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Global asymptotical behavior of the Lengyel–Epstein reaction–diffusion system[☆]

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ABSTRACT

The Lengyel–Epstein reaction–diffusion system of the CIMA reaction is revisited. We construct a Lyapunov function to show that the constant equilibrium solution is globally asymptotically stable when the feeding rate of iodide is small. We also show that for small spatial domains, all solutions eventually converge to a spatially homogeneous and time-periodic solution.

1. Introduction

In this work, we reconsider the Lengyel–Epstein reaction–diffusion system:

$$\begin{cases} u_t = \Delta u + a - u - \dfrac{4uv}{1+u^2}, & x \in \Omega, t > 0, \\ v_t = \sigma\left[c\Delta v + b\left(u - \dfrac{uv}{1+u^2}\right)\right], & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, \qquad v(x,0) = v_0(x) > 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$, with sufficiently smooth boundary $\partial\Omega$. Here $u = u(x,t)$ and $v = v(x,t)$ denote the chemical concentrations of the activator iodide (I^-) and the inhibitor chlorite (ClO_2^-), respectively, at time $t > 0$ and a point $x \in \Omega$. The parameters a and b are parameters related to the feed concentrations; c is the ratio of the diffusion coefficient; $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch, enlarging the effective diffusion ratio to σc. We shall assume accordingly that all constants $a, b, c,$ and σ are positive. Problem (1.1) is based on the well-known chlorite–iodide–malonic acid chemical (CIMA) reaction (see [4,8]). A more detailed historical account of the development of the CIMA reaction model and experiments can be found in [1]. In the past decade, both experimental and numerical studies on the system (1.1) have been conducted; however, the mathematical investigations are still very limited. See for example [6,9,11].

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In an earlier paper [11], we derived precise conditions such that the spatially homogeneous equilibrium solution and the spatially homogeneous periodic solution become Turing unstable; we also performed a detailed Hopf bifurcation analysis for both the ODE and PDE models, deriving a formula for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

The purpose of this work is to consider the global asymptotical behavior of solutions of the system (1.1). We construct a Lyapunov function to show that the constant equilibrium solution is globally asymptotically stable when $0 < a^2 \le 27$. We also show that for small spatial domains, all solutions eventually converge to a spatially homogeneous and time-periodic solution. We hope that these results will provide another step towards the understanding of the dynamics of the important Lengyel–Epstein system. The precise statements and proofs of our results are given in Section 2.

2. Global asymptotical stability

First we consider the global asymptotical stability of the unique constant equilibrium solution (u_*, v_*) of the system (1.1).

In [9,11], the local stability of (u_*, v_*) with respect to (1.1) was studied. These results can be summarized as follows. Recall that the system (1.1) has a unique equilibrium point $(u_*, v_*) = (a/5, 1 + a^2/25)$. Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be eigenvalues of the elliptic operator $-\triangle$ subject to the no-flux boundary condition on $\partial\Omega$. The Jacobian matrix of the corresponding ODE system of (1.1) evaluated at (u_*, v_*) takes on the following form:

$$J := \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{3a^2 - 125}{a^2 + 25} & -\dfrac{4a}{a^2 + 25} \\ \dfrac{2\sigma a^2 b}{a^2 + 25} & -\dfrac{5\sigma ab}{a^2 + 25} \end{pmatrix}. \tag{2.1}$$

Suppose that $\lambda_1 < J_{11}$; then there exists a positive integer (the largest one) i_m such that $\lambda_i < J_{11}$ holds for any $i \le i_m$. Define

$$\widehat{c} = \min_{1 \le i \le i_m} \frac{5ab(\lambda_i + 5)}{\lambda_i (a^2 + 25)(J_{11} - \lambda_i)}. \tag{2.2}$$

In [9, Lemma 5.1], the following stability conditions were derived:

Lemma 2.1. *Suppose that $0 < a^2 \le \frac{125 + 5\sigma ab}{3}$ is satisfied. If $\lambda_1 \ge J_{11}$ or $\lambda_1 < J_{11}$ and $0 < c < \widehat{c}$ holds, then (u_*, v_*) is locally asymptotically stable; if $\lambda_1 < J_{11}$ and $c > \widehat{c}$, then (u_*, v_*) is unstable.*

For the one-dimensional spatial domain $\Omega = (0, \pi)$, a more precise description of the stability parameter region can be obtained (see [11, Theorem 3.1]):

Lemma 2.2. *Suppose that $b > b_0 := \frac{3a^2 - 125}{5\sigma a}$ is satisfied, and that $0 < a^2 \le 75$ or $a^2 > 75$ and $0 < c < \frac{15ab}{a^2 - 75}$ hold. Then (u_*, v_*) is locally asymptotically stable; and if $a^2 > 75$ and $c > \frac{15ab}{a^2 - 75}$ hold, then (u_*, v_*) is unstable.*

The corresponding ODE system of (1.1) takes on the following form:

$$\begin{cases} \dfrac{du}{dt} = a - u - \dfrac{4uv}{1 + u^2} := F(u, v), \\ \dfrac{dv}{dt} = \sigma b \left(u - \dfrac{uv}{1 + u^2} \right) := G(u, v). \end{cases} \tag{2.3}$$

A rectangle $\mathscr{R} := (0, r_1) \times (0, r_2)$ is called an *invariant rectangle* (see [3]) if the vector field (F, G) on the boundary $\partial\mathscr{R}$ points inside, that is,

$$\begin{aligned} &F(0, v) \ge 0 \quad \text{and} \quad F(r_1, v) \le 0 \quad \text{for } 0 < v < r_2, \\ &G(u, 0) \ge 0 \quad \text{and} \quad G(u, r_2) \le 0 \quad \text{for } 0 < u < r_1. \end{aligned} \tag{2.4}$$

From Proposition 2.2 of [9], we know that (1.1) admits a region of attraction

$$\mathscr{R}_a := (0, a) \times (0, 1 + a^2) \tag{2.5}$$

which attracts all solutions of (1.1), regardless of the initial values u_0 and v_0; in particular this also implies that (1.1) admits a unique solution $(u(t, x), v(t, x))$ defined for all $x \in \Omega$ and $t > 0$.

Our global asymptotical stability result of (u_*, v_*) goes as follows:

Theorem 2.3. *Suppose that $\sigma, b, c > 0$ and $0 < a^2 \le 27$. Then for any nonnegative $u_0(x), v_0(x) \in C^2(\Omega) \bigcap C(\overline{\Omega})$, the solution $(u(t, x), v(t, x))$ of the system (1.1) converges uniformly in x to (u_*, v_*).*

Proof. 1. From the aforementioned results in [9], there exists $T > 0$ such that for any $t > T$, the solution $(u(t,x), v(t,x)) \in \mathscr{R}_a$ for all $(t,x) \in (T,\infty) \times \bar{\Omega}$. Without loss of generality, we can assume that $T = 0$.

Define the following Lyapunov function on $\mathscr{R}_a$:

$$E(t) = \int_\Omega [E_1(u(x,t)) + E_2(v(x,t))]\mathrm{d}x, \tag{2.6}$$

where $E_1(u) = \sigma b \int (u^2 - u_*^2)\mathrm{d}u = \sigma b(u^3/3 - u_*^2 u)$, $E_2(v) = 4\int (v - v_*)\mathrm{d}v = 2v^2 - 4v_* v$. Then,

$$\begin{aligned} E'(t) &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}u}[E_1(u(x,t))]\, u_t \mathrm{d}x + \int_\Omega \frac{\mathrm{d}}{\mathrm{d}v}[E_2(v(x,t))]\, v_t \mathrm{d}x \\ &= \int_\Omega \sigma b(u^2 - u_*^2)u_t \mathrm{d}x + \int_\Omega 4(v - v_*)v_t \mathrm{d}x \\ &= \int_\Omega \sigma b(u^2 - u_*^2)(\Delta u + a - u - 4\phi(u)v)\mathrm{d}x + \int_\Omega 4(v - v_*)(\sigma[c\Delta v + b(u - \phi(u)v)])\mathrm{d}x \\ &= \int_\Omega \left[\sigma b\phi(u)(u^2 - u_*^2)(f(u) - 4v) + 4\sigma b\phi(u)(v - v^*)(u^2 + 1 - v)\right]\mathrm{d}x - \sigma b I(t) \\ &= \sigma b \int_\Omega \phi(u)\left[(u^2 - u_*^2)(f(u) - 4v) + 4(v - v^*)(u^2 + 1 - v)\right]\mathrm{d}x - \sigma b I(t) \\ &= \sigma b \int_\Omega \phi(u)\left[(u^2 - u_*^2)(f(u) - f(u_*)) - 4(v - v^*)^2\right]\mathrm{d}x - \sigma b I(t), \end{aligned}$$

where $I(t) := \int_\Omega 2u|\nabla u|^2 \mathrm{d}x + \frac{4c}{b}\int_\Omega |\nabla v|^2 \mathrm{d}x$, $\phi(u) := \frac{u}{1+u^2}$, and $f(u) := \frac{(a-u)(1+u^2)}{u}$. When $0 < a^2 \le 27$ holds, $f(u)$ is a strictly decreasing function. Thus, $E'(t) \le 0$ and $E'(t) = 0$ if and only if $(u, v) = (u_*, v_*)$, that is, E decreases monotonically along a solution orbit.

2. By [3, Lemma 4], for any $\delta > 0$ the set $U_\delta := \{u(t,\cdot) : t \ge \delta\}$ or $V_\delta := \{v(t,\cdot) : t \ge \delta\}$ is relatively compact; and then by [7], the ω limit set $\omega(x,t)$ of $(u(x,t), v(x,t))$ is nonempty, compact, connected and invariant; and the trajectory approaches the limit set. The boundedness and precompactness of orbits imply that E is bounded below, so there exists $E_0 = \lim_{t\to\infty} E(u(x,t), v(x,t))$. Suppose that $(\tilde{u}(x,t), \tilde{v}(x,t)) \in \omega$; then $E(\tilde{u}(x,t), \tilde{v}(x,t)) = E_0$. Let $\omega^*(x,t)$ be the orbit starting at some point $(\tilde{u}(x,t), \tilde{v}(x,t))$ in the ω limit set of (u_0, v_0). Since that set is invariant, the entire orbit $w^*(x,t)$ must belong to it, so $E(w^*(x,t)) = E_0$. However, by the monotonicity of $E(t)$, this is impossible. Thus, $(\tilde{u}(x,t), \tilde{v}(x,t))$ must be an equilibrium of (1.1). Since $(\tilde{u}(x,t), \tilde{v}(x,t))$ is an arbitrary element of the ω limit set of an arbitrary point $(u_0(x), v_0(x))$, $\omega(x,t)$ for the semi-flow generated by (1.1) must consist entirely of equilibria. If there is an equilibrium solution $(\hat{u}(x), \hat{v}(x))$ other than (u_*, v_*), then E is strictly decreasing along $(\hat{u}, \hat{v})$, which is a contradiction. Hence (u_*, v_*) is the only possible ω limit set. □

Corollary 2.4. *Suppose that $\sigma, b, c > 0$ and $0 < a^2 \le 27$. Then (u_*, v_*) is the only equilibrium solution of (1.1), and it is globally asymptotically stable.*

The construction of the Lyapunov function here is motivated by but different from the ones in [3]. Our result apparently also applies to ODE system (2.3). For a recent survey of applications of the Lyapunov function in reaction–diffusion systems or ODEs, see [5]. We notice that the results in Theorem 2.3 are independent of parameters σ, b, c as well as the domain Ω. Recall that a is the feeding rate of the iodide. Thus when this feeding rate is very low, the chemical reaction will stabilize at the unique constant equilibrium. When the feeding rate a increases, the system becomes an activator–inhibitor system, then complex patterns have been observed (see [4,8,10]) and the existence of non-constant equilibrium solutions has been shown (see [6,9].) Global stability will be irrelevant in these situations.

Another case in which global asymptotical behavior can be shown is when the spatial domain is a small one, and here the scale of a spatial domain is measured by the principal eigenvalue of $-\Delta$ on Ω with homogeneous Neumann boundary conditions. Our next result shows that under certain conditions, every solution of system (1.1) with initial value $(u_0(x), v_0(x)) \neq (u_*, v_*)$ could converge exponentially to a time-periodic but spatially homogeneous function.

Following [2], we define γ by: $\gamma := d\lambda_1 - M$, where λ_1 is the principal eigenvalue of $-\Delta$ on Ω with homogeneous Neumann boundary conditions, $d = \min\{1, \sigma\}$, and $M = \sup_{(u,v)\in\mathscr{R}_a}\{|J(u,v)|\}$, where $J(u,v)$ is defined by

$$J(u,v) = \begin{pmatrix} -1 - 4\varphi'(u)v & -4\varphi(u) \\ \sigma b(1 - \varphi'(u)v) & -\sigma b\varphi(u) \end{pmatrix}. \tag{2.7}$$

The following lemma is due to [2]:

Lemma 2.5. *Assume that $\gamma > 0$. Let $(u(t,x), v(t,x))$ be a solution of (1.1). Then, there exist constants $c_i > 0$, $i = 1, 2, 3$, such that*

$$\|\nabla_x(u(\cdot,t), v(\cdot,t))\|_{L^2(\Omega)} \le c_1 \mathrm{e}^{-\gamma t}, \qquad \|(u(\cdot,t), v(\cdot,t)) - (\overline{u}(t), \overline{v}(t))\|_{L^2(\Omega)} \le c_2 \mathrm{e}^{-\gamma t}, \tag{2.8}$$

where $\bar{u}(t)$, $\bar{v}(t)$ are the averages of u and v over Ω respectively, satisfying

$$\begin{cases} \dfrac{d\bar{u}}{dt} = F(\bar{u}, \bar{v}) + g_1(t), & \dfrac{d\bar{v}}{dt} = G(\bar{u}, \bar{v}) + g_2(t), \\ \bar{u}(0) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0(x)dx, & \bar{v}(0) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega v_0(x)dx, \end{cases} \tag{2.9}$$

with $|g_i(t)| \le c_3 e^{-\gamma t}$ *for* $i = 1, 2$.

Straightforward calculation shows that

$$\begin{aligned} |J(u, v)| &:= \max\{|1 + 4\varphi'(u)v| + 4\varphi(u), \sigma b|1 - \varphi'(u)v| + \sigma b\varphi(u)\} \\ &< M(\sigma, b, a) := (2\sigma b + 5) + (\sigma b + 4)(1 + a^2)^2. \end{aligned} \tag{2.10}$$

Then, like in the proof of Lemma 3.1 in [2], we obtain:

Theorem 2.6. *Suppose that* $\sigma, b, c > 0$ *and a satisfies*

$$a > \frac{\sqrt{1500 + 25\sigma^2 b^2} + 5\sigma b}{6}. \tag{2.11}$$

If in addition, either

$$\sigma \ge 1, \quad \text{and} \quad \lambda_1 > M(\sigma, b, a), \tag{2.12}$$

or

$$0 < \sigma < 1, \quad \text{and} \quad \sigma\lambda_1 > M(\sigma, b, a), \tag{2.13}$$

is satisfied, where $M(\sigma, b, a)$ *is defined in (2.10), then every solution* $(u(t, x), v(t, x)) \not\equiv (u_*, v_*)$ *of the system (1.1) converges exponentially to* $(\bar{u}(t, x), \bar{v}(t, x)) \equiv (\bar{u}(t), \bar{v}(t))$*, and* $(\bar{u}(t), \bar{v}(t))$ *is a nontrivial periodic solution of (2.3).*

Proof. The condition (2.12) implies that (u_*, v_*) is unstable with respect to the ODE system (2.3), and Theorem 2.2 of [11] shows the existence of at least a stable periodic solution of (2.3). Thus the result stated follows from Lemma 2.5 and Poincaré–Bendixson theory. □

Theorem 2.6 shows that when λ_1 is large (and so the scale of Ω is small) and a is large, the system exhibits asymptotical periodic behavior. The uniqueness of a periodic solution to (2.3) under (2.11) is not known. However the system may possess multiple periodic orbits under other conditions; see Theorem 2.1 of [11].

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