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On stationary patterns of a reaction–diffusion model with autocatalysis and saturation law

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Abstract
Understanding of spatial and temporal behaviour of interacting species or reactants in ecological or chemical systems has become a central issue, and rigorously determining the formation of patterns in models from various mechanisms is of particular interest to applied mathematicians. In this paper, we study a bimolecular autocatalytic reaction–diffusion model with saturation law and are mainly concerned with the corresponding steady-state problem subject to the homogeneous Neumann boundary condition. In particular, we derive some results for the existence and non-existence of non-constant stationary solutions when the diffusion rate of a certain reactant is large or small. The existence of non-constant stationary solutions implies the possibility of pattern formation in this system. Our theoretical analysis shows that the diffusion rate of this reactant and the size of the reactor play decisive roles in leading to the formation of stationary patterns.

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1. Introduction

Natural systems exhibit an amazing diversity of structures in both living and non-living mechanisms, and thereby, in-depth understanding of spatial and temporal behaviour of

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interacting species or reactants in ecological and chemical dynamics has become a central issue. For this purpose, numerous coupled partial differential equations have been proposed by biologists, chemists and applied mathematicians to model problems arising from various disciplines such as population dynamics, genetics and chemical reactions. In these situations, one of the most interesting and natural questions for a model concerned with multi-species interactions is whether the involved species can persist or even stabilize at a co-existence steady state. In the case where the species are homogeneously distributed, this would be indicated by a constant positive solution of an ordinary differential equation system. In the spatially inhomogeneous case, the existence of non-constant time-independent positive solutions, also called stationary patterns, is an indication of the richness of the corresponding partial differential equation dynamics.

In the past decades, the existence of stationary patterns induced by diffusion has attracted the extensive attention of a great number of biologists and mathematicians, and lots of fascinating and important phenomena have been observed. The notion of pattern formation can be dated back to the original work of Turing in 1952. In his seminal paper [27], Turing proposed that diffusion can be regarded as the driving force of the spontaneous emergence of spatiotemporal structures in a variety of non-equilibrium situations. More importantly, recent study shows that stationary patterns have counterparts in natural systems which would provide a plausible way to model the mechanisms of biological growth and chemical reaction. To verify the influence of diffusion on this aspect, much work has been devoted to the investigation of the existence of stationary patterns in chemical and biological dynamics theoretically as well as numerically. Examples of such kind include the Lengyel–Epstein reaction–diffusion system of the CIMA reaction [12, 13, 17, 29] and the Gray–Scott model of autocatalytic chemical reaction [8, 11, 23, 26, 30].

In this work, we deal with a bimolecular autocatalytic reaction–diffusion model with saturation law and attempt to present some qualitative analysis for the corresponding stationary problem. First of all, let us give a brief description regarding the derivation of the system. The reaction process of the model is given by

$$\begin{aligned} &A \xrightarrow{k_1} Y, \\ &X + Y \xrightarrow{k_2} 2X \qquad \text{(autocatalysis)}, \\ &X \xrightarrow{S(k_3,k_4)} P \qquad \text{(saturation law)}, \end{aligned}$$

in which A, X, Y and P are chemical reactants and products, and the system is considered open to in-and-out-flow of A and P. In addition, k_1, k_2, k_3 and k_4 represent the reaction rates and $S(k_3, k_4)$ accounts for the Langmuir–Hinshelwood law in heterogeneous catalysis and adsorption, the Michaelis–Menten law in enzyme-controlled processes and the Holling law in ecology (see, e.g., [2, 10]). It is assumed that all three steps of the reaction process are irreversible and the concentrations of A and P are independent of time and spatial variables, that is, the concentration of these two chemicals is kept uniform throughout the reactor. Disregarding convective phenomena and considering isothermal processes only, the above scheme can be described by the nonlinear partial differential equations

$$\begin{cases} \dfrac{\partial[X]}{\partial t} - D_{[X]}\Delta[X] = k_2[X][Y] - \dfrac{k_3[X]}{1+k_4[X]}, \\ \dfrac{\partial[Y]}{\partial t} - D_{[Y]}\Delta[Y] = k_1[A] - k_2[X][Y], \end{cases}$$

where $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$ is the Laplace operator, showing the spatial dependence of the reaction, $[A]$, $[X]$ and $[Y]$ are the concentrations of A, X and Y, respectively, and $D_{[X]}$ and $D_{[Y]}$ denote the diffusion coefficients which are assumed to be positive constants.

To simplify the reaction–diffusion system, we introduce the following quantities:

$$U = \frac{k_2}{k_3}[X], \qquad V = \frac{k_3[Y]}{k_1[A]}, \qquad \bar{t} = k_3 t,$$

$$\lambda' = \frac{k_1 k_2}{k_3}[A], \qquad k = \frac{k_3 k_4}{k_2}, \qquad d_1 = \frac{D_{[X]}}{k_3}, \qquad d_2 = \frac{D_{[Y]}}{k_3}.$$

As no confusion is to be expected, we drop the upper bar on $\bar{t}$, and thus the reduced system becomes in dimensionless form

$$\begin{cases} U_t - d_1 \Delta U = \lambda' U V - \dfrac{U}{1 + kU} & \text{in } \Omega \times (0, \infty), \\ V_t - d_2 \Delta V = 1 - UV & \text{in } \Omega \times (0, \infty). \end{cases}$$

Here, the reactor $\Omega \subset \boldsymbol{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, U and V represent the concentrations of the two reactants, respectively. The constants d_1, d_2, λ' and k are assumed to be positive.

The main interest of this work is in the stationary patterns generated by the above reaction–diffusion system. This leads us to investigate the associated steady-state problem, which satisfies the following coupled elliptic system:

$$\begin{cases} -d_1 \Delta U = \lambda' U V - \dfrac{U}{1 + kU} & \text{in } \Omega, \\ -d_2 \Delta V = 1 - UV & \text{in } \Omega, \\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$, and we impose a homogeneous Neumann type boundary condition, which implies that (1.1) is a closed system and has no flux across the boundary $\partial\Omega$.

As in [6], we may suppose that $d_1 = 1$ with a scaling of the domain Ω and make a further change of variable to (1.1):

$$u = (\lambda')^{-1} U, \qquad v = d_2 V, \qquad \lambda = d_2^{-1} \lambda' \qquad \text{and} \qquad \mu = k\lambda',$$

then (1.1) is transformed to the problem

$$\begin{cases} -\Delta u = \lambda u v - \dfrac{u}{1 + \mu u} & \text{in } \Omega, \\ -\Delta v = 1 - \lambda u v & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For more details about the chemical background of this model, we refer the interested reader to [2, 6, 10, 25].

For simplicity in later discussion, we set $w = \lambda v$ and so (1.2) takes the form of

$$\begin{cases} -\Delta u = u w - \dfrac{u}{1 + \mu u} & \text{in } \Omega, \\ -\Delta w = \lambda (1 - u w) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

According to the realistic meanings of u and w, only non-negative solutions to (1.3) are of physical interest.

Since our purpose of this work is to derive some results about the existence and non-existence of non-constant positive solutions to (1.1) (equivalently (1.2)), according to the previous variable changes, it suffices to deal with (1.3). In addition, we easily observe that (1.3) has a unique constant positive solution

$$(u, w) = (1/(1-\mu),\ 1-\mu)$$

if and only if $\mu \in (0, 1)$. For the sake of simplicity, throughout this paper, we denote this trivial positive solution by (u^*, w^*).

System (1.1) (or (1.2)) and its corresponding reaction–diffusion dynamics have been studied by several researchers. For example, in [2, 10], in the case that the spatial dimension is one and the homogeneous Neumann boundary condition or the non-homogeneous Dirichlet boundary condition is imposed so that the systems possess a constant positive solution, the authors mainly focused their attention on the effect of large diffusion coefficients and obtained the existence of non-trivial steady-state positive solutions and the asymptotic behaviour of the time-dependent solutions by the use of bifurcation theory and asymptotic expansion approaches. In [25], Ruan discussed the case where the spatial dimension is not necessarily one, and the boundary condition includes the homogeneous Dirichlet type and the Robin type (not including the homogeneous Neumann type) so that the model has no constant steady-state solutions. The author employed the degree theory and bifurcation theory developed by [1, 4] and [3], respectively, to determine the existence and stability of positive steady-state solutions, and also analysed the asymptotic behaviour of time-dependent solutions. In a more recent work [6], when the parameter μ or λ is large, Du investigated (1.2) with the homogeneous Dirichlet boundary condition and obtained a quite satisfactory understanding of the number and stability of the positive solutions by establishing rather delicate *a priori* estimates for positive solutions and applying the regular and singular perturbation theory.

The main results in this paper include both the existence and non-existence for non-constant positive solutions. Our mathematical approach is based on the *a priori* estimates, topological degree theory and asymptotic analysis of the solution's behaviour. Roughly speaking, we prove that for any fixed $\mu \in (0, 1)$, (1.3) has no non-constant positive solution if either λ is small and the size of the reactor Ω is 'small', or λ is sufficiently large; on the other hand, with additional other hypotheses, (1.3) admits at least one non-constant positive solution if λ is small and the size of the reactor is 'properly large' (which is explained below). The existence of non-constant positive solutions indicates the formation of stationary patterns.

Here the size of the reactor is indicated by the first positive Neumann eigenvalue $\mu_1(\Omega)$ of the Laplace operator. It is well known that the eigenvalue problem

$$-\Delta\phi = \zeta\phi \quad \text{in } \Omega, \qquad \partial_\nu\phi = 0 \quad \text{on } \partial\Omega,$$

admits a sequence of eigenvalues $0 = \mu_0(\Omega) < \mu_1(\Omega) < \mu_2(\Omega) < \cdots$. Defining a rescaling of Ω: $\Omega_d = \{y : dy \in \Omega\}$, then it is easy to see that $\mu_1(\Omega_d) = d^2\mu_1(\Omega)$. In that sense one can say that if the first positive eigenvalue of a domain is large, then the size of the domain is small since $|\Omega_d| = d^{-n}|\Omega|$. Hence when the shape of the reactor is fixed (for example, spherical, in many common cases of chemical reactions), then the size of the domain is completely determined by the principal eigenvalue $\mu_1(\Omega)$. However, a recent result of Ni and Wang [18] shows that, in contrast to wide beliefs, it is not true in general that 'smaller domains have larger first positive eigenvalue' for the Neumann boundary problem in dimension $n \geqslant 2$. They showed that for certain $\Omega_1 \supset \Omega_2$, but $\mu_1(\Omega_1) > \mu_1(\Omega_2)$. Thus the 'size of the domain' here should be only understood under a rescaling without changing the domain's geometry when the domain is at least two dimensional. In the practical chemical reactions with spherical reactors, the size of the domain is completely determined by the radius of the reactors. The non-existence result holds for some small domains with the size of the domain also given in the sense of rescaling and in terms of eigenvalues $\mu_i(\Omega)$, and this situation is more delicate as the existence result holds true only for some large domains but not all of them. Taking the one-dimensional case as an example, our result shows that if $\Omega = (0, L)$, then the existence result holds when L belongs to the union of a sequence of non-overlapping intervals which tend to infinity. See theorem 3.4 and remark 3.2 for a more precise description on a domain being 'properly large'.

On the other hand, our non-existence/existence results are also given under conditions on the parameters λ and μ. From our previous changes of variables,

$$\lambda = \frac{k_1 k_2 [A]}{D_{[Y]}} \qquad \text{and} \qquad \mu = k_1 k_4 [A].$$

Since μ is fixed and all reaction rates k_i depend only on the nature of the chemical reactions, λ large (or small) implies that the diffusion coefficient $D_{[Y]}$ is small or large.

Finally we point out that the mathematical approaches adopted in the papers [2, 6, 10, 25] seem not to work in our case of homogeneous Neumann boundary condition and all spatial dimensions. In addition, although one can easily see that the solution to the corresponding reaction–diffusion system of (1.2) or (1.3) with continuous and non-negative initial data globally exists, we are unable to obtain more precise qualitative results about the asymptotical behaviour of the global solution. A deeper understanding of this aspect seems to be a very difficult and interesting mathematical problem and is expected to receive further investigation.

The remaining content of this paper is arranged as follows. In section 2, we collect some basic theorems for elliptic equations and then establish *a priori* upper and lower bounds for non-negative solutions to (1.3). In section 3, we study the non-existence and existence for non-constant solutions with respect to the parameter λ mainly through the topological degree argument.

2. *A priori* estimates for non-negative solutions to (1.3)

In this section, we first recall some general results for elliptic equations, which will become fundamental in obtaining *a priori* upper and lower bounds for non-negative solutions to (1.3). These results can be found in [14] (also see, e.g., [20]), but are still stated here for the reader's convenience.

Before going further, we give the simple fact that (1.3) (equivalently (1.2)) has no non-negative solution if $\mu \geqslant 1$. Indeed, assume that (u, w) is a non-negative solution to (1.3), then integrating the equations in (1.3) and adding the results, we find that

$$\int_\Omega \frac{1 + (\mu - 1)u}{1 + \mu u} \, \mathrm{d}x = 0, \tag{2.1}$$

which leads to a contradiction if $\mu \geqslant 1$. In virtue of this fact, from now on, unless specified otherwise, it is always assumed that $\mu \in (0, 1)$.

First we cite a local result for weak super solution of linear elliptic equations from [14] (also see, for example, [7, theorem 8.18]).

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$. Let Λ be a non-negative constant and suppose that $z \in W^{1,2}(\Omega)$ is a non-negative weak solution of the inequalities*

$$0 \leqslant -\Delta z + \Lambda z \quad \text{in } \Omega, \qquad \partial_\nu z \leqslant 0 \quad \text{on } \partial\Omega.$$

Then, for any $q \in [1, n/(n-2))$, there exists a positive constant C_0, depending only on q, Λ and Ω, such that

$$\|z\|_q \leqslant C_0 \inf_\Omega z.$$

Next is a Harnack inequality for weak solutions, whose strong form was obtained in [15].

Lemma 2.2. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$, and let $c(x) \in L^q(\Omega)$ for some $q > n/2$. If $z \in W^{1,2}(\Omega)$ is a non-negative weak solution of the boundary value problem*

$$\Delta z + c(x)z = 0 \quad \text{in } \Omega, \qquad \partial_\nu z = 0 \quad \text{on } \partial\Omega,$$

then there is a constant C_1, determined only by $\|c\|_q$, q and Ω such that

$$\sup_\Omega z \leqslant C_1 \inf_\Omega z.$$

Finally, we cite a maximum principle for weak solutions due to [14], which is an analogue of proposition 2.2 in [16].

Lemma 2.3. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$, and let $g \in C(\bar{\Omega} \times \boldsymbol{R})$. If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta z + g(x, z) \geqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu z \leqslant 0 \quad \text{on } \partial\Omega,$$

and if there is a constant K such that $g(x, z) < 0$ for $z > K$, then,

$$z \leqslant K \ a.e. \text{ in } \Omega.$$

For convenience we also state a counterpart of lemma 2.3 as follows.

Lemma 2.4. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$, and let $g \in C(\bar{\Omega} \times \boldsymbol{R})$. If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta z + g(x, z) \leqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu z \geqslant 0 \quad \text{on } \partial\Omega,$$

and if there is a constant K such that $g(x, z) > 0$ for $z < K$, then,

$$z \geqslant K \ a.e. \text{ in } \Omega.$$

Based on the above preparation, we are ready to derive *a priori* upper and lower bounds for all non-negative solutions to (1.3). More precisely, we have

Theorem 2.1. *For any given $\mu^* \in (0, 1)$, there exist two positive constants $\underline{C}$ and $\overline{C}$ with $\underline{C} < \overline{C}$ depending only on μ^* and Ω, such that any non-negative $W^{1,2}(\Omega)$ solution (u, w) to (1.3) satisfies*

$$\underline{C} \leqslant u(x), \quad w(x) \leqslant \overline{C}, \qquad for\ any\ x \in \bar{\Omega},$$

provided that $0 < \mu \leqslant \mu^*$.

Proof. Our argument is similar to that in the proof of theorem 3.1 in [14]. First of all, using (2.1), we note that

$$|\Omega| = \int_\Omega \frac{u}{1 + \mu u}\, \mathrm{d}x \leqslant \int_\Omega u\, \mathrm{d}x. \tag{2.2}$$

Moreover, it is easy to see that $0 \leqslant -\Delta u + u$ in Ω, and hence lemmas 2.1 and (2.2) yield

$$|\Omega| \leqslant \int_\Omega u\, \mathrm{d}x \leqslant C_0 \inf_\Omega u, \tag{2.3}$$

where C_0 takes the value corresponding to $q = 1$ in lemma 2.1 and so depends only on Ω. It follows from (2.3) that $u \geqslant \underline{C}$ in Ω.

Since

$$\Delta w + \lambda(1 - \underline{C}w) \geqslant \Delta w + \lambda(1 - uw) = 0,$$

we see that $w \leqslant 1/\underline{C}$ due to lemma 2.3.

As a consequence, together with lemma 2.2, the equation for u implies that there is a positive constant C_1 depending only on Ω, such that

$$\sup_{\Omega} u \leqslant C_1 \inf_{\Omega} u. \tag{2.4}$$

To estimate the upper bound of u, we adopt a contradiction argument. Suppose that there exists a non-negative solution sequence $\{(u_i, w_i)\}_{i=1}^{\infty}$ to (1.3) such that $\|u_i\|_\infty \to \infty$ as $i \to \infty$. Then, (2.4) implies $u_i \to \infty$ uniformly on $\bar{\Omega}$. Thanks to (2.2) and $\mu \leqslant \mu^* < 1$, we also have

$$\frac{|\Omega|}{\mu^*} \leftarrow \int_{\Omega} \frac{u_i}{1+\mu^* u_i}\,\mathrm{d}x \leqslant \int_{\Omega} \frac{u_i}{1+\mu u_i}\,\mathrm{d}x = |\Omega|.$$

An obvious contradiction. Hence, we can find the desired $\overline{C} > 0$ such that $u \leqslant \overline{C}$. Therefore, similarly to the above analysis, lemma 2.4 deduces $1/\overline{C} \leqslant w$ in Ω. The proof is complete. □

3. Existence and non-existence of non-constant positive solutions to (1.3)

This section is devoted to the non-existence and existence of non-constant positive solutions to (1.3). The main approach is the Leray–Schauder degree theory for compact operators in Banach spaces. Note that we have assumed that Ω is smooth. Thus, from theorem 2.1, the standard regularity theory for elliptic equations and embedding theorems (see, e.g., [7]), we can conclude that any weak $W^{1,2}(\Omega)$ solution (u, w) to (1.3) must be a classical one, that is, (u, w) satisfies (1.3) in $[C^2(\Omega) \cap C^1(\bar{\Omega})]^2$.

Recall that

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots$$

are the eigenvalues of the Laplace operator $-\Delta$ on Ω with homogeneous Neumann boundary condition, and denote $E(\mu_i)$ to be the eigenspace corresponding to μ_i in $W^{1,2}(\Omega)$. Let

$$\boldsymbol{X} = [W^{1,2}(\Omega)]^2,$$

and $\{\phi_{ij} : j = 1, \ldots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\boldsymbol{X}_{ij} = \{\boldsymbol{c}\phi_{ij} : \boldsymbol{c} \in \boldsymbol{R}^2\}$. Then,

$$\boldsymbol{X} = \bigoplus_{i=0}^{\infty} \boldsymbol{X}_i \qquad \text{and} \qquad \boldsymbol{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \boldsymbol{X}_{ij}.$$

3.1. Non-existence of non-constant positive solutions

In this subsection, we establish some results of non-existence of non-constant positive solutions in the following two cases: either both λ and the size of the reactor Ω are small or λ is large. From the remarks in the introduction, the size of Ω is characterized by the first positive eigenvalue μ_1.

First of all, using the integration estimates, we prove the following result:

Theorem 3.1. *Let $\mu_1 > 1$, then for any given $\mu^* \in (0, 1)$, there exists a small $\epsilon_0 > 0$ depending only on μ^* and Ω, such that (1.3) has no non-constant positive solution provided that $\mu \leqslant \mu^*$ and $\lambda \leqslant \epsilon_0$.*

Proof. Let (u, w) be a positive solution of (1.3), and for any function g, denote $\hat{g} = |\Omega|^{-1}\int_\Omega g\,\mathrm{d}x$. Then

$$\int_\Omega (u-\hat{u})\,\mathrm{d}x = \int_\Omega (w-\hat{w})\,\mathrm{d}x = 0.$$

Multiplying the first equation in (1.3) by $(u-\hat{u})$, we get

$$\begin{aligned}\int_\Omega |\nabla(u-\hat{u})|^2\,\mathrm{d}x &= \int_\Omega \left(uw - \frac{u}{1+\mu u}\right)(u-\hat{u})\,\mathrm{d}x\\ &= \int_\Omega \left(uw - u\hat{w} + u\hat{w} - \hat{u}\hat{w} + \frac{\hat{u}}{1+\mu\hat{u}} - \frac{u}{1+\mu u}\right)(u-\hat{u})\,\mathrm{d}x\\ &= \int_\Omega \left\{u(u-\hat{u})(w-\hat{w}) + \hat{w}(u-\hat{u})^2 - \frac{1}{(1+\mu\hat{u})(1+\mu u)}(u-\hat{u})^2\right\}\mathrm{d}x\\ &\leqslant \int_\Omega \{u|u-\hat{u}||w-\hat{w}| + \hat{w}(u-\hat{u})^2\}\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

In a similar manner, combined with theorem 2.1, we multiply the second equation in (1.3) by $(w-\hat{w})$ to have

$$\begin{aligned}\int_\Omega |\nabla(w-\hat{w})|^2\,\mathrm{d}x &= \lambda\int_\Omega (1-uw)(w-\hat{w})\,\mathrm{d}x\\ &= \lambda\int_\Omega (\hat{u}\hat{w} - uw)(w-\hat{w})\,\mathrm{d}x\\ &= -\lambda\int_\Omega \hat{u}(w-\hat{w})^2\,\mathrm{d}x - \lambda\int_\Omega w(u-\hat{u})(w-\hat{w})\,\mathrm{d}x\\ &\leqslant -\underline{C}\lambda\int_\Omega (w-\hat{w})^2\,\mathrm{d}x + \overline{C}\lambda\int_\Omega |u-\hat{u}||w-\hat{w}|\,\mathrm{d}x, \end{aligned} \tag{3.2}$$

where $\underline{C}$ and $\overline{C}$ are given in theorem 2.1 and both of them are independent of λ.

Thus, with (3.2) and the well-known Poincaré inequality:

$$\mu_1\int_\Omega (g-\hat{g})^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla(g-\hat{g})|^2\,\mathrm{d}x,$$

we find that

$$\begin{aligned}(\mu_1 + \underline{C}\lambda)\int_\Omega (w-\hat{w})^2\,\mathrm{d}x &\leqslant \overline{C}\lambda\int_\Omega |u-\hat{u}||w-\hat{w}|\,\mathrm{d}x\\ &\leqslant \overline{C}\lambda\left(\int_\Omega (u-\hat{u})^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega (w-\hat{w})^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

If $w \equiv \hat{w}$ which is a positive constant, then it follows from the second equation in (1.3) that u must also be a positive constant. And so, $(u, w) = (u^*, w^*)$.

We now assume that $w \not\equiv \hat{w}$. Thus, the above inequality directly leads to

$$(\mu_1 + \underline{C}\lambda)^2\int_\Omega (w-\hat{w})^2\,\mathrm{d}x \leqslant (\overline{C})^2\lambda^2\int_\Omega (u-\hat{u})^2\,\mathrm{d}x. \tag{3.3}$$

In this case, we first need to claim: for arbitrary small $\delta > 0$, there exists a small $\epsilon_0 > 0$ depending only on δ, μ^* and Ω such that if $\lambda < \epsilon_0$, then

$$\hat{w} < 1 + \delta. \tag{3.4}$$

To emphasize the dependence of solutions on λ, we use (u_λ, w_λ) instead of (u, w) here. Let $\{\lambda_i\}$ be a sequence such that $\lambda_i \to 0^+$ as $i \to \infty$, and let $(u_{\lambda_i}, w_{\lambda_i})$ be a sequence of solutions of (1.3). By applying the standard regularity theory for elliptic equations and imbedding theorems to the second equation in (1.3), together with theorem 2.1, we see that there is a subsequence of $\{\lambda_i\}$ (still denoted by $\{\lambda_i\}$) such that $w_{\lambda_i} \to \tilde{w}$ in $C^2(\bar{\Omega})$ as $i \to \infty$, where $\tilde{w}$ is a positive constant, and hence $\hat{w}_{\lambda_i} \to \tilde{w}$. Furthermore, by theorem 2.1 again and the first equation in (1.3), we can find a further subsequence of $\{u_{\lambda_i}\}$, still denoted by itself, such that $u_{\lambda_i} \to \tilde{u}$ in $C^2(\bar{\Omega})$ as $i \to \infty$, and $\tilde{u}$ is a positive solution of

$$-\Delta z = \frac{z[(\tilde{w}-1)+\mu\tilde{w}z]}{1+\mu z} \quad \text{in } \Omega, \qquad \partial_\nu z = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

Then, we integrate (3.5) over Ω to obtain

$$\int_\Omega \frac{z[(\tilde{w}-1)+\mu\tilde{w}z]}{1+\mu z}\,\mathrm{d}x = 0,$$

which obviously implies $\tilde{w} < 1$ and thus $\hat{w}_{\lambda_i} < 1+\delta$ for all large i. Therefore the claim (3.4) is true for all small λ.

By virtue of theorem 2.1, (3.1), (3.3), (3.4) and the Poincaré inequality again, for any given small $\delta > 0$ we have

$$\mu_1 \int_\Omega (u-\hat{u})^2\,\mathrm{d}x \leqslant \left[\frac{(\overline{C})^2\lambda}{\mu_1+\underline{C}\lambda} + 1 + \delta\right] \int_\Omega (u-\hat{u})^2\,\mathrm{d}x, \tag{3.6}$$

for $\lambda \leqslant \epsilon_0$. Thanks to (3.6), by taking $\delta = (\mu_1 - 1)/2$ and ϵ_0 to be smaller if necessary, it is clear that $u \equiv \hat{u}$ when $1 < \mu_1$ and $\lambda \leqslant \epsilon_0$ are satisfied, which in turn indicates $w \equiv \hat{w}$ by (3.3).

To sum up, the above analysis concludes that in any case, $(u, w) = (\hat{u}, \hat{w})$ if $1 < \mu_1$ and $\lambda \leqslant \epsilon_0$ where $\epsilon_0 > 0$ is chosen as above. Hence, it is necessary that $(u, w) = (u^*, w^*)$ and theorem 3.1 follows. □

In the interpretation of domain size by the first positive eigenvalue in the introduction, for any domain Ω, we can always choose d large so that $\mu_1(\Omega_d) > 1$ so our theorem 3.2 holds for such small domain Ω_d.

In order to prove the non-existence of non-constant positive solutions in the case of large λ, we need to make some necessary preparations. To begin with, we shall analyse the asymptotical stability of the unique positive constant solution (u^*, w^*) for the corresponding reaction–diffusion dynamics of (1.3):

$$\begin{cases} u_t - \Delta u = uw - \dfrac{u}{1+\mu u} & \text{in } \Omega\times(0,\infty), \\ w_t - \Delta w = \lambda(1-uw) & \text{in } \Omega\times(0,\infty), \\ \partial_\nu u = \partial_\nu w = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{3.7}$$

with the admissible non-negative and continuous initial data $u(x, 0)$ and $w(x, 0)$ on $\bar{\Omega}$.

The linearized problem of (3.7) at (u^*, w^*) satisfies

$$\begin{cases} u_t - \Delta u = \mu(1-\mu)u + (1-\mu)^{-1}w & \text{in } \Omega\times(0,\ \infty), \\ w_t - \Delta w = -\lambda(1-\mu)u - \lambda(1-\mu)^{-1}w & \text{in } \Omega\times(0,\ \infty), \\ \partial_\nu u = \partial_\nu w = 0 & \text{on } \partial\Omega\times(0,\ \infty). \end{cases}$$

Lemma 3.1. *Assume that $\lambda > \mu(1-\mu)^2$, then the constant solution (u^*, w^*) is uniformly asymptotically stable as the steady state of the reaction–diffusion system* (3.7).

Proof. Denote

$$\boldsymbol{L} = \begin{pmatrix} \Delta + \mu(1-\mu) & (1-\mu)^{-1} \\ -\lambda(1-\mu) & \Delta - \lambda(1-\mu)^{-1} \end{pmatrix}.$$

For each i, $i = 0, 1, 2, \ldots, \boldsymbol{X}_i$ is invariant under the operator $\boldsymbol{L}$, and ξ is an eigenvalue of $\boldsymbol{L}$ on $\boldsymbol{X}_i$ if and only if ξ is an eigenvalue of the matrix

$$\boldsymbol{A}_i = \begin{pmatrix} -\mu_i + \mu(1-\mu) & (1-\mu)^{-1} \\ -\lambda(1-\mu) & -\mu_i - \lambda(1-\mu)^{-1} \end{pmatrix}.$$

The direct calculation shows

$$\det \boldsymbol{A}_i = \mu_i^2 + [\lambda(1-\mu)^{-1} - \mu(1-\mu)]\mu_i + \lambda(1-\mu)$$

and

$$\mathrm{Tr}\, \boldsymbol{A}_i = -2\mu_i - \lambda(1-\mu)^{-1} + \mu(1-\mu),$$

where $\det \boldsymbol{A}_i$ and $\mathrm{Tr}\, \boldsymbol{A}_i$ are, respectively, the determinant and trace of $\boldsymbol{A}_i$. Under our assumption, it is easy to check that $\det \boldsymbol{A}_i > 0$ and $\mathrm{Tr}\, \boldsymbol{A}_i < 0$ for any $i \geqslant 0$. Therefore, the two eigenvalues ξ_i^+ and ξ_i^- have negative real parts. Moreover, for any $i \geqslant 0$, the following hold:

(i) If $(\mathrm{Tr}\, \boldsymbol{A}_i)^2 - 4\det \boldsymbol{A}_i \leqslant 0$, then

$\mathrm{Re}\, \xi_i^{\pm} = \frac{1}{2}\mathrm{Tr}\, \boldsymbol{A}_i \leqslant \frac{1}{2}[-\lambda(1-\mu)^{-1} + \mu(1-\mu)] < 0$;

(ii) If $(\mathrm{Tr}\, \boldsymbol{A}_i)^2 - 4\det \boldsymbol{A}_i > 0$, then

$$\mathrm{Re}\, \xi_i^- = \tfrac{1}{2}\{\mathrm{Tr}\, \boldsymbol{A}_i - \sqrt{(\mathrm{Tr}\, \boldsymbol{A}_i)^2 - 4\det \boldsymbol{A}_i}\,\} \leqslant \tfrac{1}{2}\mathrm{Tr}\, \boldsymbol{A}_i \leqslant \tfrac{1}{2}[-\lambda(1-\mu)^{-1} + \mu(1-\mu)] < 0,$$

$$\mathrm{Re}\, \xi_i^+ = \frac{1}{2}\{\mathrm{Tr}\, \boldsymbol{A}_i + \sqrt{(\mathrm{Tr}\, \boldsymbol{A}_i)^2 - 4\det \boldsymbol{A}_i}\,\} = \frac{2\det \boldsymbol{A}_i}{\mathrm{Tr}\, \boldsymbol{A}_i - \sqrt{(\mathrm{Tr}\, \boldsymbol{A}_i)^2 - 4\det \boldsymbol{A}_i}} \leqslant \frac{\det \boldsymbol{A}_i}{\mathrm{Tr}\, \boldsymbol{A}_i} < -\delta,$$

for some positive δ which is independent of i.

This shows that there exists a positive constant δ, which is independent of i, such that

$$\mathrm{Re}\, \xi_i^{\pm} < -\delta, \qquad \forall i \geqslant 0.$$

Consequently, the spectrum of $\boldsymbol{L}$ lies in $\{\xi : \mathrm{Re}\, \xi < -\delta\}$ (since the spectrum of $\boldsymbol{L}$ only consists of eigenvalues). By theorem 5.1.1 of [9, p 98] the conclusion follows and thus the proof is complete. □

In what follows, we present two other simple but useful conclusions, whose proofs are given in [24] and [22], respectively.

Lemma 3.2. *Assume that $\lambda > 0$ is a constant and $b(x)$ is a continuous positive function on $\bar{\Omega}$. Then, the following problem:*

$$-\Delta z = \lambda(1 - b^{-1}(x)z) \quad \text{in } \Omega, \qquad \partial_\nu z = 0 \quad \text{on } \partial\Omega,$$

has a unique positive solution z_λ, and $z_\lambda \to b(x)$ uniformly on $\bar{\Omega}$ as $\lambda \to \infty$.

Lemma 3.3. *Assume that $f(z)$ is a continuous real function in $[0, \infty)$ and for some positive constant a, $f(z) > 0$ in $(0, a)$ and $f(z) < 0$ in (a, ∞). Then the following problem:*

$$-\Delta z = zf(z) \quad \text{in } \Omega, \qquad \partial_\nu z = 0 \quad \text{on } \partial\Omega$$

has a unique positive classical solution $z(x) = a$.

Using lemmas 3.2 and 3.3, together with theorem 2.1 and lemma 3.2 of [22], we can provide the asymptotic behaviour of positive solutions to (1.3) as $\lambda \to \infty$:

Lemma 3.4. *Assume that (u_λ, w_λ) is a positive solution to (1.3), then $(u_\lambda, w_\lambda) \to (u^*, w^*)$ uniformly on $\bar{\Omega}$ as $\lambda \to \infty$.*

Proof. Assume that (u_λ, w_λ) is a positive solution of (1.3) and let $\lambda \to \infty$. With the help of theorem 2.1, by the first equation of (1.3), the standard theory follows that there is a sequence $\{u_{\lambda_i}\}_{i=1}^{\infty}$ of u_λ satisfying $u_{\lambda_i} \to u$ in $C^1(\bar{\Omega})$ with $u > 0$ on $\bar{\Omega}$ as $i \to \infty$.

For arbitrary small $\epsilon > 0$, consider the following two auxiliary problems:

$$-\Delta \overline{w} = \lambda_i (1 - (u - \epsilon)\overline{w}) \quad \text{in } \Omega, \qquad \partial_\nu \overline{w} = 0 \quad \text{on } \partial\Omega \tag{3.8}$$

and

$$-\Delta \underline{w} = \lambda_i (1 - (u + \epsilon)\underline{w}) \quad \text{in } \Omega, \qquad \partial_\nu \underline{w} = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

As w_{λ_i} solves

$$-\Delta w_{\lambda_i} = \lambda_i (1 - u_{\lambda_i} w_{\lambda_i}) \quad \text{in } \Omega, \qquad \partial_\nu w_{\lambda_i} = 0 \quad \text{on } \partial\Omega,$$

simple comparison arguments yield that for any large i,

$$\underline{w}_{\lambda_i} < w_{\lambda_i} < \overline{w}_{\lambda_i},$$

where $\overline{w}_{\lambda_i}$ and $\underline{w}_{\lambda_i}$ are, respectively, the unique positive solution to (3.8) and (3.9).

On the other hand, we can conclude from lemma 3.2 that

$$\underline{w}_{\lambda_i} \to 1/(u + \epsilon) \qquad \text{and} \qquad \overline{w}_{\lambda_i} \to 1/(u - \epsilon), \quad \text{uniformly on } \bar{\Omega}.$$

Therefore, in view of the arbitrariness of ϵ, we have $w_{\lambda_i} \to 1/u$ uniformly on $\bar{\Omega}$. By the equation for u_{λ_i} again, thanks to lemma 3.3, it is easily seen that

$$u_{\lambda_i} \to 1/(1 - \mu) \qquad \text{and} \qquad w_{\lambda_i} \to 1 - \mu \text{ uniformly on } \bar{\Omega}, \quad \text{as } i \to \infty.$$

Clearly, our analysis also verifies that $(u_\lambda, w_\lambda) \to (u^*, w^*)$ uniformly on $\bar{\Omega}$ as $\lambda \to \infty$. The proof is thus ended. □

We observe that all of the conclusions obtained before hold for $\mu = 0$. Therefore, from now on, our discussion includes the case of $\mu = 0$. To establish the non-existence result for non-constant positive solutions to (1.3) for any fixed $\mu \in [0, 1)$ and large λ, our strategy consists of two steps: we first prove this claim holds in the case of $\mu = 0$ and large λ via integration estimates; based on this result we then use the topological degree theory to finish the whole proof.

As the first step, we now prove that system (1.3) admits no non-constant positive solution when $\mu = 0$ and λ is large enough. That is, the following holds:

Theorem 3.2. *Let $\mu = 0$, then there exists a large $\Lambda_0 > 0$ depending only on Ω, such that (1.3) has no non-constant positive solution provided that $\lambda \geqslant \Lambda_0$.*

Proof. To the end, we shall adopt an indirect argument. Suppose that there exists a sequence $\{\lambda_i\}$ with $\lambda_i \to \infty$ as $i \to \infty$ and the corresponding non-constant positive solution (u_i, w_i) to (1.3) for $\lambda = \lambda_i$, such that our claim is false. Then, lemma 3.4 shows that $(u_i, w_i) \to (u^*, w^*) = (1, 1)$ uniformly on $\bar{\Omega}$ as $i \to \infty$.

Now, we can define

$$h_i = u_i - 1, \qquad k_i = w_i - 1,$$

and

$$\tilde{h}_i = \frac{h_i}{\|h_i\|_\infty + \|k_i\|_\infty}, \qquad \tilde{k}_i = \frac{k_i}{\|h_i\|_\infty + \|k_i\|_\infty}.$$

Thus, it is not hard to find that $(\tilde{h}_i, \tilde{k}_i)$ satisfies

$$-\Delta \tilde{h}_i = u_i \tilde{k}_i \quad \text{in } \Omega, \qquad \partial_\nu \tilde{h}_i = 0 \quad \text{on } \partial\Omega \tag{3.10}$$

and

$$-\Delta\tilde{k}_i = -\lambda_i\tilde{h}_i - \lambda_i u_i\tilde{k}_i \quad \text{in } \Omega, \qquad \partial_\nu\tilde{k}_i = 0 \text{ on } \partial\Omega. \tag{3.11}$$

Noting that the right side of (3.10) is L^∞ bounded by theorem 2.1 and $\|\tilde{h}_i\|_\infty, \|\tilde{k}_i\|_\infty \leqslant 1$, it follows from the L^p estimates and the Sobolev embedding theorems that there is a subsequence of $\tilde{h}_i$, labelled by itself again, such that $\tilde{h}_i \to \tilde{h}$ in $C^1(\bar{\Omega})$. In addition, $\|\tilde{k}_i\|_\infty \leqslant 1$ guarantees there exists a subsequence of $\tilde{k}_i$, still denoted by itself, such that $\tilde{k}_i \to \tilde{k}$ weakly in $L^2(\Omega)$ with $\tilde{k} \in L^\infty(\Omega)$.

Since $(u_i, v_i) \to (1, 1)$ uniformly on $\bar{\Omega}$ as $i \to \infty$, passing to the limit in (3.10), we have

$$-\Delta\tilde{h} = \tilde{k} \quad \text{in } \Omega, \qquad \partial_\nu\tilde{h} = 0 \quad \text{on } \partial\Omega, \tag{3.12}$$

Then, multiplying the above equation by $\tilde{h}$ and integrating, we yield from the above equation that

$$0 \leqslant \int_\Omega |\nabla\tilde{h}|^2\,dx = \int_\Omega \tilde{h}\tilde{k}\,dx.$$

Combined with (3.11), this easily follows that

$$\int_\Omega \tilde{k}_i^2\,dx \to 0 \qquad \text{and} \qquad \int_\Omega \tilde{h}\tilde{k}\,dx = 0. \tag{3.13}$$

The first limit of (3.13) implies $\tilde{k}_i \to 0$ in $L^2(\Omega)$ and so $\tilde{k} = 0$ a.e. in Ω. Moreover, by choosing a further subsequence if necessary, we have $\tilde{k}_i \to 0$ a.e. in Ω, which, in turn, deduces $\|\tilde{h}\|_\infty = 1$. Thanks to (3.12), it is necessary that

$$\tilde{h} = 1 \qquad \text{or} \qquad \tilde{h} = -1.$$

On the other hand, multiplying (3.10) and (3.11) by $\tilde{k}_i$ and $\tilde{h}_i$, respectively, integrating over Ω, and then subtracting the results, we find

$$-(\lambda_i)^{-1}\int_\Omega u_i\tilde{k}_i^2\,dx = \int_\Omega \tilde{h}_i^2\,dx + \int_\Omega u_i\tilde{h}_i\tilde{k}_i\,dx. \tag{3.14}$$

As $i \to \infty$, by (3.13) again, the left side of (3.14) converges to zero whereas the right side converges to $|\Omega|$. Hence, we reach a contradiction, and this finishes our proof. □

We next present and prove that (1.3) has no non-constant positive solution for any $\mu \in (0, 1)$ if λ is sufficiently large.

Theorem 3.3. *For any fixed $\mu^* \in (0, 1)$, there exists a large $\Lambda > 0$, which depends only on μ^* and Ω, such that (1.3) has no non-constant positive solution provided that $\mu \leqslant \mu^*$ and $\Lambda \leqslant \lambda$.*

Proof. To achieve this aim, we need to reformulate system (1.3) in the framework that the Leray–Schauder degree theory can be easily applied.

Let us denote

$$\Theta = \{(u, w) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \underline{C} < u,\ w < \overline{C}\},$$

where $\underline{C}$ and $\overline{C}$ are given by theorem 2.1. Thus, for such $\underline{C}$ and $\overline{C}$, (1.3) has no positive solution $(u, w) \in \partial\Theta$. We also define the operator

$$\boldsymbol{B}(\lambda, \mu, u, w) = (-\Delta + \boldsymbol{I})^{-1}\left(u + uw - \frac{u}{1 + \mu u},\ w + \lambda(1 - uw)\right),$$

where $(-\Delta + \boldsymbol{I})^{-1}$ stands for the inverse operator of $-\Delta + \boldsymbol{I}$ subject to Neumann boundary condition over $\partial\Omega$.

It is well known that $\boldsymbol{B}$ is a compact operator from $[\underline{\lambda}, \bar{\lambda}] \times [\underline{\mu}, \bar{\mu}] \times \Theta$ to $C(\bar{\Omega}) \times C(\bar{\Omega})$, where $\underline{\lambda}, \bar{\lambda} \in (0, \infty)$ and $\underline{\mu}, \bar{\mu} \in [0, \mu^*]$. Furthermore, $(u, w) \in \Theta$ solves (1.2) if and only if (u, w) satisfies

$$(u, w) = \boldsymbol{B}(\lambda, \mu, u, w).$$

In addition,

$$(u, w) \neq \boldsymbol{B}(\lambda, \mu, u, w), \qquad \forall\lambda \in (0, \infty), \qquad \mu \in [0, \mu^*] \quad \text{and } (u, w) \in \partial\Theta.$$

As a result, the topological degree $\deg(\boldsymbol{I} - \boldsymbol{B}(\lambda, \mu, \cdot), \Theta, 0)$ is well defined, which is also independent of $\lambda \in (0, \infty)$ and $\mu \in [0, \mu^*]$.

We recall that theorem 3.2 shows that $(\Lambda_0, 0, 1, 1)$ is the unique fixed point of $\boldsymbol{B}(\Lambda_0, 0, \cdot)$ in Θ, and thus

$$\deg(\boldsymbol{I} - \boldsymbol{B}(\Lambda_0, 0, \cdot), \Theta, 0) = \text{index}\,(\boldsymbol{I} - \boldsymbol{B}(\Lambda_0, 0, \cdot), (1, 1)).$$

In addition, lemma 3.1 claims that $(1, 1)$ is non-degenerate (namely, zero is not the eigenvalue of the linearized problem of (3.7) at $(1, 1)$) and is also linearly stable as the unique constant steady-state positive solution of (3.7) with $\mu = 0$. Hence, by the well-known Leray–Schauder degree formula (see, e.g., theorem 2.8.1 in [19] or proposition 3.1 in the forthcoming subsection), we get

$$\deg(\boldsymbol{I} - \boldsymbol{B}(\Lambda_0, 0, \cdot), \Theta, 0) = \text{index}\,(\boldsymbol{I} - \boldsymbol{B}(\Lambda_0, 0, \cdot), (1, 1)) = 1.$$

Therefore, it follows that

$$\deg(\boldsymbol{I} - \boldsymbol{B}(\lambda, \mu, \cdot), \Theta, 0) = \deg(\boldsymbol{I} - \boldsymbol{B}(\Lambda_0, 0, \cdot), \Theta, 0) = 1,$$

for any $\lambda \in (0, \infty)$ and $\mu \in [0, \mu^*]$.

According to the proof of lemma 3.1, combined with lemma 3.4, it is easy to see that there exists a large $\Lambda > 0$ such that every possible positive solution (u_λ, w_λ) of (1.3) is non-degenerate and linearly stable provided that $\lambda \geqslant \Lambda$. Hence, the fixed point index of $\boldsymbol{B}(\lambda, \mu, u_\lambda, w_\lambda)$ is well defined and is equal to 1 if $\lambda \geqslant \Lambda$. Furthermore, for such fixed λ and μ, by the compactness of $\boldsymbol{B}(\lambda, \mu, \cdot)$, we can show that there are at most finitely many such fixed points in Θ, denoted by $\{(u_i, w_i)\}_1^\ell$. Then, from the property of the Leray–Schauder degree, it follows that

$$1 = \deg(\boldsymbol{I} - \boldsymbol{B}(\lambda, \mu, \cdot), \Theta, 0) = \Sigma_1^\ell\, \text{index}\,(\boldsymbol{I} - \boldsymbol{B}(\lambda, \mu, \cdot), (u_i, w_i)) = \ell.$$

This indicates the uniqueness of positive solutions of (1.3) for $\lambda \geqslant \Lambda$ and it must be (u^*, w^*). Our proof is complete. □

Remark 3.1. It should be noted that one can apply the analysis in this subsection to the following well-known Sel'kov model:

$$\begin{cases} -\theta\Delta u = \lambda(1 - uv^p) & \text{in } \Omega, \\ -\Delta v = \lambda(uv^p - v) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where θ, λ and p are positive constants. Recently, many works have been devoted to the mathematical analysis of this model; please refer to [5, 14, 20, 28] for the details of the discussion. Note that, when $\lambda = p = 1$, (3.15) becomes problem (1.3) with $\mu = 0$. By theorem 2.1 in [28], the unique constant solution $(1, 1)$ is uniformly asymptotically stable if $p < 2$ and $\lambda(p - 1) \leqslant \mu_1$. In addition, as stated in remark 4.2 of [20], for any fixed p and λ, every positive solution to (3.15) converges to $(1, 1)$ uniformly on $\bar{\Omega}$ as $\theta \to 0$. As a result, using these results and the *a priori* estimates in [14], we can prove that (3.15) has no constant positive solution if $p < 2$, $\lambda(p - 1) \leqslant \mu_1$ and θ is small enough. This partially solves the open problem left in remark 4.2 of [20].

3.2. Existence of non-constant positive solutions

In view of the non-existence result for small parameter λ and small reactor Ω or arbitrary large λ in the previous subsection, it is particularly interesting to know whether (1.3) possesses non-homogeneous positive solutions when the parameter λ takes on small values and the size of the reactor is properly chosen. In this subsection, we shall give an affirmative answer to this question. The result of existence implies that there exists a certain range of parameters such that the two reactants in (1.3) are spatially non-homogeneously distributed. Together with theorem 3.3, our conclusion also demonstrates that the parameter λ and the size of the reactor really play critical roles in leading to the generation of non-uniform steady states (namely, stationary patterns) in (1.3) or (1.2).

For our later purpose, let us define

$$\boldsymbol{Y} = \{(u,\ w) \in [C^2(\bar{\Omega})]^2\ :\ \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega\},$$

let Θ be as in section 3.1, and also denote

$$\boldsymbol{u} = (u, w), \qquad \boldsymbol{u}^* = (u^*, w^*), \qquad \boldsymbol{Y}^+ = \{\boldsymbol{u} \in \boldsymbol{Y} : u,\ w > 0 \quad \text{on } \bar{\Omega}\}$$

and

$$\boldsymbol{G}(\boldsymbol{u}) = \begin{pmatrix} uw - u(1+\mu u)^{-1} \\ \lambda(1-uw) \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} \mu(1-\mu) & (1-\mu)^{-1} \\ -\lambda(1-\mu) & -\lambda(1-\mu)^{-1} \end{pmatrix}.$$

Thus, $D_{\boldsymbol{u}}\boldsymbol{G}(\boldsymbol{u}^*) = \boldsymbol{A}$, and (1.3) can be written as

$$-\Delta \boldsymbol{u} = \boldsymbol{G}(\boldsymbol{u}) \quad \text{in } \Omega, \qquad \partial_\nu \boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{3.16}$$

Then $\boldsymbol{u}$ is a positive solution of (3.16) if and only if

$$\boldsymbol{F}(\boldsymbol{u}) \equiv \boldsymbol{u} - (-\Delta + \boldsymbol{I})^{-1}\{\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{u}\} = 0 \qquad \text{in } \boldsymbol{Y}^+.$$

Note that $\boldsymbol{F}(\cdot)$ is a compact perturbation of the identity operator, and so the Leray–Schauder degree $\deg(\boldsymbol{F}(\cdot), \Theta, 0)$ is well defined because of $\boldsymbol{F}(\boldsymbol{u}) \neq 0$ on $\partial\Theta$. Furthermore, we observe that

$$D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*) = \boldsymbol{I} - (-\Delta + \boldsymbol{I})^{-1}(\boldsymbol{A} + \boldsymbol{I}),$$

and recall that if $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ is invertible, the index of $\boldsymbol{F}$ at $\boldsymbol{u}^*$ is defined as

$$\text{index}\ (\boldsymbol{F}(\cdot), \boldsymbol{u}^*) = (-1)^\gamma,$$

where γ is the number of negative eigenvalues of $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ [19, theorem 2.8.1].

Let $\boldsymbol{X}$ and $\boldsymbol{X}_i$ be defined as before. Since the eigenvalues of $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ and their algebraic multiplicities are the same regardless of whether it is considered as an operator in $\boldsymbol{X}$ or in $\boldsymbol{Y}$, it is convenient to use the decomposition made in section 3.1 in our discussion of the eigenvalues of $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$. A straightforward calculation shows that, for each integer $i \geqslant 0$, $\boldsymbol{X}_i$ is invariant under $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$, and ξ is an eigenvalue of $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ on $\boldsymbol{X}_i$ if and only if it is an eigenvalue of the matrix

$$\frac{1}{1+\mu_i}(\mu_i \boldsymbol{I} - \boldsymbol{A}).$$

Thus, $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ is invertible if and only if, for all $i \geqslant 0$, the matrix $\mu_i\boldsymbol{I} - \boldsymbol{A}$ is non-singular. Denote

$$\boldsymbol{H}(\sigma) = \det(\sigma\, \boldsymbol{I} - \boldsymbol{A}),$$

we also have that, if $\boldsymbol{H}(\mu_i) \neq 0$, the number of negative eigenvalues of $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}^*)$ on $\boldsymbol{X}_i$ is odd if and only if $\boldsymbol{H}(\mu_i) < 0$.

Let $m(\mu_i)$ be the algebraic multiplicity of μ_i. By the same arguments as in [17, 21, 28], we can assert the following:

Proposition 3.1. *Suppose that, for all $i \geqslant 0$, the matrix $\mu_i \boldsymbol{I} - \boldsymbol{A}$ is non-singular. Then*

$$\mathrm{index}(\boldsymbol{F}(\cdot),\ \boldsymbol{u}^*) = (-1)^{\gamma}, \qquad \text{where } \gamma = \sum_{i\geqslant 0,\, \boldsymbol{H}(\mu_i)<0} m(\mu_i).$$

Now, we analyse the sign of $\boldsymbol{H}(\sigma)$. Simple computations give that if

$$\Upsilon \equiv \lambda^2(1-\mu)^{-2} + \mu^2(1-\mu)^2 - 2\lambda(2-\mu) > 0,$$

the equation $\boldsymbol{H}(\sigma) = 0$ has exactly two different positive roots $\mu_*(\lambda)$ and $\mu^*(\lambda)$, which are, respectively, expressed as

$$\mu_*(\lambda) = \tfrac{1}{2}\{\mu(1-\mu) - \lambda(1-\mu)^{-1} - \sqrt{\Upsilon}\},$$

$$\mu^*(\lambda) = \tfrac{1}{2}\{\mu(1-\mu) - \lambda(1-\mu)^{-1} + \sqrt{\Upsilon}\}.$$

In fact, we observe that $\mu_*(\lambda)$ and $\mu^*(\lambda)$ are the two eigenvalues of $\boldsymbol{A}$. Moreover, $\boldsymbol{H}(\sigma) < 0$ if and only if $\sigma \in (\mu_*(\lambda), \mu^*(\lambda))$.

We can state the main result of this subsection as follows.

Theorem 3.4. *Assume that $\Upsilon > 0$ and let $\mu_*(\lambda) < \mu^*(\lambda)$ be the two positive eigenvalues of $\boldsymbol{A}$. If*

$$\mu_*(\lambda) \in (\mu_i, \mu_{i+1}) \quad \text{and} \quad \mu^*(\lambda) \in (\mu_j, \mu_{j+1}) \qquad \text{for some } 0 \leqslant i < j,$$

and $\sum_{k=i+1}^{j} \mathrm{m}(\mu_k)$ is odd, then (1.3) has at least one non-constant positive solution.

Proof. Let Λ be given in theorem 3.3 such that (1.3) has no non-constant positive solution and $\boldsymbol{u}^*$ is linearly stable. For $0 \leqslant t \leqslant 1$, we define

$$\boldsymbol{G}(\boldsymbol{u}; t) = \begin{pmatrix} (-\Delta + \boldsymbol{I})^{-1}\left(u + uw - \dfrac{u}{1+\mu u}\right) \\ (-\Delta + \boldsymbol{I})^{-1}(w + ((1-t)\Lambda + t\lambda)(1-uw)) \end{pmatrix}.$$

It is clear that finding the positive solution of (1.3) becomes equivalent to solving the equation

$$\boldsymbol{u} - \boldsymbol{G}(\boldsymbol{u}; 1) = 0 \qquad \text{in } \boldsymbol{Y}^+.$$

As the argument in the proof of theorem 3.3, by the homotopy invariance of degree, we have

$$\deg(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 0),\ \Theta,\ 0) = \deg(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 1),\ \Theta,\ 0). \tag{3.17}$$

Moreover, due to the choice of Λ, we also get

$$\deg(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 0),\ \Theta,\ 0) = \mathrm{index}(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 0),\ \boldsymbol{u}^*) = 1. \tag{3.18}$$

In contrast, we assume that (1.3) has no non-constant positive solution. It follows from proposition 3.1 and the previous arguments that

$$\deg(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 1),\ \Theta,\ 0) = \mathrm{index}(\boldsymbol{I} - \boldsymbol{G}(\boldsymbol{u}; 1),\ \boldsymbol{u}^*) = (-1)^{\sum_{k=i+1}^{j} m(\mu_k)} = -1. \tag{3.19}$$

Obviously, (3.17)–(3.19) arrive at a contradiction, and this implies that our theorem 3.4 holds and the proof is complete. □

Remark 3.2. In theorem 3.4, if we let $\lambda \to 0^+$, then $\mu_*(\lambda) \to 0$ and $\mu^*(\lambda) \to \mu(1-\mu)$. Consequently, theorem 3.4 shows that (1.3) admits at least one non-constant positive solution if λ is small enough and $\mu(1-\mu) \in (\mu_i, \mu_{i+1})$ for some $i \geqslant 1$ and $\sum_{k=1}^{i} m(\mu_k)$ is odd. In particular, if the number of spatial dimension is one (that is, Ω is a finite interval $(0, L)$), the multiplicity of each μ_i $(i \geqslant 0)$ is one. Therefore, in this case, we can easily capture the non-constant positive solution to (1.3) only if λ is sufficiently small and the length L of the domain Ω belongs to a sequence of non-overlapping intervals which tend to infinity. For higher spatial dimensions, one can use parameter d of Ω_d defined in the introduction to replace L in the discussion above, and again for most spatial domains Ω, the condition that $\mu(1-\mu) \in (\mu_i, \mu_{i+1})$ for some $i \geqslant 1$ and $\sum_{k=1}^{i} m(\mu_k)$ is odd holds for Ω_d when the value d^{-1} belongs to a sequence of non-overlapping intervals which tend to infinity.

Remark 3.3. We also mention that the range of the parameter λ guaranteeing the existence or non-existence of non-trivial positive solutions (namely, non-constant positive solutions) to (1.3) with the homogeneous Neumann boundary condition is sharply different from that where the homogeneous Dirichlet or Robin boundary condition is imposed on (1.3). In the latter case, [6, 25] proved that (1.3) has no (non-trivial) positive solution if λ is small while it admits positive solutions only if λ is larger than a critical value.

Finally, similarly to the analysis in [20], we can also establish the asymptotic behaviour of non-constant positive solutions to (1.3) as $\lambda \to 0^+$. To be more precise, we can claim

Theorem 3.5. *Assume that $\mu(1-\mu) \neq \mu_j$ for any $j \geqslant 1$. Then, for any sequence of non-constant positive solution (u_λ, w_λ) to (1.3), there is a subsequence $(u_{\lambda_i}, w_{\lambda_i})$ of (u_λ, w_λ) satisfying that $(u_{\lambda_i}, w_{\lambda_i})$ converges to $(\tilde{u}, \tilde{w})$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $\lambda_i \to 0^+$, where $\tilde{u}$ is a non-constant positive solution of*

$$-\Delta\tilde{u} = |\Omega|\left(\int_\Omega \tilde{u}\,\mathrm{d}x\right)^{-1}\tilde{u} - \frac{\tilde{u}}{1+\mu\tilde{u}} \quad \text{in } \Omega, \qquad \partial_\nu\tilde{u} = 0 \quad \text{on } \partial\Omega, \tag{3.20}$$

and $\tilde{w}$ is uniquely determined by

$$\tilde{w} = |\Omega|\left(\int_\Omega \tilde{u}\,\mathrm{d}x\right)^{-1}.$$

Therefore, if $\mu(1-\mu) \in (\mu_j, \mu_{j+1})$ for some fixed $j \geqslant 1$ and $\sum_{k=1}^{j} m(\mu_k)$ is odd, the non-local elliptic problem (3.20) admits at least one non-constant positive solution.

Proof. For the reader's convenience, we give the details of the proof here. Obviously, it suffices to verify the first part of theorem 3.5 since remark 3.2 and this conclude the second part of theorem 3.5. To this end, we shall proceed with a contradiction argument.

Suppose that the conclusion of the first part of theorem 3.5 is false. Then, using theorem 2.1, the standard L^p and Schauder's estimates and the embedding theorems, there exists a sequence $\{\lambda_i\}$ with $\lambda_i \to 0^+$ as $i \to \infty$ and the corresponding non-constant positive solution (u_i, w_i) to (1.3) for $\lambda = \lambda_i$, such that there is a subsequence of (u_i, w_i), still denoted by itself, satisfying $(u_i, w_i) \to (u^*, w^*)$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $i \to \infty$.

As in the proof of theorem 3.3, we let

$$h_i = u_i - u^*, \qquad k_i = w_i - w^*$$

and

$$\tilde{h}_i = \frac{h_i}{\|h_i\|_\infty + \|k_i\|_\infty}, \qquad \tilde{k}_i = \frac{k_i}{\|h_i\|_\infty + \|k_i\|_\infty}.$$

As a result, $\tilde{h}_i$ and $\tilde{k}_i$ satisfy

$$\begin{cases} -\Delta\tilde{h}_i = u_i\tilde{k}_i + w^*\tilde{h}_i - \dfrac{1}{(1+\mu u^*)(1+\mu u)}\tilde{h}_i & \text{in } \Omega, \\ -\Delta\tilde{k}_i = -\lambda_i u_i\tilde{k}_i - \lambda_i w^*\tilde{h}_i & \text{in } \Omega, \\ \partial_\nu\tilde{h}_i = \partial_\nu\tilde{k}_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

Applying the standard theory, we can assume that $(\tilde{h}_i, \tilde{k}_i) \to (\tilde{h}, \tilde{k})$ in $C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$ as $i \to \infty$. By passing to the limit in (3.21), we find that

$$-\Delta\tilde{h} = \mu(1-\mu)\tilde{h} + \tilde{k}/(1-\mu), \quad -\Delta\tilde{k} = 0 \quad \text{in } \Omega, \qquad \partial_\nu\tilde{h} = \partial_\nu\tilde{k} = 0 \quad \text{on } \partial\Omega.$$

Hence, $\tilde{k}$ is a constant and so our hypothesis $\mu(1-\mu) \neq \mu_j$ for any $j \geqslant 1$ deduces

$$\tilde{h} = -\tilde{k}/\mu(1-\mu)^2. \tag{3.22}$$

Furthermore, $\|\tilde{h}_i\|_\infty + \|\tilde{k}_i\|_\infty = 1$ for each $i \geqslant 1$ gives

$$(1 + |1/\mu(1-\mu)^2|)|\tilde{k}| = 1. \tag{3.23}$$

On the other hand, multiplying the second equation in (3.21) by w_i and integrating over Ω, and then letting $i \to \infty$, we obtain

$$(1-\mu)^2\tilde{h} + \tilde{k} = 0,$$

which, together with (3.22), implies $\tilde{h} = \tilde{k} = 0$, and thus this contradicts (3.23), and we complete the proof. □

Remark 3.4. If (3.20) possesses a unique non-constant positive solution, then the result in the first part of theorem 3.5 holds for the whole solution sequence (u_λ, w_λ). Unfortunately, we cannot determine the multiplicity of non-constant positive solutions to (3.20).

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References

[1] Amann H 1976 Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces *SIAM Rev.* **18** 620–709

[2] Bonilla L L and Velarde M G 1979 Singular perturbations approach to the limit cycle and global patterns in a nonlinear diffusion–reaction problem with autocatalysis and saturation law *J. Math. Phys.* **20** 2692–703

[3] Crandall M G and Rabinowitz P H 1973 Bifurcation, perturbation of simple eigenvalues, and linearized stability *Arch. Ration. Mech. Anal.* **52** 161–80

[4] Dancer E N 1983 On the indices of fixed points of mappings in cones and applications *J. Math. Anal. Appl.* **91** 131–51

[5] Davidson F A and Rynne B P 2000 *A priori* bounds and global existence of solutions of the steady-state Sel'kov model *Proc. R. Soc. Edinb.* A **130** 507–16

[6] Du Y H 1996 Uniqueness, multiplicity and stability for positive solutions of a pair of reaction–diffusion equations *Proc. R. Soc. Edinb.* A **126** 777–809

[7] Gilbarg D and Trudinger N S 2001 *Elliptic Partial Differential Equation of Second Order* (*Classics in Mathematics*) (Berlin: Springer) (Reprint of the 1998 edition)

[8] Gray P and Scott S K 1990 *Chemical Oscillations and Instabilies: Nonlinear Chemical Kinetics* (Oxford: Clarendon)

[9] Henry D 1981 *Geometric Theory of Semilinear Parabolic Equations* (*Lecture Notes in Mathematics* vol 840) (Berlin: Springer)

[10] Ibanez J L and Velarde M G 1978 Multiple steady states in a simple reaction–diffusion model with Michaelis–Menten (first-order Hinshelwood–Langmuir) saturation law: the limit of large separation in the two diffusion constants *J. Math. Phys.* **19** 151–6

[11] Jiang J F and Shi J P 2008 Dynamics of a reaction–diffusion system of autocatalytic chemical reaction *Discrete Contin. Dyn. Syst.* A **21** 245–58

[12] Lengyel I and Epstein I R 1991 Modeling of Turing structure in the chlorite–iodide–malonic-acid–starch reaction system *Science* **251** 650–2

[13] Lengyel I and Epstein I R 1992 A chemical approach to designing Turing patterns in reaction–diffusion system *Proc. Natl Acad. Sci. USA* **89** 3977–9

[14] Lieberman G M 2005 Bounds for the steady-state Sel'kov model for arbitrary p in any number of dimensions *SIAM J. Math. Anal.* **36** 1400–6

[15] Lin C S, Ni W M and Takagi I 1988 Large amplitude stationary solutions to a chemotaxis systems *J. Diff. Eqns* **72** 1–27

[16] Lou Y and Ni W M 1996 Diffusion, self-diffusion and cross-diffusion *J. Diff. Eqns* **131** 79–131

[17] Ni W M and Tang M X 2005 Turing patterns in the Lengyel–Epstein system for the CIMA reactions *Trans. Am. Math. Soc.* **357** 3953–69

[18] Ni W M and Wang X F 2007 On the first positive Neumann eigenvalue *Discrete Contin. Dyn. Syst.* A **17** 1–19

[19] Nirenberg L 2001 *Topics in Nonlinear Functional Analysis* (Providence, RI: American Mathematical Society)

[20] Peng R 2007 Qualitative analysis of steady states to the Sel'kov model *J. Diff. Eqns* **241** 386–98

[21] Peng R, Shi J P and Wang M X 2007 Stationary pattern of a ratio-dependent food chain model with diffusion *SIAM Appl. Math.* **67** 1479–503

[22] Peng R and Wang M X 2005 Positive steady states of the Holling–Tanner prey–predator model with diffusion *Proc. R. Soc. Edinb.* A **135** 149–64

[23] Peng R and Wang M X 2007 Pattern formation in the Gray–Scott model *Sci. China A—Math.* **50** 377–86

[24] Peng R and Wang M X Qualitative analysis on a diffusive prey–predator model with ratio-dependent functional response *Sci. China A—Math.* to be published

[25] Ruan W H 1993 Asymptotic behavior and positive steady-state solutions of a reaction–diffusion model with autocatalysis and saturation law *Nonlinear Anal., TMA* **21** 439–56

[26] Shi J P and Wang X F 2006 Hair-triggered instability of radial steady states, spread and extinction in semilinear heat equations *J. Diff. Eqns* **231** 235–51

[27] Turing A 1952 The chemical basis of morphogenesis *Phil. Trans. R. Soc.* B **237** 37–72

[28] Wang M X 2003 Non-constant positive steady states of the Sel'kov model *J. Diff. Eqns* **190** 600–20

[29] Yi F Q, Wei J J and Shi J P 2008 Diffusion-driven instability and bifurcation in the Lengyel–Epstein system *Nonlinear Anal. Real World Appl.* **9** 1038–51

[30] Zhao Y H, Shi J P and Wang Y W 2007 Exact multiplicity and bifurcation of a chemical reaction model *J. Math. Anal. Appl.* **331** 263–78