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Exact multiplicity of solutions to perturbed logistic type equations on a symmetric domain

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Abstract We apply the imperfect bifurcation theory in Banach spaces to study the exact multiplicity of solutions to a perturbed logistic type equations on a symmetric spatial domain. We obtain the precise bifurcation diagrams.

Keywords: imperfect bifurcation, exact multiplicity, perturbed logistic equation

MSC(2000): 35J60, 35J55, 35B32, 35P30, 58C25, 47J15

1 Introduction

The equation of steady state solutions of an ideal dynamical problem can often be written in an abstract form as

$$F(\lambda, u) = 0, \tag{1}$$

where λ is a parameter and u is in the state space. However imprecision and noise-induced perturbation make it more realistic to consider the perturbed version of the equation:

$$F(\varepsilon, \lambda, u) = 0. \tag{2}$$

Singularity theory and perturbation theory have been used in discussion of such imperfect bifurcation phenomena, see [1–3]. In [4, 5], we developed a theory of imperfect bifurcation theory in infinite dimensional spaces based on implicit function theorem and its variants including saddle-node, transcritical and pitchfork bifurcation theorems proved in [6, 7]. Some more general bifurcation theorems in that spirit have also been obtained by using generalized inverses of linear operators in [8–10]. Some applications of these theories to population biology, combustion theory, buckling problems have been considered in [4, 5, 11, 12].

In this paper we revisit a problem considered in [12]. We consider

$$\begin{cases} \Delta u + \lambda[f(u) - \varepsilon h(x)] = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

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where λ is a positive parameter, $\varepsilon > 0$ is a small parameter, $h(x) \in C^\alpha(\overline{\Omega})$ is a perturbation term, and Ω is a smooth bounded region in $\mathbb{R}^n$ for $n \geqslant 1$. The nonlinear function f is assumed to be a logistic type which satisfies

(f1) $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(0) > 0$, $f(u) > 0$ for $u \in (0, M)$, where $M = \infty$ or $M < \infty$, $f(M) = 0$ and $f'(M) < 0$;

(f2) $f''(u) < 0$ for $u \in \mathbb{R}$;

(f3) If $M = \infty$, then

$$\lim_{u\to\infty} \frac{f(u)}{u} = 0.$$

Some typical examples of $f(u)$ are $f(u) = au - bu^p$, $f(u) = a - be^{-u}$ and $f(u) = \ln(u + a)$ where $a, b > 0, p > 1$, see [12–14].

We denote by λ_k the k-th eigenvalue of

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

It is well-known that λ_1 is simple, and its eigenfunction ϕ_1 does not change sign. For $k \geqslant 2$, the corresponding eigenfunction ϕ_k must change sign in Ω. We also define $\lambda_k^0 = \lambda_k / f'(0)$.

In [12], precise bifurcation diagrams were shown under the condition that λ_2 is a simple eigenvalue and

$$\int_\Omega \phi_2(x)dx \cdot \int_\Omega \phi_2^3(x)dx \neq 0, \tag{5}$$

where ϕ_2 is the eigenfunction corresponding to λ_2. However when the domain Ω is symmetric, this non-degenerate condition often fails. For example, if $n = 1$ and $\Omega = (0, \pi)$, then the eigenfunction $\phi_2 = \sin(2x)$ for which $\int_0^\pi \phi_2(x)dx = 0$. The same can be said for the domains with at least one symmetry, such as rectangles and ellipses. In general we can consider a domain Ω which possesses a Steiner symmetry with respect to $x_n = 0$, and ϕ_2 is an odd function satisfying $\phi_2(x', -x_n) = -\phi_2(x', x_n)$. Hence it is necessary that

$$\int_\Omega \phi_2(x)dx = \int_\Omega \phi_2^3(x)dx = 0. \tag{6}$$

But in our result we do not need to assume that the domain is symmetric, but only (6) is satisfied, although the typical examples of (6) occurring are symmetric domains. Our first result is

Theorem 1.1. *Suppose that f satisfies* (f1)–(f3)*, λ_2 is a simple eigenvalue satisfying* (6)*, and*

$$\int_\Omega [f'''(0)\phi_2^4(x) + 3f''(0)\phi_2^2(x)\theta(x)]dx < 0, \tag{7}$$

where θ is the solution of

$$\Delta\theta + \lambda_2\theta + \lambda_2^0 f''(0)\phi_2^2 = 0. \tag{8}$$

We also assume that $\phi_1(x) > 0$, and fix a ϕ_2, and $h \in C^\alpha(\overline{\Omega})$ satisfies

$$\int_\Omega h(x)\phi_i(x)dx > 0, \quad i = 1, 2. \tag{9}$$

Let $X = C_0^{2,\alpha}(\overline{\Omega})$. *Define* $\Sigma = \{(\lambda, u) \in \mathbb{R} \times X : (\lambda, u)$ *solves* $(3)\}$, *and* $T(a, b, c) = \{(\lambda, u) : a < \lambda < b, \|u\|_X < c\}$. *Then for any small* δ_1, $\delta_2 > 0$, *there exists* $\varepsilon_1 = \varepsilon_1(\delta_1, \delta_2, f)$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$,

$$\Sigma_0 \equiv \Sigma \cap T(\lambda_1^0 - \delta_1, \lambda_2^0 + \delta_1, \delta_2) = \bigcup_{i=1}^{3} \Sigma_i,$$

where Σ_i *is a connected component of* Σ_0 (i=1, 2, 3). *Moreover*

(i) *each* Σ_i ($i = 1, 2, 3$) *is a smooth curve in* $\mathbb{R} \times X$;

(ii) Σ_1 *is exactly* $\supset$*-shaped, there is a unique degenerate solution on* Σ_1, *and each solution on* Σ_1 *is negative*;

(iii) Σ_3 *is exactly* $\subset$*-shaped, there is a unique degenerate solution on* Σ_3, *and each solution on* Σ_3 *is sign-changing*;

(iv) *there is a unique degenerate solution on* Σ_2, Σ_2 *can be parameterized as* $(\lambda(s), u(s))$, $s \in (s_1, s_4)$. *The portion of* Σ_2 *with* $s \in (s_1, s_2)$ *is exactly* $\subset$*-shaped,* $u(s)$ *is positive, and it contains a unique degenerate solution*; *the portion of* Σ_2 *with* $s \in (s_3, s_4)$ *is a monotone curve without degenerate solution,* $u(s)$ *is sign-changing, where* $s_1 < s_2 < s_3 < s_4$ (*see Figure* 1).

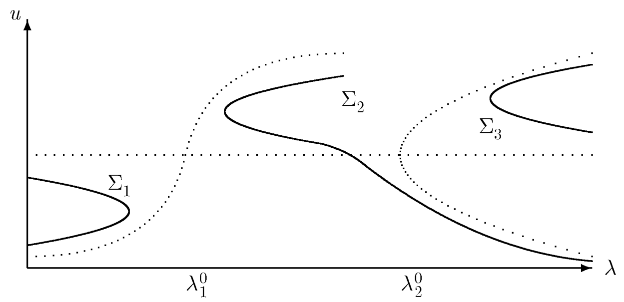


Figure 1 Precise bifurcation diagram when (7) holds for $\|u\|$ is small.
Solid curve: small $\varepsilon > 0$; Dashed curve: $\varepsilon = 0$.

If the condition $\int_\Omega h(x)\phi_2(x)dx > 0$ in Theorem 1.1 is changed to

$$\int_\Omega h(x)\phi_2(x)dx < 0, \tag{10}$$

then the diagram in Figure 1 becomes the one in Figure 2. But Figures 1 and 2 are essentially the same. When $\varepsilon = 0$, a transcritical bifurcation always occurs at $\lambda = \lambda_1$ under our assumptions on f. For a symmetric domain Ω (with respect to $x_n = 0$), a pitchfork bifurcation occurs at $\lambda = \lambda_2$ when $\varepsilon = 0$, and the two branches are indeed symmetric so that $u_1(x', x_n) = u_2(x', -x_n)$ if u_1 and u_2 are a pair of solutions with the same λ. Now for the perturbed problem, a change from (9) to (10) triggers a switch of Σ_2 following one subbranch of the unperturbed branch to the other one as indicated by Figures 1 and 2.

Next we consider

$$\begin{cases} \Delta u + \lambda[f(u) - \varepsilon^2 h(x)] = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{11}$$

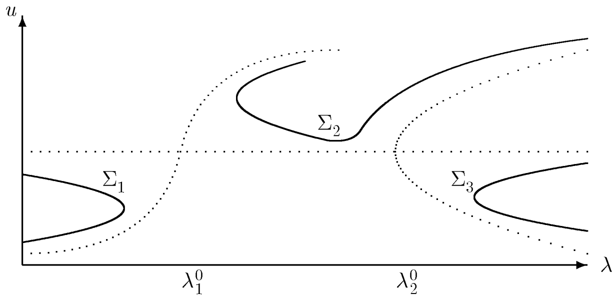


Figure 2 Precise bifurcation diagram when (10) holds for $\|u\|$ is small.
Solid curve: small $\varepsilon > 0$; Dashed curve: $\varepsilon = 0$.

Theorem 1.2. *Suppose that f satisfies* (f1)–(f3), *λ_2 is a simple eigenvalue and*

$$\int_\Omega \phi_2(x)dx \cdot \int_\Omega \phi_2^3(x)dx > 0. \tag{12}$$

We also assume that $\phi_1(x) > 0$, and fix a ϕ_2, and $h(x) \in C^\alpha(\overline{\Omega})$ satisfies $\int_\Omega h(x)\phi_i(x)dx < 0$, $i = 1, 2$.

Define $\Sigma = \{(\lambda, u) \in \mathbb{R} \times X : (\lambda, u)$ solves (11), *and $T(a, b, c) = \{(\lambda, u) : a < \lambda < b, \|u\|_X < c\}$. Then for any small δ_1, $\delta_2 > 0$, there exists $\varepsilon_1 = \varepsilon_1(\delta_1, \delta_2, f)$ such that for any $\varepsilon \in (0, \varepsilon_1)$,*

$$\Sigma_0 \equiv \Sigma \cap T(\lambda_1^0 - \delta_1, \lambda_2^0 + \delta_1, \delta_2) = \bigcup_{i=1}^{3} \Sigma_i,$$

where Σ_i is a connected component of Σ_0, and each Σ_i is a smooth and monotone curve, (i=1, 2, 3) (*see Figure* 3).

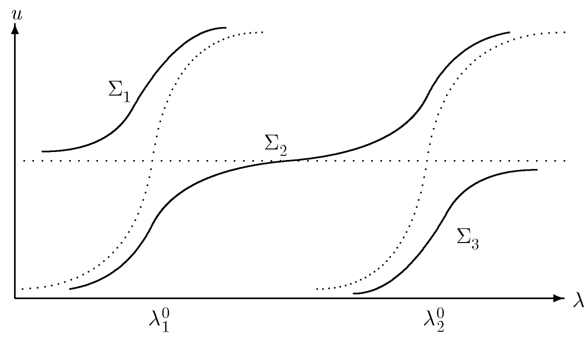


Figure 3 Precise bifurcation diagram when (12) holds for $\|u\|$ is small.
Solid curve: small $\varepsilon > 0$; Dashed curve: $\varepsilon = 0$.

The rest of the paper is organized as follows. In Section 2, we do some preliminaries. In Section 3, we give the proof of our main results. In Section 4, we discuss some examples. We use $N(L)$ and $R(L)$ to represent the null space and range space of a linear operator L respectively, and we use F_u to denote the partial derivative of a nonlinear operator F with respect to u.

2 Preliminaries

Very commonly and also generically, at a degenerate solution (λ_0, u_0), 0 is a simple eigenvalue of F_u, that is

(F1) $\dim N(F_u(\lambda_0,u_0)) = \operatorname{codim} R(F_u(\lambda_0,u_0)) = 1$, and $N(F_u(\lambda_0,u_0)) = \operatorname{span}\{w_0\}$, where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator F_u. The following theorem is of fundamental importance in many application problems:

Theorem 2.1 (Transcritical and pitchfork bifurcations)[6, Theorem 1.7]. *Let $F : \mathbb{R}\times X \to Y$ be continuously differentiable. Suppose that $F(\lambda,u_0)=0$ for $\lambda\in\mathbb{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0,u_0), F satisfies* (F1) *and*

(F2) $F_{\lambda u}(\lambda_0,u_0)[w_0] \notin R(F_u(\lambda_0,u_0))$. *Then the solutions of* (1) *near (λ_0,u_0) consists precisely of the curves $u=u_0$ and $(\lambda(s),u(s))$, $s\in I=(-\delta,\delta)$, where $(\lambda(s),u(s))$ are continuously differentiable functions such that $\lambda(0)=\lambda_0$, $u(s)=u_0+sw_0+o(s)$, and*

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0,u_0)[w_0,w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0,u_0)[w_0]\rangle}, \tag{13}$$

where $l\in Y^*$ satisfying $N(l)=R(F_u(\lambda_0,u_0))$. If it satisfies

(F3) $F_{uu}(\lambda_0,u_0)[w_0,w_0] \notin R(F_u(\lambda_0,u_0))$, *i.e., $\lambda'(0)\neq 0$, then a transcritical bifurcation occurs at (λ_0,u_0); If it satisfies*

(F3′) $F_{uu}(\lambda_0,u_0)[w_0,w_0] \in R(F_u(\lambda_0,u_0))$, *i.e., $\lambda'(0)=0$ and $\lambda''(0)\neq 0$,*

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0,u_0)[w_0,w_0,w_0]\rangle + 3\langle l, F_{uu}(\lambda_0,u_0)[w_0,\theta]\rangle}{3\langle l, F_{\lambda u}(\lambda_0,u_0)[w_0]\rangle}, \tag{14}$$

where θ is the solution of

$$F_u(\lambda_0,u_0)[\theta] + F_{uu}(\lambda_0,u_0)[w_0,w_0] = 0, \tag{15}$$

then a pitchfork bifurcation occurs at (λ_0,u_0).

In [4, 5], we consider the imperfect transcritical and pitchfork bifurcations under a small perturbation. For the sake of completeness, we summarize the main results in [4, 5] below. Define

$$H(\varepsilon,\lambda,u,w) = \begin{pmatrix} F(\varepsilon,\lambda,u) \\ F_u(\varepsilon,\lambda,u)[w] \end{pmatrix}. \tag{16}$$

Theorem 2.2 [5, Theorems 2.4 and 2.5]. *Let F be twice continuously differentiable, and $F(\varepsilon_0,\lambda,u_0)\equiv 0$ for $\lambda\in\mathbb{R}$. For $T_0=(\varepsilon_0,\lambda_0,u_0,w_0)$, $H(T_0)=(0,0)$, and T_0 satisfies* (F1), (F2), (F3) *and*

(F4) $F_\varepsilon(\varepsilon_0,\lambda_0,u_0)\notin R(F_u(\varepsilon_0,\lambda_0,u_0))$. *Then there exists $\delta>0$ such that all the solutions of $H(\varepsilon,\lambda,u,w)=(0,0)$ near T_0 are in a form of*

$$\{T_s = (\varepsilon(s),\lambda(s),u(s),w(s)) : s\in(-\delta,\delta)\}, \tag{17}$$

where $\varepsilon(0)=\varepsilon_0$, $\varepsilon'(0)=0$, $u(s)=u_0+skw_0+o(s)$, $\lambda(s)=\lambda_0+s+o(s)$,

$$\varepsilon''(0) = \frac{\langle l, F_{\lambda u}[w_0]\rangle^2}{\langle l, F_{uu}[w_0,w_0]\rangle\langle l, F_\varepsilon\rangle}, \tag{18}$$

i.e. $\varepsilon''(0)\neq 0$, where k is the unique number such that $\langle l, F_{\lambda u}[w_0]+kF_{uu}[w_0,w_0]\rangle = 0$. If $\varepsilon''(0)>0$, then there exists $\rho_1,\delta_1,\delta_2>0$, such that for $N=\{(\lambda,u)\in\mathbb{R}\times X : |\lambda-\lambda_0|\leqslant \delta_1, \|u\|\leqslant\delta_2\}$,

(A) *for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$,

$$F^{-1}(0) \cap N = \Sigma_\varepsilon^1 \cup \Sigma_\varepsilon^2, \quad \Sigma_\varepsilon^i = \{(\lambda, \overline{u_i}(\lambda)) : \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\}, \quad i = 1, 2, \tag{19}$$

and $\overline{u_i}'(\lambda) > 0$ *for* $\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]$, $i = 1, 2$;

(B) *for* $\varepsilon = \varepsilon_0$,

$$F^{-1}(0) \cap N = \{(\lambda, 0) : \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\} \cup \Sigma_0, \quad \Sigma_0 = \{(\overline{\lambda}(t), \overline{u}(t)) : t \in [-\eta, \eta]\}, \tag{20}$$

and $\overline{u}'(t) > 0$ *for* $t \in [-\eta, \eta]$;

(C) *for* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$,

$$F^{-1}(0) \cap N = \Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-, \quad \Sigma_\varepsilon^\pm = \{(\overline{\lambda}_\pm(t), \overline{u}_\pm(t)) : t \in [-\eta, \eta]\}, \tag{21}$$

$\overline{\lambda}_+(\pm\eta) = \lambda_0 + \delta_1$, $\overline{\lambda}_-(\pm\eta) = \lambda_0 - \delta_1$, $\overline{\lambda}'_\pm(0) = 0$, $\overline{\lambda}''_+(0) < 0$, $\overline{\lambda}''_-(0) > 0$, *and there is exactly one turning point on each component* $\Sigma_\varepsilon^\pm$ (*see Figure* 4).

Theorem 2.3[4, Theorems 3.2 and 4.1]**.** *Let* $F \in C^3(M, Y)$, $M = \mathbb{R} \times \mathbb{R} \times X$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0)$ *such that* $H(T_0) = (0, 0)$. *Suppose* $F(\varepsilon_0, \lambda, u_0) = 0$ *for* $|\lambda - \lambda_0| < \gamma$ *for some* $\gamma > 0$, *and* T_0 *satisfies* (F1), (F2), (F3′), (F4) *and*

(F5) $F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0, w_0] + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[\theta, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$, *where* θ *is the unique solution of* (15).

Then there exists $\delta > 0$ *such that all the solutions of* $H(\varepsilon, \lambda, u, w) = (0, 0)$ *near* T_0 *are in a form of* (17), *where* $\varepsilon(0) = \varepsilon_0$, $\varepsilon'(0) = \varepsilon''(0) = 0$, $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, $u(s) = u_0 + sw_0 + o(s)$, *and*

$$\lambda''(0) = -\frac{\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[w_0, \theta]\rangle}{\langle l, F_{\lambda u}[w_0]\rangle}, \tag{22}$$

$$\varepsilon'''(0) = 2 \cdot \frac{\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[w_0, \theta]\rangle}{3\langle l, F_\varepsilon\rangle}, \tag{23}$$

where θ *is the solution of* (15). *If* $\langle l, F_\varepsilon\rangle < 0$, $\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[w_0, \theta]\rangle < 0$, *and* $\langle l, F_{\lambda u}[w_0]\rangle > 0$, *then* $\varepsilon'''(0) > 0$ *and* $\lambda''(0) > 0$, *there exist* ρ_1, δ_1, $\delta_2 > 0$ *such that for* $N = \{(\lambda, u) \in \mathbb{R} \times X : |\lambda - \lambda_0| \leqslant \delta_1, \|u\| \leqslant \delta_2\}$, *we have*

(A) *for* $\varepsilon = \varepsilon_0$,

$$F^{-1}(0) \cap N = \{(\lambda, 0) : |\lambda - \lambda_0| \leqslant \delta_1\} \cup \Sigma_0, \quad \Sigma_0 = \{(\overline{\lambda}(t), \overline{u}(t)), |t| \leqslant \eta\}, \tag{24}$$

where $\overline{\lambda}(0) = \lambda_0$, $\overline{\lambda}'(0) = 0$, $\overline{\lambda}''(0) > 0$, $\overline{\lambda}(t) > 0$ *for* $t \in (0, \eta)$, $\overline{\lambda}(t) < 0$ *for* $t \in (-\eta, 0)$, *and* $\overline{\lambda}(\pm\eta) = \lambda_0 + \delta_1$;

(B) *for fixed* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0) \cup (\varepsilon_0, \varepsilon_0 + \rho_1)$,

$$F^{-1}(0) \cap N = \Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-, \tag{25}$$

where $\Sigma_\varepsilon^+ = \{(\overline{\lambda}_+(t), \overline{u}_+(t)),\ t \in [-\eta, +\eta]\}$ *where* $\overline{\lambda}_+(\pm\eta) = \lambda_0 + \delta_1$, $\overline{\lambda}'_+(0) = 0$, $\overline{\lambda}''_+(0) > 0$, *and* $(\overline{\lambda}_+(0), \overline{u}_+(0))$ *is the unique degenerate solution on* Σ_ε^+; *and* $\Sigma_\varepsilon^- = \{(\lambda_-, \overline{u}(\lambda_-)) : \lambda_- \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\}$ *is a monotone curve without degenerate solutions* (*see Figure* 5).

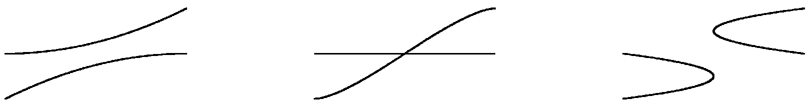
Theorem 2.4[4, Corollary 3.4 and Remark 3.5]**.** *Let* $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0)$ *such that* $H(T_0) = (0, 0)$. *We assume that there exists a neighborhood* U *of* $(\varepsilon_0, \lambda_0)$ *in* $\mathbb{R}^2$ *such that*

$F(\varepsilon_0,\lambda,u_0)\equiv 0$ *for* $(\varepsilon_0,\lambda)\in U$ *and* $F_\varepsilon(\varepsilon_0,\lambda_0,u_0)=0$. *Suppose that* T_0 *satisfies* (F1), (F2), (F3) *and*

(F6) $(\mathbf{v}H_1\mathbf{v}^{\mathrm T})/\langle l,F_{uu}[w_0,w_0]\rangle<0$, *where* $\mathbf{v}=(\langle l,F_{\lambda u}[w_0]\rangle,\langle l,F_{\varepsilon\lambda}\rangle)$, *and*

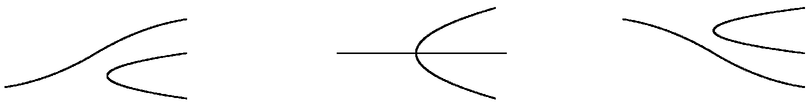
$$H_1\equiv\begin{pmatrix}\langle l,F_{\varepsilon\varepsilon}\rangle & -\langle l,F_{\varepsilon u}[w_0]\rangle\\ -\langle l,F_{\varepsilon u}[w_0]\rangle & \langle l,F_{uu}[w_0,w_0]\rangle\end{pmatrix},\tag{26}$$

then the set of degenerate solutions near T_0 *is the singleton* $\{T_0\}$ (*see Figure* 6).



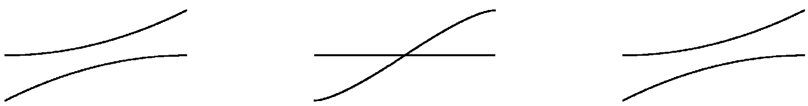
$\varepsilon<\varepsilon_0$ $\varepsilon=\varepsilon_0$ $\varepsilon>\varepsilon_0$

Figure 4 Symmetry breaking of transcritical bifurcation.



$\varepsilon<\varepsilon_0$ $\varepsilon=\varepsilon_0$ $\varepsilon>\varepsilon_0$

Figure 5 Symmetry breaking of pitchfork bifurcation.



$\varepsilon<\varepsilon_0$ $\varepsilon=\varepsilon_0$ $\varepsilon>\varepsilon_0$

Figure 6 Non-typical symmetry breaking of pitchfork bifurcation.

3 Proofs of main results

Proof of Theorem 1.1. First we show that the bifurcation diagram when $\varepsilon=0$ is as described by the dashed curves in Figure 1. From our assumptions, λ_1 and λ_2 are both simple eigenvalues. Thus we can apply Theorem 2.1 to conclude that $(\lambda_1^0,0)$ is a bifurcation point where a transcritical bifurcation occurs and $(\lambda_2^0,0)$ is also a bifurcation point where a pitchfork bifurcation occurs. To show that, we define

$$F^0(\lambda,u)=\Delta u+\lambda f(u),\tag{27}$$

where $\lambda\in\mathbb{R}$ and $u\in X$, and the range space of F^0 is $Y=C^\alpha(\overline{\Omega})$. Then $u=0$ is a trivial solution for any λ. We consider the linearized operator of F^0 at $\lambda=\lambda_i^0$, where it is not invertible. For $i=1,2$, $N(F_u^0(\lambda_i^0,0))=\mathrm{span}\{\phi_i\}$; $R(F_u^0(\lambda_i^0,0))=\{v\in Y:\int_\Omega v\phi_i dx=0\}$ and $F_{\lambda u}^0(\lambda_i^0,0)[\phi_i]=f'(0)\phi_i\notin R(F_u^0(\lambda_i^0,0))$. Thus (F1), (F2) are satisfied, Theorem 2.1 is applicable. Moreover, $F_{uu}^0(\lambda_1^0,0)[\phi_1,\phi_1]=\lambda_1^0f''(0)\phi_1^2\notin R(F_u^0(\lambda_1^0,0))$, thus (F3) is satisfied at $(\lambda_1^0,0)$ and $F_{uu}^0(\lambda_2^0,0)[\phi_2,\phi_2]=\lambda_2^0f''(0)\phi_2^2\in R(F_u^0(\lambda_2^0,0))$, thus (F3′) is satisfied at $(\lambda_2^0,0)$, since $\int_\Omega\phi_2^3(x)dx=0$. Thus from Theorem 2.1, the solutions of (27) near $(\lambda_i^0,0)$ are on two curves $\Sigma^0=\{(\lambda,0)\}$ and $\Sigma^i=\{(\lambda_i(s),v_i(s)):|s|\leqslant\delta\}$, where $\lambda_i(0)=\lambda_i^0$, $v_i(s)=s\phi_i+o(|s|)$.

Moreover, from (13),

$$\lambda_1'(0) = -\frac{\lambda_1^0 \int_\Omega f''(0)\phi_1^3(x)dx}{2\int_\Omega f'(0)\phi_1^2(x)dx} > 0,$$

as $\phi_1 > 0$ and the assumptions on f; and on the other hand, $\lambda_2'(0) = 0$, and from (14)

$$\lambda_2''(0) = -\lambda_2^0 \frac{\int_\Omega [f'''(0)\phi_2^4(x) + 3f''(0)\phi_2^2(x)\theta(x)]dx}{3\int_\Omega f'(0)\phi_2^2(x)dx} > 0,$$

where θ satisfies (8). Thus a transcritical bifurcation occurs at $(\lambda_1^0, 0)$, and a pitchfork bifurcation occurs at $(\lambda_2^0, 0)$, as the dashed curves shown in Figure 1.

Next we apply Theorem 2.2 to show that when $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some small $\varepsilon_0 > 0$, the local bifurcation picture near $(\lambda_1^0, 0)$ is as in Figure 4. Indeed, we define

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda[f(u) - \varepsilon h(x)]. \tag{28}$$

Since $F(0, \lambda, u) = F^0(\lambda, u)$, $F_\varepsilon = -\lambda h(x)$ and according to (9), we have that (F1), (F2), (F3), (F4) are satisfied at $(0, \lambda_1^0, 0, \phi_1)$. From Theorem 2.2, the degenerate solutions of (3) form a curve $\{(\varepsilon(s), \lambda(s), u(s), w(s)) : |s| \leqslant \delta\}$, $\lambda'(0) = 1$, $\varepsilon'(0) = 0$, and from (18)

$$\varepsilon''(0) = -\frac{[f'(0)]^2(\int_\Omega \phi_1^2 dx)^2}{(\lambda_1^0)^2 f''(0)(\int_\Omega \phi_1^3 dx)\cdot(\int_\Omega h(x)\phi_1 dx)} > 0. \tag{29}$$

Hence there are two degenerate solutions near $(\lambda_1^0, 0)$ when $\varepsilon > 0$, and no degenerate solution when $\varepsilon < 0$. Therefore we obtain the parts of the bifurcation diagram in Figure 1 when λ is near $(\lambda_1^0, 0)$. Moreover, since the solutions near $(\lambda_1^0, 0)$ are all in a form $(\lambda, tk\phi_1 + o(|t|))$, where $k = -\frac{f'(0)\int_\Omega \phi_1^2 dx}{\lambda_1^0 f''(0)\int_\Omega \phi_1^3 dx} > 0$, the solutions on the $\supset$-branch are all negative, and the ones on the $\subset$-branch are all positive.

Now we apply Theorem 2.3 to show that the local bifurcation picture near $(\lambda_2^0, 0)$ is as in Figure 5. Indeed, (F1), (F2), (F3$'$), (F4) and (F5) are satisfied at $(0, \lambda_2^0, 0, \phi_2)$ according to (7). From Theorem 2.3, the degenerate solution of (3) forms a curve $\{(\varepsilon(s), \lambda(s), u(s), w(s)) : |s| \leqslant \delta\}$, $\varepsilon'(0) = \varepsilon''(0) = \lambda'(0) = 0$, and from (22),(23)

$$\lambda''(0) = -\lambda_2^0 \frac{\int_\Omega [f'''(0)\phi_2^4(x) + 3f''(0)\phi_2^2(x)\theta]dx}{\int_\Omega f'(0)\phi_2^2(x)dx} > 0,$$

$$\varepsilon'''(0) = -2\frac{\int_\Omega [f'''(0)\phi_2^4(x) + 3f''(0)\phi_2^2(x)\theta]dx}{\int_\Omega h(x)\phi_2(x)dx} > 0.$$

Hence there is a unique degenerate solution near $(\lambda_2^0, 0)$ when $\varepsilon > 0$, and there is also a unique degenerate solution when $\varepsilon < 0$. Therefore we obtain the parts of bifurcation diagram in Figure 1 when λ is near λ_2^0. Moreover, since the solutions near $(\lambda_2^0, 0)$ are all in form $(\lambda, t\phi_2 + o(|t|))$, the solutions on monotone branch and $\subset$-branch are sign-changing solutions.

To be more precise, we select $\widetilde{\delta_1} > 0$ and $\varepsilon_2 > 0$ such that when $\varepsilon \in (0, \varepsilon_2)$, (3) has exactly two degenerate solutions in the cube $C_1 = \{(\lambda, u) : |\lambda - \lambda_1^0| \leqslant \widetilde{\delta_1}, \|u\|_X \leqslant \widetilde{\delta_1}\}$, and a unique degenerate solution in the cube $C_2 = \{(\lambda, u) : |\lambda - \lambda_2^0| \leqslant \widetilde{\delta_1}, \|u\|_X \leqslant \widetilde{\delta_1}\}$. Then the portion of the bifurcation diagram in C_1 (resp. C_2) cube is exactly same as the third diagram in Figure 4 (resp. Figure 5). The curve of degenerate solutions $\{(\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)), i = 1, 2\}$ satisfies

$\lambda_i(0) = \lambda_i^0$, $\lambda_1'(0) > 0$, $\lambda_2'(0) = 0$, $\lambda_2''(0) > 0$, thus $\lambda_1(s) < \lambda_1^0$ for $s < 0$ and $\lambda_1(s) > \lambda_1^0$ for $s > 0$, $\lambda_2(s) > \lambda_2^0$ for all $s \in \mathbb{R}$.

Let $\widetilde{\delta_2} = \widetilde{\delta_1}/2$. For $\lambda \in [\lambda_1^0 + \widetilde{\delta_2}, \lambda_2^0 - \widetilde{\delta_2}]$, the trivial solution $(\lambda, 0)$ for (3) with $\varepsilon = 0$ is nondegenerate. We choose $\widetilde{\delta_3} > 0$ such that the solutions on the line $(\lambda, 0)$ are the only solutions of (3) when $\varepsilon = 0$ in the cube $\{(\lambda, u) : \lambda_1^0 + \widetilde{\delta_2} < \lambda < \lambda_2^0 - \widetilde{\delta_2}, \|u\|_X \leqslant \widetilde{\delta_3}\}$. Thus by the implicit function theorem, there exists $\varepsilon_3 > 0$ such that when $\varepsilon \in (0, \varepsilon_3)$, for each $\lambda \in [\lambda_1^0 + \widetilde{\delta_2}, \lambda_2^0 - \widetilde{\delta_2}]$, (3)has exactly one solution $(\lambda, u(\lambda))$ such that $\|u(\lambda)\|_X \leqslant \widetilde{\delta_3}$, and they are all nondegenerate. From the nondegeneracy of the solutions, we can see that the curve $(\lambda, u(\lambda))$ joins the lower branch of $\subset$-branch in C_1. We can use the Morse index of the solutions to conclude that $(\lambda, u(\lambda))$ joins the lower branch but not the upper branch, since the solutions on the upper branch have Morse index 0 and the ones on the lower branch have Morse index 1. All solutions on $(\lambda, u(\lambda))$ have Morse index 1 since they are perturbations of $(\lambda, 0)$ when $\varepsilon = 0$, which have Morse index 1. Similarly, $(\lambda, u(\lambda))$ joins the monotone branch in C_2, and here the terms "lower" and "upper" branches are not appropriate as the solutions are not ordered. But $(\lambda, u(\lambda))$ will connect with the branch with Morse index 1 and smaller X-norm. Therefore the $\subset$-branch in C_1 and the monotone branch in C_2 are connected. Let $\varepsilon_1 = \min(\varepsilon_2, \varepsilon_3)$, $\delta_2 = \min(\widetilde{\delta_1}, \widetilde{\delta_3})$, and $\delta_1 = \widetilde{\delta_1}$. Then we obtain the results claimed in Theorem 1.1.

Proof of Theorem 1.2. We assume $\int_\Omega \phi_2^3(x)dx > 0$, and the case when $\int_\Omega \phi_2^3(x)dx < 0$ can be shown similarly. First similar to the proof of Theorem 1.1, $F^0(\lambda, 0) = 0$, (F1), (F2) and (F3) are satisfied at $(\lambda_i^0, 0)$, $i = 1, 2$. We apply Theorem 2.1 to conclude that a transcritical bifurcation occurs at $(\lambda_i^0, 0)$. Thus the bifurcation diagram when $\varepsilon = 0$ is described by the dashed curves in Figure 3.

Next we apply Theorem 2.4 to conclude that when $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, the local bifurcation picture near $(\lambda_i^0, 0)$ $(i = 1, 2)$ is as in Figure 6. Indeed, we define

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda[f(u) - \varepsilon^2 h(x)]. \tag{30}$$

From $F_\varepsilon(0, \lambda_i^0, 0) = 0$, $F_{\varepsilon u}(0, \lambda_i^0, 0)[\phi_i] = 0$, $F_{\varepsilon\varepsilon}(0, \lambda_i^0, 0) = -2\lambda_i^0 h(x)$, $F_{\lambda u}(0, \lambda_i^0, 0)[\phi_i] = f'(0)\phi_i$, $F_{uu}(0, \lambda_i^0, 0)[\phi_i, \phi_i] = \lambda_i^0 f''(0)\phi_i^2$, $i = 1, 2$, we have that (F1), (F2), (F3) are satisfied at $(0, \lambda_i^0, 0, \phi_i)$. And the expression in (F6) is simplified as

$$-2 \cdot \frac{(f'(0))^2(\int_\Omega \phi_i^2(x)dx)^2 \int_\Omega h(x)\phi_i(x)dx}{f''(0)\int_\Omega \phi_i^3(x)dx} < 0.$$

Therefore, we can apply Theorem 2.4 to obtain the results here.

4 Example

We demonstrate our results by considering some concrete examples that (6) holds. As remarked in the introduction, (6) is satisfied by many symmetric domains such as rectangle (with unequal sides) and ellipses (with unequal axes). For these domains, we can apply Theorems 1.1 and 1.2 and abstract theorems in [4, 5] to obtain precise bifurcation diagrams.

For simplicity, we assume that $f(u) = u - u^2$. First we consider the case that $\Omega = R = (0, a) \times (0, b) \in \mathbb{R}^2$ with $a < b$. When $\varepsilon = 0$, (3) is logistic equation and $f(u)$ satisfies (f1)–(f3). Then the first eigenvalue and corresponding eigenfunction of R are

$$\lambda_1 = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \quad \phi_1 = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right), \tag{31}$$

and the second eigenvalue and corresponding eigenfunction of R are

$$\lambda_2 = \pi^2\left(\frac{1}{a^2} + \frac{4}{b^2}\right), \quad \phi_2 = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right). \tag{32}$$

Thus λ_i $(i = 1, 2)$ is a simple eigenvalue, ϕ_2 is an odd function and satisfies (6).

Next we verify that (7) holds, i.e. $\int_R \phi_2^2\theta dxdy > 0$, where θ satisfies $\Delta\theta + \lambda_2\theta - 2\lambda_2\phi_2^2 = 0$ since $f'(0) = 1$, $f''(0) = -2$, $f'''(0) = 0$. From simple calculations, we obtain

$$\phi_2^2 = \frac{1}{4}\left(1 - \cos\frac{2\pi x}{a} - \cos\frac{4\pi y}{b} + \cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}\right),$$

$$\theta = \frac{1}{2} - \frac{1}{2}\frac{b^2+4a^2}{4a^2-3b^2}\cos\frac{2\pi x}{a} - \frac{1}{2}\frac{4a^2+b^2}{b^2-12a^2}\cos\frac{4\pi y}{b} - \frac{1}{6}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b},$$

$$\int_R \phi_2^2\theta dxdy = \frac{ab}{8}\left[\frac{(4a^2+b^2)^2}{(4a^2-3b^2)(12a^2-b^2)} + \frac{11}{12}\right].$$

When $\frac{a^2}{b^2} \in (0, \frac{1}{12}) \cup (\frac{3}{4}, 1)$, $\int_R \phi_2^2\theta dxdy > 0$ holds. If we also assume $\int_R h(x, y)\phi_i dxdy > 0$, we can apply Theorem 1.1 and obtain the variation diagrams near $\varepsilon = 0$ shown in Figure 1. We demonstrate our results by considering some concrete examples that (6) holds. As remarked in the introduction, (6) is satisfied by many symmetric domains such as rectangle (with unequal sides) and ellipses (with unequal axes). For these domains, we can apply Theorems 1.1 and 1.2 and abstract theorems in [4, 5] to obtain precise bifurcation diagrams.

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