Optimal Spatial Harvesting Strategy and Symmetry-Breaking

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Abstract A reaction-diffusion model with logistic growth and constant effort harvesting is considered. By minimizing an intrinsic biological energy function, we obtain an optimal spatial harvesting strategy which will benefit the population the most. The symmetry properties of the optimal strategy are also discussed, and related symmetry preserving and symmetry breaking phenomena are shown with several typical examples of habitats.

Keywords Reaction-diffusion · Harvesting · Optimal spatial effort function · Symmetry breaking

1 Introduction

Reaction-diffusion equations have been used extensively in modeling the spatiotemporal behavior of a species of organism [6, 18, 19, 22, 26]. The most widely used model is to assume a logistic growth rate, and the density function $u(x, t)$ of the species satisfies

$$\frac{\partial u}{\partial t} = \mathcal{D} \Delta u + au - bu^2, \quad x \in \Omega, \ t > 0, \tag{1.1}$$

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where $a, b, \mathcal{D} > 0$ [5, 23, 25], and Ω is the habitat of the population. When the species is a renewable natural resource and it is harvested by the human being, then the equation can be adjusted to

$$\frac{\partial u}{\partial t} = \mathcal{D}\Delta u + au - bu^2 - h(x, u), \quad x \in \Omega, \ t > 0, \tag{1.2}$$

and $h(x, u)$ is the harvesting density per unit time [4, 10, 20, 23].

For the harvesting term in the equation, one often assumes a constant harvesting effort, and the harvesting rate is proportional to the population density and the effort:

$$h(x, u) = E(x)u, \quad \text{and} \quad E(x) \geq 0, \ x \in \partial\Omega. \tag{1.3}$$

The harvesting effort $E(x)$ may differ geographically. We assume that the total effort is a constant:

$$\int_{\Omega} E(x)dx = \beta \cdot |\Omega|, \tag{1.4}$$

for an average effort $\beta > 0$, and $|\Omega|$ is the area (or Lebesgue measure for $\Omega \subset \mathbf{R}^n$ with $n \geq 3$) of the habitat. On the other hand, it is reasonable to assume that $E(x)$ is non-negative and bounded, *i.e.*

$$0 \leq E(x) \leq M, \quad x \in \Omega, \tag{1.5}$$

where $M \geq \beta$ is the maximum allowable harvesting effort. For related discussions, see [4, 10, 20].

When the boundary of the habitat is assumed to be hostile, then the population density under constant effort harvesting is described by

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^2 - E(x)u, & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.6}$$

Here we have used dimensionless variables to simplify the equations, and also for simplicity, we assume the logistic growth is homogeneous for the whole habitat. The diffusion constant is defined as ε^2 to indicate the diffusion scale ε. The existence and uniqueness of equilibrium solutions to (1.6) have been studied in [5, 20, 23, 24] and many others, and it is known that for $\varepsilon > 0$ is small, there exists a unique positive stable equilibrium solution $u_{\varepsilon, E}(x)$ for any fixed $E(x)$ if some admissible conditions on E are satisfied.

In this paper, we consider an optimization problem of finding the optimal $E(x)$ to minimize the energy function

$$J_\varepsilon(u, E) = \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} u^2 dx + \frac{1}{3} \int_{\Omega} u^3 dx + \frac{1}{2} \int_{\Omega} E(x)u^2 dx. \tag{1.7}$$

To be more precise, we can show that the unique equilibrium solution $u_{\varepsilon, E}$ minimizes $J_\varepsilon(u, E)$ among all functions in an appropriately defined function space when $E(x)$

is fixed, and we want to find an optimal $E_*(x)$ satisfying (1.4) so that

$$A_{\varepsilon,\Omega} = J_\varepsilon(u_{\varepsilon,E_*}, E_*) = \inf_E J_\varepsilon(u_{\varepsilon,E}, E) = \inf_{E,u} J_\varepsilon(u, E), \qquad (1.8)$$

is achieved, where the infimum is taken over all possible $E(x)$ satisfying (1.4, 1.5) and all possible density functions $u(x)$.

Such questions have interesting biological implications. The energy function like J_ε is well-known as the Lyapunov functions in mathematics, and in physics, it represents the prototypical total system energy in the sense that $(\varepsilon^2/2)\int_\Omega |\nabla u|^2 dx$ is a scaled kinetic energy, and the other part is the system potential energy. In particular, one can show that the energy of a solution of (1.6) decreases with respect to t and the solution tends to a limit equilibrium solution. For all nonnegative initial distributions u_0 ($\not\equiv 0$), this limit is a stable equilibrium solution. For a biological system, this energy function can be regarded as a measurement of the wellbeing of the entire species with respect to the exterior environment, and the decreasing of the energy and evolution of the population distribution indicate the gradual adaption of the population toward a better state for the wellbeing of the species. The energy minimizer is an optimal distribution of the population for the given natural environment and possible human interferences (like harvesting). From this view, choosing a better harvesting pattern $E(x)$ to lower the energy also increases the wellbeing of the species.

The spatial harvesting effort function $E(x)$ is a human controllable strategy subject to the natural constraint (1.4). (In the following, we call $E(x)$ the spatial harvesting strategy.) When the natural laws of growth and diffusion make the energy settling at the lowest possible level, it is the job of the controller of such strategy—the human being, to provide optimal ones to decrease the biological energy function and improve the living state of the harvested species, without sacrificing the benefit of the harvesting (so that constraint (1.4) is satisfied.) One can also regard the energy function J_ε as an intrinsic quantity of conservation biology which measures the healthiness of the species, and it may be crucial for the long term sustainability of the population.

Our main result (Theorem 2.4) is that the best strategy exists when minimizing the energy function J_ε among all admissible E and u, and the optimal strategy satisfies

$$E_*(x) = \begin{cases} 0, & x \in \Omega_0, \\ M, & x \in \Omega \setminus \Omega_0, \end{cases} \qquad (1.9)$$

where Ω_0 is a subregion of Ω, and the area of Ω_0 is $|\Omega_0| = (M - \beta)|\Omega|/M$. Thus the conclusion of our approach is that a no-harvesting zone (where $E(x) = 0$) should be designed, and the area of the no-harvesting zone Ω_0 should be as large as possible under the constraints M and β. On the other hand, in the zone which allows harvesting, the effort should be put to the maximum value. This provides new evidence of validity of the no-fishing zone which has been in heavy debates among commercial fisheries and regulators.

Our another result (Proposition 2.3) shows that if the diffusion constant ε is fixed, but the size of the habitat is decreasing due to destruction, then the optimal strategy also provides the smallest minimal patch size for the survival of the species, which again is helpful for the conservation of the species. Proposition 2.3 is an adaption of an earlier result in Cantrell and Cosner [5] (see Theorem 3.9), in which the optimized

spatial growth rate function is a "best" environment for the species. The purpose of our paper is to show that such best environment not only exists for linearized problem as in [5] but also for nonlinear problem like (1.6).

We also show that while the size of the non-harvesting zone Ω_0 is determined by the optimization, the location and the geometry of the zone is an interesting mathematical question. First, Ω_0 preserves Steiner symmetry (which will be defined in Sect. 2.3), thus Ω_0 usually locates in the central part of the habitat; secondly, while the area of Ω_0 is fixed, the optimal strategy favors smaller perimeter of Ω_0 especially when the diffusion constant is small (see details in Sect. 2.4). In particular, when the habitat is exactly a ball, then Ω_0 is a smaller concentric ball, and the harvesting will occur on the boundary side.

An interesting phenomenon is that not all symmetries are preserved for the no-harvesting zone Ω_0. In Sect. 2.3, we show that symmetry breaking occurs for an annulus with increasing inner radius and fixed width and also for a symmetric dumbbell-shaped habitat. Both lead to some practical no-harvesting zone design principles, see details in Sect. 2.3. Note that such symmetry breaking phenomena were first found in [8, 9] for the eigenvalue problem of Schrödinger operator, see also related work in [14, 15]. After this work was done, we learned about recent work of Lou and Yanagida [17] for one-dimensional Neumann boundary value problem. On the other hand, Du and Shi [11] studied a predator-prey system with a protect zone for the prey, and some related optimization problems were proposed (see the concluding remarks of [11]).

In a recent study, Neubert [20] considered the same reaction-diffusion equation (1.6) in one-spatial dimension, but to find optimal $E(x)$ to maximize the total yield:

$$Y_\varepsilon(u, E) = \int_\Omega E(x)u(x)dx. \tag{1.10}$$

The similarity between the two is that both approaches first choose the equilibrium solution $u_{\varepsilon, E}$ of (1.6) as the candidates of the next optimization problem, but in [20] (1.10) is maximized, while we minimize (1.7). While the goals of the two approaches are different, it is interesting to notice that the optimizer $E(x)$ in [20] when ε is not large (but larger than the critical patch size), is the same as ours. Thus combining the result of [20], we conclude that if the habitat $\Omega = (0, l)$, when $L_0 \leq l \leq L_1$ for some constants L_0, L_1, the strategy of setting no-harvesting zone in the center of Ω will both minimize the intrinsic energy J_ε and maximize the yield Y_ε—so we get two birds with one stone.

However, the optimal strategy in [20] when l is large is a "chattering control" with infinite sequences of reserves alternating with areas of intense fishing. As pointed in [20], such strategy would be impossible to implement, and we can see that such strategy also makes the intrinsic energy J_ε very large, which contradicts the assumption that the species prefers a lower energy level. We also point out that our results here are for higher dimensional general habitats in which the domain geometry also plays important role.

We will state our main mathematical results and explain more biological implications in four subsections of Sect. 2, and the proofs of the mathematical results will be given in Sect. 3. In the following we use standard mathematical notations such

as L^p spaces $L^p(\Omega)$, and Sobolev space $H_0^1(\Omega)$. For the definitions of these spaces, see Sect. 3.1. The habitat Ω is a bounded domain in $\mathbf{R}^n$ with Lipschitz continuous boundary for $n \geq 1$, and $|\Omega|$ is the n-dimensional Lebesgue measure of Ω.

2 Main Mathematical Results and Biological Implications

2.1 Basic Setup and Equilibrium Solutions

The population under a constant effort harvesting satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^2 - E(x)u, & x \in \Omega,\ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (2.1)$$

where $u(x, t)$ is the population density, $\varepsilon^2 > 0$ is the diffusion constant, $E(x) \in L^\infty(\Omega)$ is the harvesting effort such that $E(x) \geq 0$, and $u_0(x) \in C(\overline{\Omega})$ is the initial population distribution. From well-known theory of parabolic equations, there exists a unique solution $u(x, t)$ of (2.1), and $u(x, t) > 0$ for $(x, t) \in \Omega \times (0, \infty)$. Define biological energy function

$$J_\varepsilon(u, E) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \int_\Omega u^2 dx + \frac{1}{3} \int_\Omega u^3 dx + \frac{1}{2} \int_\Omega E(x)u^2 dx, \quad (2.2)$$

for $u \in H_0^1(\Omega)$ and $u \geq 0$. (To be more careful, we shall define the energy $J_\varepsilon(u, E) = J_\varepsilon(u_+, E)$, where $u_+ = \max\{u, 0\}$. But for the problem we consider here, the maximum principle holds, so we just avoid these technicalities here.) Then for a solution $u(x, t)$ of (2.1),

$$\begin{aligned} \frac{dJ_\varepsilon(u(\cdot, t))}{dt} &= \int_\Omega \left[-\varepsilon^2 \Delta u - u + u^2 + E(x)u \right] \frac{\partial u}{\partial t} dx, \\ &= -\int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx \leq 0, \end{aligned} \quad (2.3)$$

for $t \in (0, \infty)$. Moreover, it can be shown that $J_\varepsilon(u(x, t), E) \geq -C$ for a constant $C > 0$, and by standard theories, there exists a function $u_\infty(x) \in H_0^1(\Omega)$ such that

$$\lim_{t \to \infty} ||u(x, t) - u_\infty(x)||_{H_0^1(\Omega)} = 0. \quad (2.4)$$

The function $u_\infty(x)$ is necessarily an equilibrium solution of (2.1), *i.e.* it satisfies

$$\varepsilon^2 \Delta u + u - u^2 - E(x)u = 0, \quad x \in \Omega,\ u(x) = 0,\ x \in \partial\Omega. \quad (2.5)$$

From (2.3) and (2.4), we also find that

$$J_\varepsilon(u_\infty(\cdot), E) = \inf_{t \geq 0} J_\varepsilon(u(\cdot, t), E). \quad (2.6)$$

For the equilibrium solution equation, we have the following result:

Proposition 2.1 *Suppose that the set $\Omega_E = \{x \in \Omega : E(x) < 1\}$ is of positive measure.*

1. *There exists a positive number $\varepsilon_1 = \varepsilon_1(\Omega, E)$ such that (2.5) has only the trivial solution $u = 0$ when $\varepsilon \geq \varepsilon_1$, and when $0 < \varepsilon < \varepsilon_1$, there exists a unique positive solution $u_{\varepsilon,E}(x)$ of (2.5), and for any $x \in \Omega$, $0 < u_{\varepsilon,E}(x) < 1$;*
2. *The set of positive solutions to (2.5) can be parameterized as $S = \{(\varepsilon, u_{\varepsilon,E}) : 0 < \varepsilon < \varepsilon_1\}$, $\lim_{\varepsilon \to \varepsilon_1^-} u_{\varepsilon,E} = 0$, and $\varepsilon \mapsto u_{\varepsilon,E}$ is differentiable;*
3. *For any $0 < \varepsilon < \varepsilon_1$, $u_{\varepsilon,E}$ is globally asymptotically stable in the sense that, for any $u_0(x) \in L^2(\Omega)$ and $u_0(x) \geq 0$, the solution $u(x,t)$ of (2.1) satisfies (2.4) with $u_\infty = u_{\varepsilon,E}$;*
4. *When $\varepsilon \to 0$, then $u_{\varepsilon,E}(x) \to 1 - E(x)$ uniformly for x in any compact subset of Ω_E, and $u_{\varepsilon,E}(x) \to 0$ uniformly for x in any compact subset of $\Omega \backslash \Omega_E$.*

The results in Proposition 2.1 are well-known, and proof can be found in [5, Sect. 2] or [25, Sect. 2]. So we omit the proof here. In the following when we fix $\varepsilon > 0$, we will drop the subscript ε in $u_{\varepsilon,E}$ when no confusion. Similarly, we will drop the variable E in $J_\varepsilon(u, E)$ when no confusion. From (2.5), u_E satisfies the following integral identity:

$$\varepsilon^2 \int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx + \int_\Omega u^3 dx + \int_\Omega E(x)u^2 dx = 0. \tag{2.7}$$

From (2.7), we notice that

$$J_\varepsilon(u_E) = -\frac{1}{6} \int_\Omega u_E^3(x)dx < 0. \tag{2.8}$$

The admissible set for the harvesting effort function $E(x)$ is

$$C_{\beta,M} = \left\{ E(x) \in L^\infty(\Omega) : M \geq E(x) \geq 0, \int_\Omega E(x)dx = \beta|\Omega| \right\}, \tag{2.9}$$

where M is the maximum harvesting effort, and it is necessary that $M \geq \beta > 0$. Next we have a characterization of u_E in term of the energy function.

Proposition 2.2 *For fixed $E(x) \in C_{\beta,M}$, if $0 < \varepsilon < \varepsilon_1(\Omega, E)$, then*

$$J_\varepsilon(u_E) = \inf_{u \in H_0^1(\Omega)} J_\varepsilon(u). \tag{2.10}$$

Notice that ε_1 depends on E, and for some large $\varepsilon > 0$, there is no any strategy $E(x) \in C_{\beta,M}$ which can make the species survive, *i.e.* the only equilibrium solution is $u = 0$. Hence we shall identify the largest ε for which a non-trivial spatial strategy exists and the population will persist under such strategy.

Proposition 2.3 *There exists $\varepsilon_2 = \varepsilon_2(\Omega, M, \beta)$ so that*

$$\varepsilon_2 = \sup_{E \in C_{\beta,M}} \varepsilon_1(\Omega, E), \tag{2.11}$$

and for $0 < \varepsilon < \varepsilon_2$,

$$A_{\varepsilon,\Omega} \equiv \inf_{E \in C_{\beta,M}} J_\varepsilon(u_E) = \inf_{E \in C_{\beta,M}, u \in H_0^1(\Omega)} J_\varepsilon(u) < 0. \tag{2.12}$$

Moreover, ε_2 can be determined by

$$\varepsilon_2^2 = \sup_{\phi \in H_0^1(\Omega), \phi \not\equiv 0, E \in C_{\beta,M}} \frac{\int_\Omega [1 - E(x)]\phi^2(x)dx}{\int_\Omega |\nabla \phi(x)|^2 dx}, \tag{2.13}$$

and the supremum is achieved by $(\phi_s(x), E_s(x)) \in H_0^1(\Omega) \times C_{\beta,M}$ satisfying

$$\varepsilon_2^2 \Delta \phi_s + [1 - E_s(x)]\phi_s = 0, \quad x \in \Omega, \ \phi_s(x) = 0, \ x \in \partial\Omega, \tag{2.14}$$

and

$$E_s(x) = M[1 - \chi_{\Omega_s}(x)] = \begin{cases} 0, & x \in \Omega_s, \\ M, & x \notin \Omega_s, \end{cases} \tag{2.15}$$

for a subregion Ω_s of Ω, and there exists $t > 0$ such that

$$\Omega_s = \{x \in \Omega : \phi_{E_s}(x) > t\}. \tag{2.16}$$

In Proposition 2.3, χ_{Ω_s} is the characteristic function of Ω_s, defined by $\chi_{\Omega_s}(x) = 1$ if $x \in \Omega$ and $= 0$ otherwise. From (2.9), one can conclude that

$$|\Omega_s| = \frac{M - \beta}{M}|\Omega|. \tag{2.17}$$

For $\varepsilon < \varepsilon_2$, even though for some $E_1 \in C_{\beta,M}$, $\varepsilon > \varepsilon_1(\Omega, E_1)$, but there exists some $E_2 \in C_{\beta,M}$, such that $\varepsilon < \varepsilon_1(\Omega, E_2)$, then an optimal strategy does exist. It is well-known that the maximum diffusion constant $\varepsilon_1(\Omega, E)$ is related to the minimal patch size of the habitat (see [6, 25, 26]). When the diffusion coefficient of the habitat is a constant, but the habitat size is variable (more likely, shrinking), then the parameter ε is proportional to $\mathcal{D}/L$, where $\mathcal{D}$ is the diffusion coefficient, and L is the habitat diameter (assuming that the geometry of the habitat nearly takes the same shape when shrinking.) Thus $[\varepsilon_1(\Omega, E)]^{-1}$ can be interpreted as the minimal patch size for fixed harvesting pattern E. Notice that harvesting always increases the natural minimal patch size (the natural one should be given by (2.13) with $E(x) \equiv 0$). But Proposition 2.3 shows that there exists a best strategy $E_s \in C_{\beta,M}$ which minimizes the minimal patch size. That is of importance for the population, since more far away from the minimal patch size, the smaller chance of extinction. When the habitat size is larger than the critical one, E_s may not necessarily still be the optimal strategy, but in some case (like Ω is a ball), it still is (see the next subsection).

Proposition 2.3 essentially is same as Theorem 3.9 in [5], but we give a new proof based on more efficient and general variational principle Lemma 3.2. In the remark after Theorem 3.9 in [5], it is also indicated that the optimal control is a "bang-bang" type, and it may inherit some symmetries from Ω. Here we indeed show this from a variational approach, and in our main results next section, we also implement this idea for the nonlinear problem (2.5).

2.2 Existence of Optimal Spatial Harvesting Strategy

Our main existence of optimal strategy is

Theorem 2.4 *Suppose that $0 < \varepsilon < \varepsilon_2$ (which is defined in Proposition 2.3), then for any bounded domain Ω, and $M, \beta > 0$, there exists $E_{\varepsilon,*}(x) \in C_{\beta,M}$ such that*

$$A_{\varepsilon,\Omega} = J_\varepsilon(u_{E_{\varepsilon,*}}), \qquad (2.18)$$

where $u_{E_{\varepsilon,*}}$ is the unique solution of (2.5) associated with $E_{\varepsilon,*}$. Moreover, there exists a subregion $\Omega_{0,\varepsilon}$ of Ω such that

$$E_{\varepsilon,*}(x) = M[1 - \chi_{\Omega_{0,\varepsilon}}(x)] = \begin{cases} 0, & x \in \Omega_{0,\varepsilon}, \\ M, & x \notin \Omega_{0,\varepsilon}, \end{cases} \qquad (2.19)$$

and there exists $t > 0$ such that

$$\Omega_{0,\varepsilon} = \{x \in \Omega : u_{E_{\varepsilon,*}}(x) > t\}. \qquad (2.20)$$

Again from (2.9), one can conclude that

$$|\Omega_{0,\varepsilon}| = \frac{M - \beta}{M} |\Omega|. \qquad (2.21)$$

Theorem 2.4 has profound biological implications. The subregion $\Omega_{0,\varepsilon}$ is where the harvesting effort is to be put as zero, thus it gives a optimal design for the no-harvesting zone. Theorem 2.4 indicates that, to maximize the living state of the harvested species, the best harvesting strategy is to set up a no-harvesting zone, and secondly, the no-harvesting zone should be as large as possible. And on the other hand, in the fishing zone, the effort should be put as largest possible. However it only shows the existence of such subregion $\Omega_{0,\varepsilon}$, and the location of $\Omega_{0,\varepsilon}$ is not clear. Also, in general, the subregion $\Omega_{0,\varepsilon}$ is not unique even up to a zero measure set. One example will be shown in Sect. 2.3 that the habitat Ω is symmetric but $\Omega_{0,\varepsilon}$ is not, thus the non-uniqueness holds.

Our second result is on the preservation of the Steiner symmetry. We recall that a set G is *Steiner symmetric* with respect to a hyperplane P if for any $x \in G$, the line segment connecting x and the reflected point x^* with respect to P is contained in G.

Theorem 2.5 *Let $\Omega_{0,\varepsilon}$ be as defined in Theorem 2.4. If Ω is Steiner symmetric with respect to a hyperplane P, then $\Omega_{0,\varepsilon}$ is Steiner symmetric with respect to a hyperplane P.*

In particular now we can determine the optimal harvesting strategy $E_*(x)$ and the optimal no-harvesting zone when Ω is a ball $B_R(0) = \{x \in \mathbf{R}^n : |x| < R\}$ for $R > 0$.

Corollary 2.6

1. *If Ω is Steiner symmetric with respect to each $x_i = 0$ $(1 \le i \le n)$, then $\Omega_{0,\varepsilon}$ is Steiner symmetric with respect to each $x_i = 0$, and $\Omega_{0,\varepsilon}$ is connected and star-shaped.*

2. *If $\Omega = B_R(0)$, then the no-harvesting zone is a ball*

$$\Omega_{0,\varepsilon} \equiv \Omega_0 = \{x \in \mathbf{R}^n : |x| < r\}, \tag{2.22}$$

where r can be determined by the constrains M, β and the spatial dimension n (but independent of ε).

Notice that Corollary 2.6 also covers the case of $n = 1$ and $\Omega = (-1, 1)$, and $\Omega_0 = (-r, r)$. Here r is independent of ε since the size of the no-harvesting zone is determined by (2.21), which is free of ε. In this case Ω_s in Proposition 2.3 is also defined by (2.22).

2.3 Symmetry Breaking

In general, the spatial configuration of the no-harvesting zone $\Omega_{0,\varepsilon}$ is not easy to obtain. In particular, although Steiner symmetry is retained for the fishing zone, other symmetries of Ω may not be preserved by the fishing zone. Here we present several such examples.

One case is that the habitat is an "expanding annulus":

$$\Omega_a = \{x \in \mathbf{R}^n : a < |x| < a + 1\}. \tag{2.23}$$

This can be viewed as a circular island $I = B_a(0)$ surrounded by a fishery area of one unit length wide, thus the parameter a is the ratio between the spatial scale of the island and the spatial scale of the fishery zone.

Theorem 2.7 *Suppose that $n \geq 2$, and Ω_a is defined as in (2.23). Fix the size of the no-harvesting zone γ, $M \geq 1$, then there exists $\varepsilon_3 = \varepsilon_3(\gamma, M) > 0$ such that when $0 < \varepsilon < \varepsilon_3$, there exists a large $a_0 > 0$ such that for any $a > a_0$ any optimal harvesting strategy E_a for Ω_a is not radially symmetric.*

Notice that when the annulus is expanding, the volume of Ω_a is also increasing. Indeed,

$$|\Omega_a| = \frac{\omega_n}{n}[(a+1)^n - a^n], \tag{2.24}$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$. Thus when the formula (2.21) still holds, the percentage of the no-harvesting zone is decreasing when a is increasing, and

$$\beta = M\left(1 - \frac{\gamma}{|\Omega_a|}\right), \tag{2.25}$$

where γ is the fixed area of the no-harvesting zone. Recall that the optimal strategy for one-spatial dimension habitat is to harvest near the two endpoints, and protect the species in the middle. For annulus with smaller a, its radial counterpart may still be the optimal strategy. But our result above shows, when the no-harvesting zone is relatively small compared to the overall size of the annulus habitat, then the better way is to just set up one or several no-harvesting zones of ball shape (see the proof

of Theorem 2.7 for the mathematical arguments to support this claim.) One could heuristically argue that in this case a radially symmetric protect zone has too large perimeter compared to the width of the whole fishing area, thus the energy of radial zone is too large to be optimal. As we will see in next subsection, the harvesting zones with spherical shape are not coincidental, and the circular one is the best shape when the size of no-harvesting zone is small or the diffusion constant is small.

Next we consider a "dumbbell"-shaped habitat, which is a region consisting of two large separate spherical habitats and a thin channel connecting them. This simulates a two spatial diffusive patches B_1 and B_2 with additional dispersal between them through a thin channel H. To be more precise, let $x = (x_1, x')$ be the coordinate that $x' = (x_2, x_3, \ldots, x_n)$, define

$$B_1 = \{x : |x - (-2R, 0)| < R\}, \qquad B_2 = \{x : |x - (2R, 0)| < R\},$$
$$H = \{x = (x_1, x') : |x'| < h, |x_1| < 2R\}, \tag{2.26}$$

for small $h > 0$, and

$$\Omega_{R,h} = B_1 \cup H \cup B_2. \tag{2.27}$$

Theorem 2.8 *Suppose that $n \geq 2$, and $\Omega_{R,h}$ is defined as in (2.26) and (2.27). Fix the size of the no-harvesting zone γ, $M \geq 1$, then there exists $\varepsilon_4 = \varepsilon_4(\gamma, M) > 0$ for all $0 < \varepsilon < \varepsilon_4$, there exists a large $R_0 > 0$ such that for all $R > R_0$, there is a small $h_0 > 0$ such that for any $h < h_0$ any optimal harvesting strategy $E_{R,h}$ for $\Omega_{R,h}$ is not symmetric with respect to $\{x_1 = 0\}$.*

In particular, we show that setting up a single ball-shaped no-harvesting zone in one of the patch is a better way than setting up two equal smaller size in each patch. This will be very useful in designing the no-harvesting zone, since often the habitat consists several favorite components for the species. With the constraint on the total area of the protecting zones, shall we set up one large zone, or several smaller ones? Theorem 2.8 suggests that in the case of large components connected by thin channel and the maximum harvesting rate is high ($M > 1$), the first option is better for the population. Note that when $M > 1$, the population is close to zero in the other patch without no-harvesting zone, thus one can consider this approach is to "abandon" one patch. But remember this is because of the limit on the total size of no-harvesting zone, thus to not waste the other patch, one should lower the maximum harvesting rate, or increase the size of the no-harvesting zone.

2.4 Asymptotic Limit of the Optimal Strategy

Equation (2.5) becomes a singular perturbation problem when $\varepsilon \to 0$. The equilibrium solutions of singular perturbation problem with similar energy function but double-well potential (Cahn-Hillard equation, or constrained Allen-Cahn equation) have been studied extensively in the last two decades, see for example, [1, 3, 7, 13]. Although the forms of the equations are different, many results of these studies can be carried over to this optimization problem. Here we will state some results without proofs, and we will discuss the biological implications and their applications in no-harvesting zone design.

From Proposition 2.1, the equilibrium population distribution u_E has the following limit when $\varepsilon \to 0$,

$$u_\infty(x) = \begin{cases} 1, & x \in \Omega_0, \\ 0, & x \notin \Omega_0 \end{cases} \quad \text{if } M \geq 1,$$

$$u_\infty(x) = \begin{cases} 1, & x \in \Omega_0, \\ 1 - M, & x \notin \Omega_0 \end{cases} \quad \text{if } M < 1. \tag{2.28}$$

Thus for any subregion Ω_0 satisfying the area constraint, the potential energy part in $J_\varepsilon(u_E)$ tends to a constant since the area of Ω_0 has been determined by (2.21). Hence the minimization of total energy J_ε will very much depend on the minimization of the kinetic energy

$$\frac{\varepsilon^2}{2} \int_\Omega |\nabla u_E|^2 dx. \tag{2.29}$$

From (2.28) the gradient is very small in the interior of either Ω_0 and $\Omega \backslash \Omega_0$, but it can be large on the interface between the two. Here we can define the level set $N = \{x : u_E(x) = 1/2\}$ (when $M \geq 1$), or $N = \{x : u_E(x) = (2 - M)/2\}$ (when $M < 1$), to be the interface. Then the question of locating the optimal no-harvesting zone becomes the question of finding the optimal interface, since except around the interface, the values of $u_E(x)$ would be determined by (2.28). From the classical results on isoperimetric problem, the optimal shape to minimize the kinetic energy and still maintain the integral constrain (2.21) is a sphere enclosing the constrained volume. Finally one need to optimize the center of the sphere in minimizing the total energy, that is a more delicate question. But at least in some special cases, it has been shown that the most central point of Ω (the maximum point of $d(x, \partial\Omega)$, the distance function to the boundary of Ω) is the best location. For example, in Corollary 2.6, it is shown that when the habitat is symmetric with respect to each x_i-axis, then the center of the no-harvesting zone is necessarily at the origin point. For a different variational problem, it was shown in [21] that the maximum point of the least energy solution is close to the maximum point of the distance function. In summary, when the diffusion constant ε is small, then the no-harvesting zone should be close to a sphere, which locates in the most central part of the habitat.

Another singular limit can be taken when $M \to \infty$ (but fix β and ε). In this case, the limit of (2.5) becomes

$$\begin{cases} \varepsilon^2 \Delta u + u - u^2 = 0, & x \in \Omega \backslash \Omega_0, \\ u(x) = 0, & x \in \Omega_0 \cup \partial\Omega, \end{cases} \tag{2.30}$$

while Ω_0 is an optimal no-harvesting zone which satisfies (2.21). Therefore the minimizing of $J_\varepsilon(u, E)$ for $(u, E) \in H_0^1(\Omega) \times C_{\beta,M}$ is reduced to minimizing

$$\widetilde{J}_\varepsilon(u, \Omega_0) = \frac{1}{2} \int_{\Omega_0} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega_0} u^2 dx + \frac{1}{3} \int_{\Omega_0} u^3 dx, \tag{2.31}$$

for $u \in H_0^1(\Omega_0)$ and subregion Ω_0 satisfying (2.21). The minimizer of $\widetilde{J}_\varepsilon$ is also achieved at a sphere in Ω. A similar optimization problem is considered in [14].

2.5 Discussions

The model (1.2) is a simplification of the more complicated ones. As pointed out in [20], life stage structure, spatial heterogeneity, current advection and others can make the situation much more complicated, but qualitative nature of the optimal solution should be similar to that of (1.2) and (1.6). We have found that the optimal no-harvesting zone is always near the center of the habitat, that is partly due to the zero boundary condition (hostile exterior environment.) For other boundary conditions, the location and the geometry of the optimal no-harvesting zone might be different. Spatial heterogeneity also can play an important role in the selection of optimal no-harvesting zone. For some marine fish, it is known that the breeding beds are only small specific areas in the whole ocean, and these areas are usually near the boundary of the whole habitat (see the example of Atlantic bluefin tuna [2]). In the future work, we will investigate the effect of heterogeneous birth/death rates and crowding function, as well as the effect of long distance migration of the marine fish.

We also discuss the symmetry of the no-harvesting zones. In reality, mathematical symmetry of the domain is rarely found for the habitat of real population. However our arguments show that in general, one larger no-harvesting zone is better than several smaller ones, and the perimeter of the no-harvesting zone should be chosen as small as possible. These principles would be valuable when designing the no-harvesting zones.

3 Proofs of the Mathematical Results

3.1 Notations

In the following, Ω is a bounded smooth domain in $\mathbf{R}^n$ with $n \geq 1$. $L^p(\Omega)$ is the space of real-valued measurable functions $f : \Omega \to \mathbf{R}$ such that $\int_\Omega |f|^p dx < \infty$ for $p > 0$, and the norm of the space is

$$||f||_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}. \tag{3.1}$$

$L^\infty(\Omega)$ is the space of real-valued measurable functions $f : \Omega \to \mathbf{R}$ such that $f(x)$ is bounded except a possible zero measure set, and

$$||f||_\infty = \sup_{|O|=0} \{\sup |f(x)| : x \in \Omega \backslash O\}. \tag{3.2}$$

The Sobolev space $W_0^{1,p}(\Omega)$ is the space of L^p functions such that their weak (first order) derivatives also belong to $L^p(\Omega)$, and their boundary values are zero. In this paper we only need the space $W_0^{1,2}(\Omega) = H_0^1(\Omega)$, and the norm of $H_0^1(\Omega)$ is

$$||f||^2_{H_0^1(\Omega)} = ||f||_2^2 + |||\nabla f|||_2^2. \tag{3.3}$$

3.2 Lemmas from Previous Work

We recall the following two lemmas from [15]:

Lemma 3.1 (Lemma 1 of [15]) *Let Ω be a bounded domain in $\mathbf{R}^n$. Suppose that a sequence $\{f_j\}_{j=1}^\infty \subset L^\infty(\Omega)$ satisfies*

$$0 \le f_j(x) \le N \quad (a.e.\ x \in \Omega), \qquad \int_\Omega f_j(x)dx = \alpha \tag{3.4}$$

for some positive constants N and α. Then there exists a subsequence $\{f_{jk}\}_{k=1}^\infty$ and $f \in L^\infty(\Omega)$ such that f_{jk} converges to f weakly in $L^2(\Omega)$ and

$$0 \le f(x) \le N \quad (a.e.\ x \in \Omega), \qquad \int_\Omega f(x)dx = \alpha. \tag{3.5}$$

Lemma 3.2 (Lemma 2 of [15]) *Suppose that $u \in L^1(\Omega)$ and satisfies $u(x) \ge 0$ almost everywhere in Ω. Then for a given positive constant N and α there exists a maximizer η^* to the following maximizing problem*:

$$\sup\left\{ \int_\Omega \eta(x)u(x)dx : 0 \le \eta(x) \le N, \int_\Omega \eta(x)dx = \alpha \right\}. \tag{3.6}$$

Moreover there exists $t \ge 0$ such that any maximizer η^* can be written by $\eta^*(x) = M\chi_D$ for some subset D of Ω satisfying

$$\{x \in \Omega : u(x) > t\} \subset D \subset \{x \in \Omega : u(x) \ge t\}. \tag{3.7}$$

In fact the constant t can be determined by $t = \inf\{s : |\{u(x) > s\}| < \alpha\}$ and $M|D| = \alpha$.

Note that Lemma 3.2 is a variant of Theorem 1.14 (Bathtub Principle) in [16], and the proofs of both lemmas can be found in [15]. The following result on the Steiner symmetry is Theorem 3.6 of [12].

Lemma 3.3 *Suppose that Ω is Steiner symmetric with respect to the hyperplane $x_1 = 0$. Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfying $u(x) > 0$ in Ω and suppose that*

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega \left[f_1(u) + f_2(u) \right] \phi dx \tag{3.8}$$

for every $\phi \in H_0^1(\Omega)$. Here $f_1(t)$ is locally Lipschitz continuous and $f_2(t)$ is nondecreasing and is zero on an interval $[0, h]$ for some $h > 0$. Then we have

$$u(-x_1, x') = u(x_1, x'), \quad (x_1, x') \in \Omega, \tag{3.9}$$

and

$$\frac{\partial u}{\partial x_1}(x_1, x') < 0, \quad (x_1, x') \in \{(x_1, x') \in \Omega : x_1 > 0\}. \tag{3.10}$$

3.3 Proof of Proposition 2.2

Proof When $0 < \varepsilon < \varepsilon_1$, it is well-known that $u_{\varepsilon,E}$ is a local minimizer of J_ε, and $u = 0$ is a saddle critical point. So we only need to show that the energy functional J_ε is bounded from below, then the existence of the global minimizer follows from standard direct method of calculus of variations, and since u_E is the unique non-negative critical point, then the global minimum must achieve at u_E.

To show J_ε is bounded from below, define $\Omega_1 = \{x : u(x) \leq 3/2\}$. For any $u \geq 0$, since $E(x) \geq 0$, then

$$J_\varepsilon(u) \geq \frac{1}{6} \int_\Omega u^2(2u - 3)dx \geq \frac{1}{6} \int_{\Omega_1} u^2(2u - 3)dx$$

$$\geq \frac{1}{6} \left(\frac{3}{2}\right)^2 (-3)|\Omega| = -\frac{9}{8}|\Omega|. \tag{3.11}$$

$\square$

3.4 Proof of Proposition 2.3

Proof It is well-known that for fixed $E(x) \in C_{\beta,M}$, $\varepsilon_1(\Omega, E)$ can be characterized by

$$\varepsilon_1^2 = \sup_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega [1 - E(x)]\phi^2(x)dx}{\int_\Omega |\nabla\phi(x)|^2 dx}, \tag{3.12}$$

and the maximizer ϕ_E satisfies

$$\varepsilon_1^2 \Delta\phi_E + [1 - E(x)]\phi_E = 0, \quad x \in \Omega, \ \phi_E(x) = 0, \ x \in \partial\Omega. \tag{3.13}$$

Now we maximize among $E \in C_{\beta,M}$. The supremum exists since

$$\int_\Omega [1 - E(x)]\phi^2(x)dx \leq |M - 1| \int_\Omega \phi^2(x)dx \leq \lambda_1^{-1}(\Omega)|M - 1| \int_\Omega |\nabla\phi(x)|^2 dx, \tag{3.14}$$

from the Poincaré inequality. We take a minimizing sequence $E_j \in C_{\beta,M}$, and assuming $\int_\Omega |\nabla\phi_{E_j}(x)|^2 dx = 1$ such that $\int_\Omega [1 - E_j(x)]\phi_{E_j}^2(x)dx \to \varepsilon_2^2$. Since $\{\phi_{E_j}\}$ is bounded in $H_0^1(\Omega)$, then there is a subsequence of $\{\phi_{E_j}\}$ (for simplicity, still denoted by $\{\phi_{E_j}\}$) which converges to ϕ weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$ for $p < 2n/(n - 2)$, weakly in $L^{2n/(n-2)}(\Omega)$, and $\phi_{E_j} \to \phi$ almost everywhere in Ω. From Lemma 3.1, we can choose a further subsequence (still denoted by $\{\phi_{E_j}\}$) such that $E_j \to E_s$ weakly in $L^2(\Omega)$ for some $E_s \in L^\infty(\Omega)$ satisfying

$$0 \leq E_s(x) \leq M \quad (\text{a.e. } x \in \Omega), \quad \int_\Omega E_s(x)dx = \beta|\Omega|. \tag{3.15}$$

Then

$$\int_\Omega [1 - E_s(x)]\phi^2(x)dx = \varepsilon_2^2, \tag{3.16}$$

and it is necessary that $\phi = \phi_{E_s}$ (solution of (3.12)) otherwise $\int_\Omega [1 - E_s(x)] \phi_{E_s}^2(x) dx > \varepsilon_2^2$.

Finally we rewrite $\int_\Omega [1 - E_s(x)] \phi_{E_s}^2(x) dx$ into

$$\int_\Omega [M - E_s(x)] \phi_{E_s}^2(x) dx + \int_\Omega (1 - M) \phi_{E_s}^2(x) dx, \tag{3.17}$$

and we maximize the integral $\int_\Omega [M - E(x)] \phi_{E_s}^2(x) dx$ for $E \in C_{\beta,M}$. The maximizer must be E_s otherwise we have a contradiction again. From Lemma 3.2, the maximizer $E_s = M(1 - \chi_{\Omega_s})$ for some subregion Ω_s of Ω, and

$$\{x \in \Omega : \phi_{E_s}(x) > t\} \subset \Omega_s \subset \{x \in \Omega : \phi_{E_s}(x) \geq t\}. \tag{3.18}$$

We claim that $\Omega_s = \{x \in \Omega : \phi_{E_s}(x) > t\}$ up to a difference of zero measure set. Suppose not, there exists a closest subset A of Ω_s such that $|A| > 0$, $\phi_{E_s}(x) = t$ for any $x \in A$. Then there exists $x_0 \in A$ such that $\nabla \phi_{E_s}(x_0) = \Delta \phi_{E_s}(x_0) = 0$, and from the equation, $\phi_{E_s}(x_0) = 0$, which is a contradiction since $t > 0$. Thus the claim is proved, and this completes the proof of Proposition 2.3. □

3.5 Proof of Theorem 2.4

Proof Since $0 < \varepsilon < \varepsilon_2$, there exists $E \in C_{\beta,M}$ such that (2.5) has a positive solution $u_E = u_{\varepsilon,E}$. From $0 < u_E(x) < 1$ and (2.8), we have

$$J_\varepsilon(u_E) = -\frac{1}{6} \int_\Omega u_E^3(x) dx \geq -\frac{1}{6} |\Omega|. \tag{3.19}$$

Thus $A_{\varepsilon,\Omega} \geq -(1/6)|\Omega|$. We take a minimizing sequence $E_j \in C_{\beta,M}$ such that $J_\varepsilon(u_{E_j}) \to A_{\varepsilon,\Omega}$. From $0 < u_E(x) < 1$ and (2.8), we conclude that $\{u_{E_j}\}$ is bounded in $H_0^1(\Omega)$ and $L^p(\Omega)$ for any $p > 0$. Thus there is a subsequence of $\{u_{E_j}\}$ (for simplicity, still denoted by $\{u_{E_j}\}$) which converges to u weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$ for $p < (2n)/(n-2)$, weakly in $L^{2n/(n-2)}(\Omega)$, and $u_{E_j} \to u$ almost everywhere in Ω. From Lemma 3.1, we can choose a further subsequence (still denoted by $\{u_{E_j}\}$) such that $E_j \to E_*$ weakly in $L^2(\Omega)$ for some $E_* \in L^\infty(\Omega)$ satisfying

$$0 \leq E_*(x) \leq M \quad (\text{a.e. } x \in \Omega), \quad \int_\Omega E_*(x) dx = \beta |\Omega|. \tag{3.20}$$

Moreover, from the weak convergence of $\{u_{E_j}\}$ and $\{E_j\}$, we have

$$\int_\Omega (\varepsilon^2 \nabla u \cdot \nabla \phi \, dx + u\phi - u^2\phi - E_*(x)u\phi) dx = 0, \tag{3.21}$$

for any $\phi \in H_0^1(\Omega)$. Thus u is a weak solution of (2.5) with $E(x) = E_*(x)$, and from the elliptic regularity theorems and maximum principle, $u = u_{E_*} \in W_{loc}^{2,p}(\Omega) \cap C_0^{1,\gamma}(\Omega) \cap C(\overline{\Omega})$ for $p > 0$ and $\gamma \in (0, 1)$, and $u(x) > 0$ for $x \in \Omega$. From the weakly

lower semi-continuity of the Dirichlet integral and dominant convergence theorem,

$$J_\varepsilon(u_{E_*}) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla u_{E_*}|^2 dx - \frac{1}{2}\int_\Omega u_{E_*}^2 dx + \frac{1}{3}\int_\Omega u_{E_*}^3 dx$$
$$+ \frac{1}{2}\int_\Omega E_*(x)u_{E_*}^2 dx \le A_{\varepsilon,\Omega}. \tag{3.22}$$

We rewrite $J_\varepsilon(u_{E_*})$ into

$$J_\varepsilon(u_{E_*}) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla u_{E_*}|^2 dx - \frac{1}{2}\int_\Omega (1-M)u_{E_*}^2 dx + \frac{1}{3}\int_\Omega u_{E_*}^3 dx$$
$$- \frac{1}{2}\int_\Omega [M - E_*(x)]u_{E_*}^2 dx.$$

Then the function $M \ge M - E_*(x) \ge 0$ for $x \in \Omega$, and $M - E_*(x) \in L^\infty(\Omega)$. From Lemma 3.2, the maximizing problem:

$$\sup\left\{ \int_\Omega \eta(x)u_{E_*}^2 dx : 0 \le \eta(x) \le M, \int_\Omega \eta(x)dx = (M-\beta)|\Omega| \right\}, \tag{3.23}$$

has a maximizer η which is in form of $M\chi_{\Omega_{0,\varepsilon}}$ for some subregion $\Omega_{0,\varepsilon}$ of Ω. Then $M - E_*(x)$ must be identical to $\eta(x)$, otherwise

$$\frac{\varepsilon^2}{2}\int_\Omega |\nabla u_{E_*}|^2 dx - \frac{1}{2}\int_\Omega u_{E_*}^2 dx + \frac{1}{3}\int_\Omega u_{E_*}^3 dx + \frac{1}{2}\int_\Omega [M - \eta(x)]u_{E_*}^2 dx < A_{\varepsilon,\Omega},$$

which contradicts with (2.12). Moreover from Lemma 3.2, there exists $t > 0$ such that

$$\{x \in \Omega : u_{E_*}(x) > t\} \subset \Omega_{0,\varepsilon} \subset \{x \in \Omega : u_{E_*}(x) \ge t\}. \tag{3.24}$$

We claim that $\Omega_{0,\varepsilon} = \{x \in \Omega : u_{E_*}(x) > t\}$ up to a difference of zero measure set. Suppose not, there exists a closest subset A of $\Omega_{0,\varepsilon}$ such that $|A| > 0$, $u_{E_*}(x) = t$ for any $x \in A$. Then there exists $x_0 \in A$ such that $\nabla u_{E_*}(x_0) = \Delta u_{E_*}(x_0) = 0$, and from the equation, $u_{E_*}(x_0) - u_{E_*}^2(x_0) = 0$, which is a contradiction since $0 < u_{E_*}(x) < 1$ for any $x \in \Omega$. Thus

$$\Omega_{0,\varepsilon} = \{x \in \Omega : u_{E_*}(x) > t\}. \tag{3.25}$$

$\square$

3.6 Proof of Theorem 2.5

Proof Suppose that Ω is Steiner symmetric with respect to the hyperplane $x_1 = 0$. Let (E_*, u_{E_*}) be the solution of optimization problem in Theorem 2.4. Then

$$-\varepsilon^2 \Delta u_{E_*} = (1-M)u_{E_*} - u_{E_*}^2 + M\chi_{\Omega_{0,\varepsilon}}u_{E_*}. \tag{3.26}$$

Let $f_1(u) = (1-M)u - u^2$ and let $f_2(u) = M\chi_{\Omega_{0,\varepsilon}}u = MH(t-u)u$, where $H(x)$ is the Heaviside function such that $H(x) = 0$ when $x \le 0$, and $H(x) = 1$ when $x > 0$.

Then we apply Lemma 3.3 to conclude that

$$u_{E_*}(-x_1, x') = u_{E_*}(x_1, x'), \quad (x_1, x') \in \Omega, \tag{3.27}$$

and

$$\frac{\partial u_{E_*}}{\partial x_1}(x_1, x') < 0, \quad (x_1, x') \in \{(x_1, x') \in \Omega : x_1 > 0\}. \tag{3.28}$$

In particular $\Omega_{0,\varepsilon}$ is Steiner symmetric with respect to a hyperplane $x_1 = 0$. □

3.7 Proof of Theorem 2.7

Proof We choose k mutually disjoint balls $B_j(x_{j,a}, r_0)$, $(j = 1, \ldots, k)$, so that $r_0 < 1/2$, $B_j(x_{j,a}, r_0) \subset \Omega_a$, and $|B_j(x_{j,a}, r_0)| = r_0^n \omega_n / n = \gamma/k$. Define

$$D_0 = \Omega_a \setminus \bigcup_{j=1}^{k} B_j(x_{j,a}, r_0). \tag{3.29}$$

Then $E_0(x) = M\chi_{D_0}(x) \in C_{\beta,M}$.

We first prove that $\varepsilon_2(\Omega_a, M, \beta(M, \gamma))$ (defined in Proposition 2.3) is bounded from below for all $a \geq a_1$ for some $a_1 > 0$. Let a_1 be a number such that Ω_a could contain k mutually disjoint balls with radius r_0. For $D_0^c = \bigcup_{j=1}^{k} B_j(x_{j,a}, r_0)$, there exists $\varepsilon_1(D_0^c, 0)$ defined as in (3.12), and there exists an associated principal eigenfunction $\phi_1 \in H_0^1(D_0^c) \subset H_0^1(\Omega_a)$. Then

$$\varepsilon_2^2(\Omega_a, M, \beta(M, \gamma)) = \sup_{\phi \in H_0^1(\Omega), \phi \not\equiv 0, E \in C_{\beta,M}} \frac{\int_\Omega [1 - E(x)]\phi^2(x)dx}{\int_\Omega |\nabla\phi(x)|^2 dx}$$

$$\geq \frac{\int_{D_0^c} \phi_1^2(x)dx}{\int_{D_0^c} |\nabla\phi_1(x)|^2 dx} = \varepsilon_1^2(D_0^c, 0) > 0. \tag{3.30}$$

Thus the lower bound of $\varepsilon_2(\Omega_a, M, \beta(M, \gamma))$ is independent of a since $\varepsilon_1(D_0^c, 0)$ is independent of a.

Now we fix $\varepsilon < \varepsilon_1(D_0^c, 0)$. Let $v(x)$ be the unique solution of

$$\varepsilon^2 \Delta v + v - v^2 = 0, \quad x \in D_0^c, \ v(x) = 0, \ x \in \partial D_0^c, \tag{3.31}$$

and let u_{E_0} be the unique solution of (2.5) with $E_0(x) = M\chi_{D_0}(x) \in C_{\beta,M}$. From the comparison principle, we have

$$u_{E_0}(x) > v(x), \quad x \in D_0^c. \tag{3.32}$$

Hence

$$J_\varepsilon(u_{E_0}) = -\frac{1}{6}\int_{\Omega_a} u_{E_0}^3(x)dx \leq -\frac{1}{6}\int_{D_0^c} v^3(x)dx < -\delta_0, \tag{3.33}$$

for a constant δ_0 independent of $a > a_1$.

In the remaining part of the proof, we will prove that if the optimal strategy $E_a(x)$ and associated $\Omega_{0,a}$ are radially symmetric for $a > a_1$, then $J_{\varepsilon,a}(u_{E_*}) \to 0$ as $a \to \infty$. Therefore a contradiction is reached with (3.33), and it proves that there exists $a_0 > a_1$ such that the optimal strategy $E_a(x)$ and associated $\Omega_{0,a}$ are not radially symmetric for $a > a_0$.

Suppose that the optimal strategy $E_a(x)$ and associated $\Omega_{0,a}$ are radially symmetric. We express $\Omega_{0,a}$ in a polar coordinate system:

$$\Omega_{0,a} = \{(r, \omega) \in (a, a+1) \times S^{n-1} : r \in D_1, \omega \in S^{n-1}\}, \tag{3.34}$$

for a subset D_1 of $(a, a+1)$. Since $|\Omega_{0,a}| = \gamma$, then $|D_1| \le Ca^{1-n}$ for some constant $C > 0$. Since $u = u_{E_a}$ is also radially symmetric, it satisfies

$$\begin{cases} \varepsilon^2(r^{n-1}u')' + r^{n-1}u - r^{n-1}u^2 - Mr^{n-1}(1 - \chi_{D_1}(r))u = 0, & a < r < a+1, \\ u(a) = u(a+1) = 0. \end{cases}$$
$$\tag{3.35}$$

For any $r \in (a, a+1)$, we have

$$a^{n-1}u'(a) - r^{n-1}u'(r) + \int_a^r s^{n-1}u^2(s)ds$$
$$= \int_a^r s^{n-1}[(1-M)u(s) + M\chi_{D_1}(s)u(s)]ds, \tag{3.36}$$

and

$$r^{n-1}u'(r) - (a+1)^{n-1}u'(a+1) + \int_r^{a+1} s^{n-1}u^2(s)ds$$
$$= \int_r^{a+1} s^{n-1}[(1-M)u(s) + M\chi_{D_1}(s)u(s)]ds, \tag{3.37}$$

For r satisfying $u'(r) \le 0$, we use (3.36), $0 < u(s) < 1$ and $M \ge 1$ to conclude

$$r^{n-1}|u'(r)| \le M \int_a^r s^{n-1}\chi_{D_1}(s)ds, \tag{3.38}$$

and for r satisfying $u'(r) \ge 0$, we use (3.37), and similarly we have

$$r^{n-1}|u'(r)| \le M \int_r^{a+1} s^{n-1}\chi_{D_1}(s)ds. \tag{3.39}$$

Hence

$$a^{n-1}|u'(r)| \le \sup_{a \le r \le a+1} \left| r^{n-1}u'(r) \right| \le M \int_a^{a+1} s^{n-1}\chi_{D_1}(s)ds$$
$$\le M(a+1)^{n-1}|D_1| \le CM \frac{(a+1)^{n-1}}{a^{n-1}}, \tag{3.40}$$

and for any $r \in [a, a+1]$, we have

$$|u'(r)| \leq C_1 a^{1-n}, \quad \text{and} \quad |u(r)| \leq C_2 a^{1-n}. \tag{3.41}$$

From (2.8) and (3.41), since $n \geq 2$, then

$$\left| J_{\varepsilon,a} \right| = \left| -\frac{1}{6} \int_{\Omega_a} u^3(x) dx \right| \leq C_3 \frac{(a+1)^n}{a^{3n-3}} \to 0 \quad (a \to \infty), \tag{3.42}$$

which completes the proof. $\qquad\square$

3.8 Proof of Theorem 2.8

First we show that for large R and small h, there is a uniform bound for $\varepsilon_2(\Omega_{R,h}, M, \beta(M, \gamma))$. Indeed such a bound is determined by γ. Suppose that R is large enough so that a ball with measure γ can be inscribed in B_R. Then

$$\varepsilon_2^2(\Omega_{R,h}, M, \beta(M, \gamma)) \geq \varepsilon_1^2(B_\rho, 0) > 0. \tag{3.43}$$

So similar to the proof of Theorem 2.7, a lower bound for $\varepsilon_2(\Omega_{R,h}, M, \beta(M, \gamma))$ is independent of R and h since $\varepsilon_1(B_\rho, 0)$ is. Thus ε_4 in the theorem can be found. In the following we will fix $\varepsilon < \varepsilon_4$.

For large R and a fixed $\gamma > 0$, define

$$A_{\gamma, B_R} \equiv A_{\varepsilon, \gamma, B_R} = \inf_{E \in C_{\beta_1, M, R}} J_\varepsilon(u_E), \tag{3.44}$$

where

$$C_{\beta_1, M, R} = \left\{ E(x) \in L^\infty(B_R) : M \geq E(x) \geq 0, \int_\Omega E(x) dx = \beta_1 |\Omega| \right\}, \tag{3.45}$$

and β_1 is defined as

$$\beta_1 = M \left(1 - \frac{\gamma}{|B_R|} \right). \tag{3.46}$$

The proof of Theorem 2.7 is mainly based on

Proposition 3.4 *There exists $R_0 > 0$ such that for any $R > R_0$,*

$$A_{\gamma, B_R} < 2A_{\gamma/2, B_R}. \tag{3.47}$$

Since A_{γ, B_R} is achieved when the no-harvesting zone Ω_0 is radial, we define

$$A_R^\rho (= A_{\gamma, B_R}) = J_\varepsilon(u_R^\rho), \tag{3.48}$$

where u_R^ρ ($\rho \leq R$) is the radial solution of

$$\varepsilon^2 \Delta u + u - u^2 - M(1 - \chi_{B_\rho})u = 0, \quad x \in B_R, \ u = 0, \ x \in \partial B_R. \tag{3.49}$$

We recall some estimates for A_R^ρ and u_R^ρ using ideas from [15]: (Part (1) and (2) are proved in Lemma 6 of [15], and Part (3) is proved in Lemma 9 of [15])

Lemma 3.5

1. *If $\Omega_1 \subset \Omega_2$, then $A_{\varepsilon,\Omega_1} \geq A_{\varepsilon,\Omega_2}$.*
2. *If $\rho_1 \leq \rho_2$, then $A_R^{\rho_1} \geq A_R^{\rho_2}$.*
3. *Assuming $M \geq 1$, then there exists such that*

$$0 < u_R^\rho(x) \leq \frac{C}{|x|^2}, \quad \text{for } x \in B_R \backslash \overline{B_\rho}. \tag{3.50}$$

Proof of Proposition 3.4 We prove (3.47) by showing (assuming $|B_\rho| = \gamma/2$)

$$\lim_{R\to\infty} A_{\gamma,B_R} = \lim_{R\to\infty} A_R^{2^{1/n}\rho} < 2 \lim_{R\to\infty} A_R^\rho = 2 \lim_{R\to\infty} A_{\gamma/2,B_R}. \tag{3.51}$$

First for any $R > \rho$,

$$\frac{\varepsilon^2}{2} \int_{B_R} |\nabla u_R^\rho|^2 dx + \frac{1}{3} \int_{B_R} \left(u_R^\rho\right)^3 dx + \frac{M-1}{2} \int_{B_R} \left(u_R^\rho\right)^2 dx$$

$$= A_R^\rho + \frac{M}{2} \int_{B_\rho} \left(u_R^\rho\right)^2 dx \leq A_\rho^\rho + \frac{M}{2}|B_\rho| < \infty, \tag{3.52}$$

Thus if we extend u_R^ρ to $\mathbf{R}^n$ by zero-extension, then for any $R > \rho$,

$$||\nabla u_R^\rho||_{L^2(\mathbf{R}^n)} + ||u_R^\rho||_{L^3(\mathbf{R}^n)} \leq C. \tag{3.53}$$

Then there exists a subsequence of $\{u_R^\rho\}$, say $\{u_{R_j}^\rho\}$ converges weakly in $L^3(\mathbf{R}^n) \cap H^1_{loc}(\mathbf{R}^n)$ to a limit u_∞, and u_∞ satisfies

$$\varepsilon^2 \Delta u + u - u^2 - Mu + M\chi_{|B_\rho}u = 0, \quad x \in \mathbf{R}^n. \tag{3.54}$$

Note that for $R_1 < R_2$, we have $u_{R_1}^\rho(x) < u_{R_2}^\rho(x)$, thus $u_\infty(x) > u_R^\rho(x) > 0$ for any $x \in \mathbf{R}^n$. Also from Lemma 3.5 part 3,

$$0 < u_\infty(x) \leq \frac{C}{|x|^2}, \quad \text{for } x \in \mathbf{R}^n \backslash \overline{B_\rho}. \tag{3.55}$$

Hence

$$J_{\varepsilon,\infty}^\rho(u_\infty) \equiv \frac{\varepsilon^2}{2} \int_{\mathbf{R}^n} |\nabla u_\infty|^2 dx + \frac{1}{3} \int_{\mathbf{R}^n} u_\infty^3 dx$$

$$+ \frac{M-1}{2} \int_{\mathbf{R}^n} u_\infty^2 dx - \frac{M}{2} \int_{B_\rho} u_\infty^2 dx$$

$$\leq \lim_{R\to\infty} A_R^\rho. \tag{3.56}$$

Define $\overline{u}_\infty(x) = u_\infty(2^{-1/n}x)$. Then

$$J_{\varepsilon,\infty}^{2^{1/n}\rho}(\overline{u}_\infty) = \varepsilon^2 \left(2^{-1/n} - 1\right) \int_{\mathbf{R}^n} |\nabla u_\infty|^2 + 2J_{\varepsilon,\infty}^\rho(u_\infty) < 2J_{\varepsilon,\infty}^\rho(u_\infty), \tag{3.57}$$

and (3.56) and (3.57) together implies

$$J_{\varepsilon,\infty}^{2^{1/n}\rho}(\overline{u}_\infty) < 2 \lim_{R\to\infty} A_R^\rho. \qquad (3.58)$$

On the other hand, since $0 < \overline{u}_\infty < \min(1, C2^{2/n}|x|^{-2})$ for $x \in \mathbf{R}^n$, then $v_L(x) \equiv [\overline{u}_\infty - L^{-1}]_+ = 0$ when $|x| \geq 2^{1/n}(CL)^{1/2} = R_L$. In particular $v_L \in H_0^1(B_{R_L})$. From the definition of A_R^ρ, we have

$$A_{R_L}^{2^{1/n}\rho} \leq J_{\varepsilon,\infty}^{2^{1/n}\rho}(v_L). \qquad (3.59)$$

From the monotonicity of A_R^ρ,

$$\lim_{R\to\infty} A_R^{2^{1/n}\rho} \leq \lim_{L\to\infty} A_{R_L}^{2^{1/n}\rho} \leq \limsup_{L\to\infty} J_{\varepsilon,\infty}^{2^{1/n}\rho}(v_L). \qquad (3.60)$$

It is easy to check that

$$J_{\varepsilon,\infty}^{2^{1/n}\rho}(v_L) \to \frac{\varepsilon^2}{2} \int_{\mathbf{R}^n} |\nabla \overline{u}_\infty|^2 dx + \frac{1}{3} \int_{\mathbf{R}^n} \overline{u}_\infty^3 dx$$
$$+ \frac{M-1}{2} \int_{\mathbf{R}^n} \overline{u}_\infty^2 dx - \frac{M}{2} \int_{B_{2^{1/n}\rho}} \overline{u}_\infty^2 dx$$
$$\equiv J_{\varepsilon,\infty}^{2^{1/n}\rho}(\overline{u}_\infty). \qquad (3.61)$$

From (3.58) and (3.61), we complete the proof of (3.51). $\square$

Now we complete the proof of Theorem 2.8.

Proof of Theorem 2.8 From Proposition 3.4, there exists $R_0 > 0$ such that (3.47) holds. We fix $R > R_0$, and then

$$2A_{\gamma/2, B_R} - A_{\gamma, B_R} = \delta(R) > 0. \qquad (3.62)$$

If the no-harvesting zone Ω_0 for $\Omega_{R,h}$ is symmetric with respect to $\{x_1 = 0\}$, then $\gamma_1 = |\Omega_0 \cap B_1| = |\Omega_0 \cap B_2| = \gamma_2$, and $2\gamma_1 \leq \gamma$. As $h \to 0$, $2\gamma_1 = \gamma + o(h^{n-1})$. From Lemma 3.5 part 1,

$$A_{\gamma, B_1} \geq A_{\gamma, \Omega_{R,h}} \geq A_{\gamma/2, B_1} + A_{\gamma/2, B_2} - C_1 h^k = 2A_{\gamma/2, B_1} - C_1 h^k, \qquad (3.63)$$

for h is small and some positive constants C_1, k. (For details on the estimates of small h, see [15, pp. 271–273].) For small $h > 0$, (3.62) and (3.63) leads to a contradiction. Therefore Ω_0 cannot be symmetric with respect to $\{x_1 = 0\}$. $\square$

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