1234.

[26] J. Miller, *Statistical mechanics of Euler equations in two dimensions*, Phys. Rev. Lett., **65** (1990), 2137–2140.

[27] M. Mu and T. G. Shepherd, *On Arnol'd's second nonlinear stability theorem for two-dimensional quasi-geostrophic flow*, Geophys. Astrophys. Fluid Dyn., **75** (1994), 21–37.

[28] P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations,"CBMS Regional Conference Series in Mathematics, **65**, American Mathematical Society, Providence, 1986.

[29] R. Robert and J. Sommeria, *Statistical equilibrium states for two-dimensional flows*, J. Fluid Mech., **229** (1991), 291–310.

[30] R. Salmon, G. Holloway and M. C. Hendershott, *The equilibrium statistical mechanics of simple quasigeostrophic models*, J. Fluid Mech., **75** (1976), 691–703.

[31] T. G. Shepherd, *Non-ergodicity of inviscid two-dimensional flow on a beta-plane and on the surface of a rotating sphere*, J. Fluid Mech., **184** (1987), 289–302.

[32] T. G. Shepherd, *Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics*, Adv. Geophys., **32** (1990), 287–338.

[33] T. G. Shepherd, *Applications of Hamiltonian theory to GFD*, in "1993 GFD Summer School Lecture Notes,"Woods Hole Oceanographic Institution Technical Report WHOI-94-12, (1994), 113–152.

[34] Yu N. Skiba, *Spectral approximation in the numerical stability study of non-divergent viscous flows on a sphere*, Num. Meth. for PDE., **14** (1998), 143–157.

[35] Yu N. Skiba, *On the spectral problem in a linear stability study of flows on a sphere*, J. Math. Anal. Appl., **270** (2002), 165–180.

[36] R. J. Stanton and A. Weinstein, *On the L^4 norm of spherical harmonics*, Math. Proc. Cambridge Philos. Soc., **89** (1981), 343–358.

[37] M. Struwe, "Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems," Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., **34**. Springer-Verlag, Berlin, 2000.

[38] S. Yoden and M. Yamada, 2-*dimensional turbulence on rotating spherical surfaces*, (Japanese) in "Problems of Topology in Fluid Dynamics" (Kyoto, 1992). Sūrikaisekikenkyūsho Kōkyūroku No. **817** (1992), 140–148.

[39] S. Yoden and M. Yamada, *A numerical experiment on 2D decaying turbulence on a rotating sphere*, J. Atmos. Sci., **50** (1993), 631–644.

[40] W. R. Young, *Selective decay of enstrophy and the excitation of barotropic waves in a channel*, J. Atmos. Sci., **44** (1987), 2804–2812.

Received March 2008; revised December 2008.

E-mail address: limc@rpi.edu
E-mail address: shij@math.wm.edu