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Structure of the solution set for a class of semilinear elliptic equations with asymptotic linear nonlinearity[☆]

Jia Duo[a,b], Junping Shi[c,a], Yuwen Wang[a,*]

[a] *Y.Y.Tseng Functional Analysis Research Center and School of Mathematics and Computer Sciences, Harbin Normal University, Harbin, Heilongjiang, 150025, PR China*
[b] *Department of Mathematics and Mechanics, Heilongjiang Institute of Science and Technology, Harbin, Heilongjiang, 150027, PR China*
[c] *Department of Mathematics, College of William and Mary, Williamsburg, VA, 23187-8795, USA*

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Abstract

We consider a semilinear elliptic equation with asymptotic linear nonlinearity applying bifurcation theory and spectral analysis. We obtain the exact multiplicity of the positive solutions and a very precise structure of the solution set, which improves the previous knowledge of the problem.

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1. Introduction

In this paper, we study the exact multiplicity of the solutions of a semilinear elliptic equation with asymptotic linear nonlinearity:

$$\begin{cases} \Delta u + \lambda\sqrt{(u-b)^2+\varepsilon} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded $C^{2,\alpha}$ domain in $\mathbf{R}^n$ with $n \geq 1$, and λ, ε and b are positive real numbers. For convenience, we define $f(u,\varepsilon) = \sqrt{(u-b)^2+\varepsilon}$. In [5] the existence of minimal solutions and non-minimal solutions has been studied, as well as the asymptotic behavior of the solutions, and the main results of [5] are:

Theorem 1.1 (*[5]*). *Suppose $\varepsilon > 0$; then there exists exactly one solution of (1) for all $\lambda \in (0, \lambda_1)$. Here λ_1 is the principal eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition.*

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* Corresponding author.
E-mail address: wangyuwen1950@yahoo.com.cn (Y. Wang).

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Theorem 1.2 (*[5]*). *For any $\varepsilon > 0$, there exists $\lambda^*(\varepsilon) > \lambda_1$ such that if $\lambda \in (\lambda_1, \lambda^*(\varepsilon))$, then (1) has at least two solutions.*

Theorem 1.3 (*[5]*). *The solution set S_ε of (1) contains an unbounded component C_ε. If $u_\lambda \in C_\varepsilon$ and $\|u_\lambda\|_\infty \longrightarrow \infty$, then $\lambda \longrightarrow \lambda_1$.*

The results above in [5] have not completely determined the exact multiplicity of positive solutions of (1). In this article we give a more satisfactory answer to the problem by studying the degenerate solution and bifurcation problem, and consequently we obtain the exact multiplicity of the solutions depending on λ and the precise structure of the solution set of (1), at least in some parameter ranges. We use an idea of Amann [1], as well as some techniques from [8,9]. Our results will be described in the following sections. First we recall some abstract settings of the problems as well as some basic tools in bifurcation theory which will be used in the paper.

Let $F : \mathbf{R}^+ \times \mathbf{R}^+ \times X \to Y$ be defined by

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda\sqrt{(u-b)^2 + \varepsilon}$$

where $u \in X = \{u \in C^{2,\alpha}(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ and $Y = C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$. It is easy to show that $F_u(\varepsilon, \lambda, u) = \Delta + \lambda f_u(u, \varepsilon)I$. In the following, we use $R(L)$ to denote the range of L, and $N(L)$ to denote the null space of L.

Definition 1.4. If $F(\varepsilon^*, \lambda^*, u^*) = 0$ and $N(F_u(\varepsilon^*, \lambda^*, u^*)) \neq \{0\}$, then we call $(\varepsilon^*, \lambda^*, u^*)$ a degenerate solution of (1).

Theorem 1.5 (*[2] Implicit Function Theorem*). *Let $(\lambda^*, u^*) \in \mathbf{R} \times X$, and let F be a continuously differentiable mapping of an open neighborhood V of (λ^*, u^*) into Y. Suppose $F(\lambda^*, u^*) = 0$ and $F_u(\lambda^*, u^*)$ is a linear homeomorphism. Then the solutions of $F(\lambda, u) = 0$ near (λ^*, u^*) form a curve $(\lambda, u(\lambda))$, where $u(\lambda) = u^* + (\lambda - \lambda^*)\omega^* + z(\lambda)$, $\omega^* = -[F_u(\lambda^*, u^*)]^{-1}(F_\lambda(\lambda^*, u^*))$ and $\lambda \to z(\lambda) \in X$ is a continuously differentiable function near $\lambda = \lambda^*$ with $z(\lambda^*) = z'(\lambda^*) = 0$.*

Theorem 1.6 (*[2] Crandall–Rabinowitz Theorem*). *Let $(\lambda^*, u^*) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ^*, u^*) into Y. Suppose that the null space $N(F_u(\lambda^*, u^*)) = \operatorname{span}\{\omega^*\}$ is one-dimensional and* codim $R(F_u(\lambda^*, u^*)) = 1$, *where $R(F_u)$ is the range space, and $F_\lambda(\lambda^*, u^*) \notin R(F_u(\lambda^*, u^*))$. Let Z be a complement of* span$\{\omega^*\}$ *in X. Then the solutions of $F(\lambda, u) = F(\lambda^*, u^*)$ near (λ^*, u^*) form a curve $(\lambda(s), u(s)) = (\lambda^* + \lambda(s), u^* + s\omega^* + z(s))$, where $s \to (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\lambda(0) = \lambda'(0) = 0$, $z(0) = z'(0) = 0$. Moreover, if F is k-times continuously differentiable, so are $\lambda(s)$ and $z(s)$.*

Next we discuss the properties of degenerate solutions of (1). In the following we always assume that λ_i is the i-th eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition.

2. The properties of degenerate solutions

Assuming that a degenerate solution of (1) exists, we have the following property:

Proposition 2.1. *Fix $\varepsilon = \varepsilon^* > 0$; if (λ^*, u^*) is a degenerate solution of (1), then $\|u^*\|_\infty > b$.*

Proof. The linearized form of (1) is

$$\begin{cases} \Delta\omega + \lambda^* f_u(u^*, \varepsilon^*)\omega = 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and

$$f_u(u^*, \varepsilon) = \frac{u^* - b}{\sqrt{(u^* - b)^2 + \varepsilon^*}}.$$

Suppose that $\|u^*\|_\infty \leq b$; then $f_u(u^*(x), \varepsilon^*) \leq 0$ for any $x \in \Omega$, and by the weak maximum principle, (2) has only the solution $\omega = 0$, i.e. $N(F_u(\varepsilon^*, \lambda^*, u^*)) = \{0\}$. Thus if $(\varepsilon^*, \lambda^*, u^*)$ is a degenerate solution of (1), i.e. $N(F_u(\varepsilon^*, \lambda^*, u^*)) \neq \{0\}$, then $\|u^*\|_\infty > b$ is inevitable. $\square$

In the following if the parameter ε is fixed, we may omit it in the expression. For convenience, we use $F_u(\lambda^*, u^*)$ to denote the linearized operator $F_u(\varepsilon, \lambda^*, u^*)$. From Theorem 1.1, if (λ^*, u^*) is a bifurcation point of (1), then $\lambda^* > \lambda_1$. Here we prove this fact by using an estimate of the eigenvalue of a linearized equation.

Proposition 2.2. *Fix $\varepsilon > 0$; if (λ^*, u^*) is a degenerate solution of (1), then $\lambda^* > \lambda_1$.*

Proof. Eq. (2) has a nontrivial solution if and only if $\mu_k(\lambda^* f_u(u^*, \varepsilon)) = 1$ for some $k \in \mathbf{N}$, where $\mu_k(g)$ is the k-th eigenvalue of

$$\begin{cases} -\Delta\omega = \mu_k(g)g(x)\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $g \in C^\alpha(\overline{\Omega})$. By the Courant–Fischer minimax principle, $\forall k \in \mathbf{N}$, $\mu_k(g)$ is a strictly decreasing function of g (see [4]). Since $1 > f_u(u^*(x), \varepsilon)$, $\forall x \in \Omega$, then

$$1 = \mu_k(\lambda^* f_u(u^*, \varepsilon)) > \mu_k(\lambda^*) = \frac{\mu_k(1)}{\lambda^*} = \frac{\lambda_k}{\lambda^*} \geq \frac{\lambda_1}{\lambda^*},$$

and therefore $\lambda^* > \lambda_1$. □

Next we use the implicit function theorem to show that when $\lambda \in (0, \lambda_1)$, the solution set of (1) can be parameterized by λ, and the parameterized solution curve is strictly increasing. To prove that, first we recall two results from [3,5].

Lemma 2.3 ([3]). *Let $L_1 u = -\sum_{i,j=1}^n b_{ij}(x)u_{x_i x_j} + \sum_{i=1}^n c_i(x)u_{x_i} + d(x)u$, having real coefficients in $C^\alpha(\overline{\Omega})$, where $(b_{ij}(x))$ is symmetric and positive definite in $\overline{\Omega}$, let μ_1 be the least $\mu \in \mathbf{R}$ such that $L_1\omega = \mu\omega$ has a nontrivial solution $\omega \in C_0^{2,\alpha}(\overline{\Omega})$, and let $v \in C_0^{2,\alpha}(\overline{\Omega})$ be such that $L_1 v \geq 0$ in Ω. Then if $\mu_1 > 0$, we have $v \geq 0, \forall x \in \Omega$; while if $\mu_1 = 0$, then $L_1 v = 0, \forall x \in \overline{\Omega}$.*

Lemma 2.4 ([5]). *Fix $\varepsilon > 0$; if $\{u_\lambda\}$ is a family of solutions of (1) such that $\|u_\lambda\|_\infty$ tends to infinity, then λ converges to λ_1.*

Theorem 2.5. *Fix $\varepsilon > 0$; when $\lambda \in (0, \lambda_1)$ the solution set of (1) is a sufficiently smooth curve $\{(\lambda, u(\lambda)) : 0 < \lambda < \lambda_1\}$, $\lim_{\lambda \to 0^+} u(\lambda) = 0$ (limit taken in X), and $u(\lambda)(x)$ is strictly increasing with respect to λ for any $x \in \Omega$.*

Proof. It is clear that (0, 0) is the solution of (1) and when $(\lambda, u) = (0, 0)$, the linearized equation (2) becomes

$$\begin{cases} \Delta\omega = 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and thus $F_u(0, 0) = \Delta$. It is well known that $\Delta : X \longrightarrow Y$ is a linear homeomorphism; then by the implicit function theorem, $\exists\delta > 0$ such that if $\lambda \in (0, \delta)$, then the solutions of (1) near (0, 0) form a unique solution curve $(\lambda, u(\lambda)) \in (0, \delta) \times X$, where $u(\lambda) = \lambda\omega_0 + z(\lambda)$, $\omega_0 = -[F_u(0, 0)]^{-1}(F_\lambda(0, 0))$, and $\lambda \to z(\lambda) \in X$ is a continuously differentiable function near $\lambda = 0$, with $z(0) = z'(0) = 0$. Since ω_0 satisfies

$$\begin{cases} -\Delta\omega_0 = F_\lambda(0, 0) = \sqrt{b^2 + \varepsilon} > 0 & \text{in } \Omega, \\ \omega_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

by the maximum principle, $\omega_0 > 0, \forall x \in \Omega$, and thus $u(\lambda)$ is positive in Ω in a right neighborhood of $\lambda = 0$. From Proposition 2.2, any $(\lambda, u(\lambda))$ is non-degenerate for $\lambda \in (0, \lambda_1)$; hence the curve can be further extended. From Lemma 2.4, $(\lambda, u(\lambda))$ is bounded for $0 < \lambda \leq \lambda_1 - \delta$ for any $\delta > 0$. We claim that $\|u(\lambda)\|_\infty$ is also bounded as $\lambda \to \lambda_1^-$. If not, then $\|u(\lambda)\|_\infty \to \infty$ as $\lambda \to \lambda_1^-$; then a subcritical bifurcation from infinity occurs at $\lambda = \lambda_1$. However f satisfies that $f'(\infty) = \lim_{u\to\infty} f(u, \varepsilon)/u > f(u, \varepsilon)/u$ for all $u > 0$; then from Proposition 3.4 of [6] the bifurcation form infinity here should be supercritical (i.e. the curve is on the right hand side of λ_1 near infinity). That is a contradiction. Hence the curve $(\lambda, u(\lambda))$ can be defined for $\lambda \in (0, \lambda_1]$, and the curve is sufficiently smooth by the implicit function theorem since $f(u, \varepsilon) \in C^\infty$.

Finally we prove that $u(\lambda)$ is strictly increasing with respect to λ. Since $u(\lambda)$ is a differentiable function in λ, $\partial u(\lambda)/\partial\lambda$ satisfies

$$\begin{cases} \Delta \dfrac{\partial u}{\partial \lambda} + \lambda f_u(u,\varepsilon)\dfrac{\partial u}{\partial \lambda} + f(u,\varepsilon) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \lambda} = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e. $F_u(\lambda, u(\lambda))[\partial u(\lambda)/\partial\lambda] = -f(u,\varepsilon) < 0$. The principal eigenvalue $\mu_1(Q_1)$ of the linear operator $Q_1 = -F_u(\lambda, u(\lambda)) = -\Delta - \lambda f_u(u(\lambda),\varepsilon)$ satisfies $\mu_1(Q_1) \geq \mu_1(Q_2) = 0$ where $Q_2 = -\Delta - \lambda_1$ since $f_u < 1$ and $\lambda \leq \lambda_1$. Hence $\mu_1(Q_1) > 0$, and $\partial u(\lambda)/\partial\lambda > 0$ by Lemma 2.3. Therefore $u(\lambda)$ is strictly increasing. □

From Proposition 2.2, when $\lambda = \lambda_1$, the minimal solution $u(\lambda)$ is still non-degenerate; thus we may extend the solution curve in Theorem 2.5 to $(\lambda_1, \lambda_1 + \delta)$, where δ is a positive constant. Next we analyze the direction of the solution curve at a degenerate solution.

Theorem 2.6. *Fix $\varepsilon > 0$, and assume (λ^*, u^*) is a degenerate solution of (1), and the solution space of (2) is one-dimensional and is generated by $\omega^* > 0$; then:*

(1) *The solutions of (1) near (λ^*, u^*) form a curve $(\lambda^* + \lambda(s), u^* + s\omega^* + z(s))$, where $\lambda(0) = \lambda'(0) = 0$, $z(0) = z'(0) = 0$ and $s \longmapsto (\lambda(s), z(s)) \in \mathbf{R}^+ \times Z$ is a continuously differentiable function near $s = 0$, where $X = \operatorname{span}\{\omega^*\} \bigoplus Z$.*
(2) *The solution curve near (λ^*, u^*) is C^∞, and $\lambda''(0) < 0$, i.e. "the solution curve turns to the left".*

Proof. From the assumptions, the solution space of (2) is one-dimensional, and $N(F_u(\lambda^*, u^*)) = \operatorname{span}\{\omega^*\}$. Since $F_u(\lambda^*, u^*) = \Delta + \lambda^* f_u(u^*,\varepsilon)I$ is a Fredholm operator of index 0, we have $\dim N(F_u(\lambda^*, u^*)) = \operatorname{codim} R(F_u(\lambda^*, u^*)) = 1$. And $g \in R(F_u(\lambda^*, u^*))$ if and only if $\int_\Omega g(x)w^*(x)\mathrm{d}x = 0$. This also implies $F_\lambda(\lambda^*, u^*) = f(u^*,\varepsilon) \notin R(F_u(\lambda^*, u^*))$ since $f > 0$ and $w^* > 0$. Now applying Theorem 1.6, we concluded that near (λ^*, u^*) the solutions of (1) form a curve $(\lambda^* + \lambda(s), u^* + s\omega^* + z(s))$ with s near $s = 0$, and $\lambda(0) = \lambda'(0) = 0$, $z(0) = z'(0) = 0$. Since f is C^∞, then by the implicit function theorem, the solution curve near (λ^*, u^*) is also C^∞.

For the sign of $\lambda''(0)$, we recall from [6] that

$$\lambda''(0) = \frac{-\lambda^* \int_\Omega f_{uu}(u^*,\varepsilon)\omega^{*3}\mathrm{d}x}{\int_\Omega f(u^*,\varepsilon)\omega^*\mathrm{d}x}.$$

Now from

$$f_{uu}(u^*,\varepsilon) = \frac{\varepsilon}{\sqrt{((u^*-b)^2+\varepsilon)^3}} > 0,$$

$f(u^*,\varepsilon) > 0$, and ω^* and ω^{*3} having the same sign in Ω, we obtain $\lambda''(0) < 0$. □

3. Exact multiplicity of positive solutions

Since the curve of minimal solutions can be extended beyond $\lambda = \lambda_1$, we can define $\overline{\lambda} = \sup\{\lambda^* : (\lambda, u(\lambda))$ exists and is non-degenerate for $\lambda \in (0, \lambda^*)\}$, where $u(\lambda)$ is the minimal solution of (1).

Lemma 3.1. *Fix $\varepsilon > 0$, and assume $\overline{\lambda}$ is as defined above; then $\overline{u} = \lim_{\lambda\to\overline{\lambda}_-} u(\lambda)$ exists in the topology $C^{2,\alpha}(\overline{\Omega})$, and $(\overline{\lambda}, \overline{u})$ is a degenerate solution of (1).*

Proof. We first show that $\overline{\lambda} < \infty$. Indeed $f(u,\varepsilon)/u \geq k > 0$ for some $k > 0$; thus

$$\lambda_1 \int_\Omega u(x)\phi_1(x)\mathrm{d}x = \lambda \int_\Omega f(u,\varepsilon)\phi_1(x)\mathrm{d}x \geq \lambda k \int_\Omega u(x)\phi_1(x)\mathrm{d}x, \tag{4}$$

where ϕ_1 is the positive eigenfunction corresponding to the principal eigenvalue λ_1. Hence, $\lambda_1 < \overline{\lambda} \leq \lambda_1/k$. Thus there exists a sequence $\lambda^{(n)} \to \overline{\lambda}$, and corresponding $u(\lambda^{(n)})$ satisfying the following equation:

$$\begin{cases} \Delta u(\lambda^{(n)}) + \lambda^{(n)} f(u(\lambda^{(n)}),\varepsilon) = 0 & \text{in } \Omega, \\ u(\lambda^{(n)}) = 0 & \text{on } \partial\Omega. \end{cases}$$

From Lemma 2.4, there exists $M > 0$ such that $\|u(\lambda^{(n)})\|_\infty \le M$. By the standard elliptic estimates, $u(\lambda^{(n)})$ is bounded in $C^{2,\alpha}(\overline{\Omega})$. By a rather routine process, we can show that a subsequence of $u(\lambda^{(n)})$ converges to a classical solution of (1) when $\lambda = \overline{\lambda}$. Since $u(\lambda)$ is strictly increasing, this solution is clearly not trivial. By the maximality of $\overline{\lambda}$ and the implicit function theorem, $(\overline{\lambda}, \overline{u})$ is a degenerate solution of (1). □

Next we show that the positive solution of (1) not only exists at $\lambda = \overline{\lambda}$ but also is unique. We recall a lemma from [3].

Lemma 3.2 *([3]). Let $h \in C^2(\overline{\Omega} \times \mathbf{R})$, and let $u, v \in C_0^{2,\alpha}(\overline{\Omega})$ satisfy*

$$-\Delta u = h(x, u), \qquad -\Delta v \ge h(x, v).$$

Suppose also that $h_{zz}(x, u(x) + \tau(v(x) - u(x))) > 0$ for $\tau \in [0, 1]$ and $x \in \Omega$; let k_1 be the principal eigenvalue of $-\Delta - h_u(x, u)$. Then if $k_1 > 0$ we have $v \ge u, \forall x \in \Omega$, while if $k_1 = 0$, $v = u, \forall x \in \overline{\Omega}$.

Lemma 3.3. *Fix $\varepsilon > 0$; Eq. (1) has a unique solution $\overline{u} = \lim_{\lambda \to \overline{\lambda}-} u(\lambda)$ at $\lambda = \overline{\lambda}$.*

Proof. Suppose there exists another solution $\overline{v} \ne \overline{u}$ at $\lambda = \overline{\lambda}$; then $\overline{v}$ satisfies

$$\begin{cases} -\Delta \overline{v} = \overline{\lambda} f(\overline{v}, \varepsilon) & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $(\overline{\lambda}, \overline{u})$ is a degenerate solution of (1), then the principal eigenvalue $\mu_1(\overline{u})$ of the eigenvalue problem

$$\begin{cases} -\Delta \omega - \overline{\lambda} f_u(\overline{u}, \varepsilon)\omega = \mu\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

is zero. Since $\forall \tau \in [0, 1], \forall x \in \Omega$,

$$f_{uu}(\overline{u} + \tau(\overline{v} - \overline{u}), \varepsilon) = \frac{\varepsilon}{\sqrt{((\overline{u} + \tau(\overline{v} - \overline{u}) - b)^2 + \varepsilon)^3}} > 0,$$

then $\overline{v} \equiv \overline{u}$ by Lemma 3.2, and (1) has a unique solution $\overline{u} = \lim_{\lambda \to \overline{\lambda}-} u(\lambda)$ at $\lambda = \overline{\lambda}$. □

By using this lemma, we can now obtain the exact multiplicity of positive solutions.

Theorem 3.4. *Fix $\varepsilon > 0$, and suppose that $\overline{\lambda} < \lambda_2$; then (1) has exactly one positive solution for $\lambda \in (0, \lambda_1] \cup \{\overline{\lambda}\}$, has exactly two positive solutions for $\lambda \in (\lambda_1, \overline{\lambda})$, and has no positive solution for $\lambda \in (\overline{\lambda}, \infty)$.*

Proof. From Theorem 1.1 and Lemma 3.3, when $\lambda \in (0, \lambda_1] \cup \{\overline{\lambda}\}$, (1) has exactly one positive solution. To prove that (1) has exactly two positive solutions for $\lambda \in (\lambda_1, \overline{\lambda})$, we first prove the fact that $\exists \delta > 0$, when $\lambda \in (\overline{\lambda} - \delta, \overline{\lambda})$, for which (1) has exactly two positive solutions.

We consider again the eigenvalue problem (3). Since $(\overline{\lambda}, \overline{u})$ is a degenerate solution of (1), the corresponding linearized equation

$$\begin{cases} -\Delta \omega = \overline{\lambda} f_u(\overline{u}, \varepsilon)\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

has nontrivial solution $\omega \in X$, and (5) has a nontrivial solution if and only if $\mu_k(\lambda^* f_u(u^*, \varepsilon)) = 1$ for some $k \in \mathbf{N}$. Like in the proof of Proposition 2.2, we have

$$1 = \mu_k(\overline{\lambda} f_u(\overline{u}, \varepsilon)) > \mu_k(\overline{\lambda} \cdot 1) = \frac{\mu_k(1)}{\overline{\lambda}} = \frac{\lambda_k}{\overline{\lambda}}.$$

So $\overline{\lambda} > \lambda_k$, and from the assumption $\overline{\lambda} < \lambda_2$, hence $k = 1$. Consequently, the eigenfunction $\overline{\omega}$ can be chosen as positive, and $N(F_u(\overline{\lambda}, \overline{u})) = \text{span}\{\overline{\omega}\}$. From Theorem 2.6, the solutions of (1) near $(\overline{\lambda}, \overline{u})$ form a curve $(\overline{\lambda} + \lambda(s), \overline{u} + s\overline{\omega} + z(s))$, where $s \longmapsto (\lambda(s), z(s)) \in \mathbf{R}^+ \times Z$ is a continuously differentiable function near $s = 0$, and $\lambda(0) = \lambda'(0) = 0, z(0) = z'(0) = 0, \lambda''(0) < 0$. When $\lambda \in (0, \overline{\lambda}]$, $u_1 = u(\lambda)$ is the minimal solution of (1). Since the solution is unique at $\lambda = \overline{\lambda}$, and the solution curve turns to the left, then $\exists \delta > 0$, when $\lambda \in (\overline{\lambda} - \delta, \overline{\lambda})$, for which (1) has a unique non-minimal solution $u_2 > \overline{u} > u(\lambda) = u_1$, i.e. (1) has exactly two positive solutions u_1 and u_2.

Let Σ be the connected component of the solution set of (1) which contains $(\overline{\lambda}, \overline{u})$. Then Σ contains the curve of the minimal solutions $\{(\lambda, u_1(\lambda)) : \lambda \in (0, \overline{\lambda}]\}$ and $\{(\lambda, u_2(\lambda)) : \lambda \in (\overline{\lambda} - \delta, \overline{\lambda})\}$ from the last paragraph. Since (1) has a unique solution at $\lambda = \overline{\lambda}$, thus $\Sigma \subset (0, \overline{\lambda}] \times X$. We claim that $(\overline{\lambda}, \overline{u})$ is the only degenerate solution of (1) on Σ. Suppose there exists another $(\lambda^*, u^*) \neq (\overline{\lambda}, \overline{u})$ and $(\lambda^*, u^*) \in \Sigma$ such that (λ^*, u^*) is a degenerate solution of (1); then $\lambda^* \in (\lambda_1, \overline{\lambda})$ and $u^* > u_*$, where u_* is the minimal solution of (1) at $\lambda = \lambda^*$. From the arguments in the last paragraph, at the degenerate solution (λ^*, u^*), the linearized equation has a positive eigenfunction ω^*, and $N(F_u(\lambda^*, u^*)) = \text{span}\{\omega^*\}$. Thus the corresponding eigenvalue problem

$$\begin{cases} -\Delta\omega - \lambda^* f_u(u^*, \varepsilon)\omega = \mu\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

has the principal eigenvalue $\mu_1(u^*) = 0$. Moreover $\forall \tau \in [0, 1]$, $\forall x \in \Omega$,

$$f_{uu}(u^* + \tau(u_* - u^*), \varepsilon) = \frac{\varepsilon}{\sqrt{((u^* + \tau(u_* - u^*) - b)^2 + \varepsilon)^3}} > 0,$$

and then by Lemma 3.2, $u^* \equiv u_*$, which is a contradiction. Thus the only degenerate solution on Σ is $(\overline{\lambda}, \overline{u})$.

Hence if we extend the curve of non-minimal solutions $\{(\lambda, u_2(\lambda)) : \lambda \in (\overline{\lambda} - \delta, \overline{\lambda})\}$ to the left, it will not have any more bifurcation points. From Theorem 1.1 and Lemma 2.4, the curve can be extended for all $\lambda \in (\lambda_1, \overline{\lambda})$, and $\|u_2(\lambda)\|_\infty \to \infty$ as $\lambda \to \lambda_1^+$. Therefore

$$\Sigma = \{(\lambda, u_1(\lambda)) : \lambda \in \lambda \in (0, \overline{\lambda}]\} \bigcup \{(\lambda, u_2(\lambda)) : \lambda \in (\lambda_1, \overline{\lambda})\}.$$

Finally we prove that (1) has no positive solutions other than the ones on Σ. Suppose that (λ_0, u_0) is a positive solution of (1) not on Σ. Then $\lambda_0 \neq \overline{\lambda}$ from Lemma 3.3. Case 1: $\lambda_0 > \overline{\lambda}$. Then $u = u_0$ is a supersolution of (1) when $0 < \lambda < \lambda_0$, and $u = 0$ is always a strict subsolution for any $\lambda > 0$; thus (1) has a positive solution (λ, u^λ) for any $\lambda \in (\overline{\lambda}, \lambda_0)$. Moreover u^λ is bounded from above by u_0, and hence like in previous arguments, a subsequence of $\{u^\lambda\}$ converges to a limit as $\lambda \to \overline{\lambda}^+$. The limit also satisfies (1); hence it must be $\overline{u}$ from Lemma 3.3. However from Theorem 2.6, (1) has no positive solution (λ, u) near $(\overline{\lambda}, \overline{u})$ but $\lambda > \overline{\lambda}$. That is a contradiction; therefore (1) has no positive solution when $\lambda > \overline{\lambda}$. Case 2: $0 < \lambda_0 < \overline{\lambda}$. From arguments above, u_0 cannot be degenerate, as a contradiction can be reached by Lemma 3.2. Thus u_0 must be non-degenerate, and from the implicit function theorem, (λ_0, u_0) is on a smooth curve $(\lambda, u_3(\lambda))$. Since the solutions are a priori bounded from Lemma 2.4, the curve $(\lambda, u_3(\lambda))$ can be extended to $\lambda = \overline{\lambda}$. However from Theorem 2.6 and proofs above, $u_3(\lambda)$ must be one of $u_1(\lambda)$ and $u_2(\lambda)$. Thus there are no solutions of (1) except the ones on Σ. □

Notice that our proof shows that $\overline{\lambda} = \sup\{\lambda : (1) \text{ has nonnegative solution}\}$ even if $\overline{\lambda} \geq \lambda_2$, since Lemma 3.3 does not require $\overline{\lambda} < \lambda_2$. When $\varepsilon > 0$ is fixed and $\overline{\lambda} < \lambda_2$, Theorem 3.4 shows the exact multiplicity of the positive solutions of the Eq. (1), and this ensures the precise structure of the solution set. Theorem 3.4 implies that the set of positive solutions of (1) consists of a C^2-curve Σ, which joins $(0, 0)$ and (λ_1, ∞). This curve can be decomposed into two parts, namely the "lower" part, joining $(0, 0)$ and $(\overline{\lambda}, \overline{u})$, and the "upper" part, joining $(\overline{\lambda}, \overline{u})$ and (λ_1, ∞). The lower part contains all the minimal solutions and the upper part contains all the non-minimal solutions and each of the two parts can be parameterized by λ. Fig. 1 exhibits the bifurcation diagram of the solution set.

4. Stability of solutions

The stability of the solutions to Eq. (1) is an important issue, especially when we consider the corresponding parabolic equation:

$$\begin{cases} u_t = \Delta u + \lambda f(u, \varepsilon) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The stability of a solution (λ^*, u^*) can be defined through the eigenvalue problem

$$\begin{cases} \Delta\omega + \lambda^* f_u(u^*, \varepsilon)\omega = -\mu\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

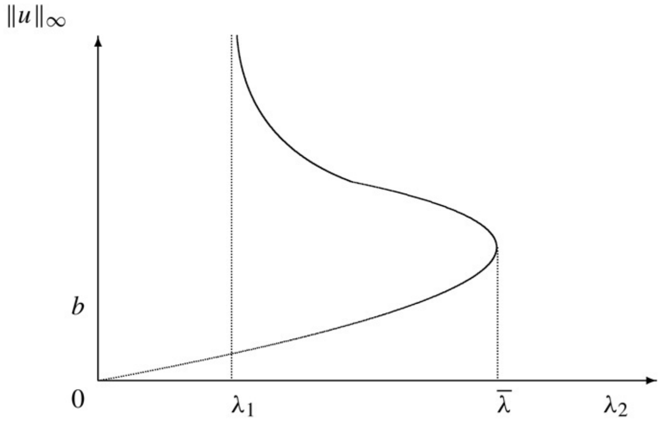

Fig. 1. Precise bifurcation diagram when $\overline{\lambda} < \lambda_2$.

It is well known that the eigenvalue problem has a sequence of real eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots \to \infty$, and μ_1 is the principal eigenvalue with a positive $\psi_1 > 0$. Let (λ^*, u^*) be a solution of (1). If $\mu_1(u^*) > 0$, then we say that (λ^*, u^*) is stable; if $\mu_1(u^*) < 0$, it is unstable; and when $\mu_1(u^*) = 0$, it is neutrally stable. If (λ^*, u^*) is unstable, then the number of negative eigenvalues of the eigenvalue problem (counting the multiplicity) is the Morse index $\mathrm{M}(u^*)$ of (λ^*, u^*).

It is well known that the minimal solution $u_1(\lambda)$ is stable when $\lambda \in (0, \overline{\lambda})$, and the degenerate solution $(\overline{\lambda}, \overline{u})$ is neutrally stable. The following two results concern the instability of non-minimal positive solutions.

Proposition 4.1. *Any non-minimal positive solution of (1) is unstable.*

Proof. Following Theorem 3.4, let $(\lambda, u_2(\lambda))$ be the non-minimal solution of (1) with $\lambda \in (\lambda, \overline{\lambda})$. We show that $(\lambda, u_2(\lambda))$ is unstable, i.e. $\mu_1(u_2(\lambda)) < 0$.

Assume there exists $\lambda^* \in (\lambda_1, \overline{\lambda})$, $u_2(\lambda^*) = u^*$ such that $\mu_1(u^*) > 0$; then $u_* < u^*$, where u_* is the minimal solution of equation (1) in $\lambda = \lambda^*$. But $\forall \tau \in [0, 1]$, $\forall x \in \Omega$,

$$f_{uu}(u^* + \tau(u_* - u^*), \varepsilon) = \frac{\varepsilon}{\sqrt{((u^* + \tau(u_* - u^*) - b)^2 + \varepsilon)^3}} > 0,$$

and then by Lemma 3.2, $u_* \geq u^*$, which is a contradiction with $u_* < u^*$.

Assume there exists $\lambda^* \in (\lambda_1, \overline{\lambda})$, $u_2(\lambda^*) = u^*$ such that $\mu_1(u^*) = 0$. Then for $\lambda = \lambda^*$, there exists a minimal solution $u_* < u^*$. We reach a contradiction again from Lemma 3.2. Therefore any non-minimal solution is unstable. □

Proposition 4.2. *The Morse index of a non-minimal solution (λ, u) is 1 if $\lambda < \lambda_2$.*

Proof. Proposition 4.1 shows that $\mu_1(u_2(\lambda)) < 0$. Now we consider the sign of $\mu_2(u_2(\lambda))$. The variational characterization of μ_2 is

$$\mu_2(u_2(\lambda)) = \min_T \max_{w \in T, w \not\equiv 0} \frac{\int_\Omega (|\nabla w|^2 - \lambda f_u(u_2(\lambda), \varepsilon) w^2) \mathrm{d}x}{\int_\Omega w^2 \mathrm{d}x},$$

where the minimum is taken over all two-dimensional subspaces T of $H_0^1(\Omega)$. Since $f_u(u_2(\lambda), \varepsilon) < 1$ and $\lambda < \lambda_2$, we have

$$\begin{aligned} \mu_2(u_2(\lambda)) &= \min_T \max_{w \in T, w \not\equiv 0} \frac{\int_\Omega (|\nabla w|^2 - \lambda f_u(u_2(\lambda), \varepsilon) w^2) \mathrm{d}x}{\int_\Omega w^2 \mathrm{d}x} \\ &> \min_T \max_{w \in T, w \not\equiv 0} \frac{\int_\Omega (|\nabla w|^2 - \lambda w^2) \mathrm{d}x}{\int_\Omega w^2 \mathrm{d}x} \\ &= \lambda_2 - \lambda > 0 \end{aligned}$$

and thus $\mu_2(u_2(\lambda)) > 0$ and the Morse index of the non-minimal solutions is 1. □

By using Proposition 4.2, even when $\overline{\lambda} \geq \lambda_2$, we have the following exact multiplicity result:

Theorem 4.3. *Suppose $\overline{\lambda} \geq \lambda_2$; then (1) has exactly two positive solutions for $\lambda \in (\lambda_1, \lambda_2)$.*

Proof. From the proof of Theorem 3.4, (1) has no degenerate solution in (λ_1, λ_2) unless it is $(\overline{\lambda}, \overline{u})$, but we assume $\overline{\lambda} \geq \lambda_2$. On the other hand, when $\lambda \in (\lambda_1, \lambda_2)$, (1) has at least two positive solutions, the minimal one $u_1(\lambda)$, and the non-minimal one $u_2(\lambda)$. Assume there is another one $u_3(\lambda_*)$ for some $\lambda_* \in (\lambda_1, \lambda_2)$; then it is non-degenerate and with Morse index 1 from Propositions 4.1 and 4.2. It can be extended by using the implicit function theorem as it is non-degenerate and bounded as long as $\lambda \in (\lambda_1, \lambda_2)$. On the other hand, (1) has a unique solution when $\lambda \in (0, \lambda_1]$; thus $\|u_3(\lambda)\|_\infty \to \infty$ as $\lambda \to \lambda_1^+$. But the bifurcation from infinity theorem implies that there is only one curve going to infinity as $\lambda \to \lambda_1^+$. Therefore another curve $(\lambda, u_3(\lambda))$ does not exist, and (1) has exactly two positive solutions when $\lambda \in (\lambda_1, \lambda_2)$. □

5. Dependence on the parameter ε

In this section we discuss the dependence of the solution set of (1) on the parameter ε. First we recall the following lemma from [5]:

Lemma 5.1 *([5]). Let u_ε be a minimal solution of (1). Consider*

$$\begin{cases} \Delta u + \lambda f(u, \varepsilon) + h(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $h \in C(\mathbf{R}, \mathbf{R})$ and $\lambda f(u, \varepsilon) + h(u) > 0, \forall u \in [0, \infty)$. Then there exists $\delta \equiv \delta(u_\varepsilon) > 0$ such that $\|h\|_\infty < \delta$, (7) has a minimal solution.

Our main result in this section is

Proposition 5.2. *Let $\overline{\lambda}(\varepsilon) = \sup\{\lambda : (1) \text{ has a nonnegative solution}\}$. Then $\overline{\lambda}'(\varepsilon) < 0$ for $\varepsilon \in (0, \infty)$, $\lim_{\varepsilon \to 0^+} \overline{\lambda}(\varepsilon) = \infty$, and $\lim_{\varepsilon \to \infty} \overline{\lambda}(\varepsilon) = \lambda_1$.*

Proof. First we prove that $\overline{\lambda}(\varepsilon)$ is differentiable by using the implicit function theorem for the pair of equations

$$\begin{cases} F(\varepsilon, \lambda, u) = 0, \\ F_u(\varepsilon, \lambda, u)[\omega] = 0. \end{cases} \tag{8}$$

Indeed we will apply Theorems 2.1 and 2.3 of [7]. Let $T = (\overline{\varepsilon}, \overline{\lambda}, \overline{u}, \overline{\omega})$ be a solution of (8). And we know that $\overline{\omega} > 0$, $N(F_u(\overline{\varepsilon}, \overline{\lambda}, \overline{u})) = \text{span}\{\overline{\omega}\}$. Like in the proofs in [7], we can also easily verify that

$$F_\lambda(\overline{\varepsilon}, \overline{\lambda}, \overline{u}) \notin R(F_u(\overline{\varepsilon}, \overline{\lambda}, \overline{u})), \qquad F_{uu}(\overline{\varepsilon}, \overline{\lambda}, \overline{u})[\overline{\omega}, \overline{\omega}] \notin R(F_u(\overline{\varepsilon}, \overline{\lambda}, \overline{u})).$$

Hence the conditions in Theorem 2.1 of [7] are met; we obtain that there exists $\delta_1 > 0$ such that all the solutions of [7] near T have the form

$$\{T_\varepsilon = (\varepsilon, \lambda(\varepsilon), u(\varepsilon), \omega(\varepsilon)) : \varepsilon \in (\overline{\varepsilon} - \delta_1, \overline{\varepsilon} + \delta_1)\},$$

where $u(\overline{\varepsilon}) = \overline{u}$, $\omega(\overline{\varepsilon}) = \overline{\omega}$ and $\lambda(\overline{\varepsilon}) = \overline{\lambda}$, and $\lambda(\cdot), u(\cdot), \omega(\cdot)$ are continuously differentiable. Differentiating

$$\Delta u(\varepsilon) + \lambda(\varepsilon) f(u(\varepsilon), \varepsilon) = 0, \tag{9}$$

we have

$$\Delta u_\varepsilon + \overline{\lambda} f_u(\overline{u}, \overline{\varepsilon}) u_\varepsilon + \lambda'(\overline{\varepsilon}) f(\overline{u}, \overline{\varepsilon}) + \overline{\lambda} f_\varepsilon(\overline{u}, \overline{\varepsilon}) = 0, \tag{10}$$

and we also have

$$\int_\Omega (\Delta u_\varepsilon + \overline{\lambda} f_u(\overline{u}, \overline{\varepsilon}) u_\varepsilon) \overline{\omega} \mathrm{d}x = \int_\Omega (\Delta \overline{\omega} + \overline{\lambda} f_u(\overline{u}, \overline{\varepsilon}) \overline{\omega}) u_\varepsilon \mathrm{d}x = 0.$$

Multiplying (10) by $\overline{\omega}$, and integrating over Ω we obtain

$$\int_\Omega (\lambda'(\overline{\varepsilon}) f(\overline{u}, \overline{\varepsilon}) + \overline{\lambda} f_\varepsilon(\overline{u}, \overline{\varepsilon})) \overline{\omega} \mathrm{d}x = 0,$$

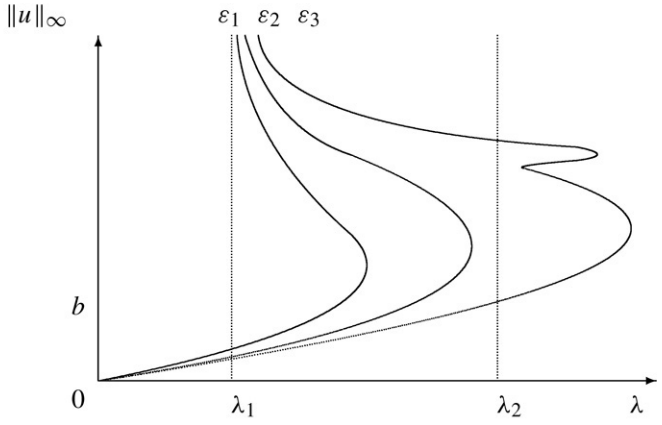

Fig. 2. Variation of the bifurcation diagrams when ε varies. Here $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

and by $\int_\Omega f(\overline{u}, \overline{\varepsilon})\overline{\omega}\mathrm{d}x > 0$, we have

$$\lambda'(\overline{\varepsilon}) = -\overline{\lambda}\frac{\int_\Omega f_\varepsilon(\overline{u}, \overline{\varepsilon})\overline{\omega}\mathrm{d}x}{\int_\Omega f(\overline{u}, \overline{\varepsilon})\overline{\omega}}\mathrm{d}x,$$

and $f_\varepsilon > 0, \overline{\omega} > 0$; then $\lambda'(\overline{\varepsilon}) < 0$.

Next we show $\lim_{\varepsilon\to 0^+}\overline{\lambda}(\varepsilon) = \infty$. We fix $\lambda > 0$, and prove that (1) has a positive solution with this λ when ε is small enough. Clearly $u = 0$ is a subsolution of (1) for any $\varepsilon > 0$ and λ. Now we look for a supersolution of (1). Consider the equivalent form of (1)

$$\Delta u + \lambda f(u, 0) + \lambda(f(u, \varepsilon) - f(u, 0)) = 0.$$

Let $g(u) = \lambda(f(u, \varepsilon) - f(u, 0))$, and $f(u, 0) = |u - b|$. Since $\sqrt{(u-b)^2 + \varepsilon} \le |u - b| + \sqrt{\varepsilon}$, we have $g(u) \le \lambda\sqrt{\varepsilon}$, and

$$\Delta u + \lambda f(u, \varepsilon) \le \Delta u + \lambda f(u, 0) + \lambda\sqrt{\varepsilon}.$$

Therefore the minimal solution of the equation

$$\begin{cases}\Delta u + \lambda f(u, 0) + \lambda\sqrt{\varepsilon} = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{11}$$

is a supersolution of (1).

First we show the existence of a minimal solution of the equation

$$\Delta u + \lambda f(u, 0) = 0, \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega. \tag{12}$$

Indeed, $u = 0$ and $u = b$ are a subsolution and supersolution of (12) respectively; hence by using the monotone iteration method we obtain a minimal solution of (12). Let $h(u) = \lambda\sqrt{\varepsilon}$; then there exists ε_0 small enough such that when $\varepsilon \in (0, \varepsilon_0)$, $h(u) = \lambda\sqrt{\varepsilon} < \delta = \lambda\sqrt{\varepsilon_0}$. From Lemma 5.1, (11) has a minimal solution which is also a supersolution of (1); thus by using the monotone iteration method, (1) has a nonnegative solution for this fixed λ when $0 < \varepsilon < \varepsilon_0$. Therefore, $\overline{\lambda}(\varepsilon) \to \infty$ when $\varepsilon \to 0^+$.

Finally we prove $\lim_{\varepsilon\to\infty}\overline{\lambda}(\varepsilon) = \lambda_1$. Since $f(u, \varepsilon) = \sqrt{(u-b)^2 + \varepsilon}$, there exists $a > 0$ such that $f(u, \varepsilon) > ku, \forall u \ge 0$; we have $\min_{u\in(0,\infty)} f(u, \varepsilon)/u \ge k$. When $u = b + \varepsilon/b$, $f(u, \varepsilon)/u$ achieves the minimum value, i.e.

$$k \le \min_{u\in(0,\infty)}\frac{f(u, \varepsilon)}{u} = \sqrt{\frac{\varepsilon}{b^2 + \varepsilon}},$$

then $k \to 1$ when $\varepsilon \to \infty$. Let $\phi_1 > 0$ be the positive eigenfunction of the principal eigenvalue λ_1. From (4), if (1) has a positive solution (λ, u), then $\lambda \le \lambda_1/k$, $\lambda_1 < \overline{\lambda}(\varepsilon) \le \lambda_1/k$; therefore $\overline{\lambda}(\varepsilon) \to \lambda_1$ when $\varepsilon \to \infty$. □

Proposition 5.2, Theorems 3.4 and 4.3 together give a rather complete portrait of the variation of the bifurcation diagrams in two parameters (ε, λ) (see Fig. 2). When ε is large, $\overline{\lambda}$ is just a little beyond λ_1, and the results in

Theorem 3.4 must hold; thus the bifurcation diagram is exactly $\supset$-shaped. This shape is kept as long as $\overline{\lambda} < \lambda_2$. When ε is decreased so that $\overline{\lambda} \geq \lambda_2$, the bifurcation diagrams may have more wiggles on the portion $\lambda > \lambda_2$, but the branch of stable minimal solutions is a monotone one without any turning points for the whole portion $0 < \lambda < \overline{\lambda}$, and the branch bifurcating from infinity at $\lambda = \lambda_1$ is also monotone for the portion $\lambda_1 < \lambda < \lambda_2$.

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