DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **21**, Number **1**, May **2008**

Website: http://aimSciences.org

pp. **245–258**

DYNAMICS OF A REACTION-DIFFUSION SYSTEM OF AUTOCATALYTIC CHEMICAL REACTION

Jifa Jiang

Department of Mathematics, Tongji University
Shanghai 200092, China

Junping Shi

Department of Mathematics, College of William and Mary
Williamsburg, VA 23187-8795, USA, and
School of Mathematics, Harbin Normal University
Harbin, Heilongjiang 150080, China

Dedicated to Professor Edward Norman Dancer on the occasion of his 60th birthday

Abstract. The precise dynamics of a reaction-diffusion model of autocatalytic chemical reaction is described. It is shown that exactly either one, two, or three steady states exists, and the solution of dynamical problem always approaches to one of steady states in the long run. Moreover it is shown that a global codimension one manifold separates the basins of attraction of the two stable steady states. Analytic ingredients include exact multiplicity of semilinear elliptic equation, the theory of monotone dynamical systems and the theory of asymptotically autonomous dynamical systems.

1. **Introduction.** The model representation for an isothermal autocatalytic chemical reaction is

$$A + pB \to (p+1)B, \tag{1}$$

and the reaction rate is kab^p, where a and b are the concentrations of the reactant A and the autocatalyst B, and $p \geq 1$ is the order of the reaction with respect to the autocatalytic species [8]. The equations describing the reaction and diffusion of the two reactants A and B in a bounded region are

$$\frac{\partial a}{\partial t} = D_A \Delta a - kab^p, \;\; \frac{\partial b}{\partial t} = D_B \Delta b + kab^p, \;\; t > 0, \;\; x \in \Omega, \tag{2}$$

where D_A and D_B are the diffusion coefficients of A and B respectively, and Ω is a bounded reaction zone in $\mathbf{R}^n$. Following [8] (see page 240–247), we examine the behavior of an open system in which the transport of reactants and products relies on molecular diffusion processes. Outside the reaction zone is an external reservoir where the chemicals A and B have fixed concentrations. The reservoir provides

2000 *Mathematics Subject Classification.* Primary: 35J55; Secondary: 35B40, 80A32.

Key words and phrases. autocatalytic chemical reaction, asymptotic autonomous system, convergence to equilibrium.

The first author is partially supported by the Chinese NSF grants 10671143 and 10531030; the second author is partially supported by United States NSF grants DMS-0314736 and EF-0436318, Chinese NSF grant 10671049, and Longjiang scholar grant from Department of Education of Heilongjiang Province, China.

a source of reactants which can diffuse across the boundary $\partial\Omega$ into Ω, and it is either a source or a sink for the intermediate and final products. Thus the boundary conditions of A and B can be taken as

$$a(x,t) = a_0 > 0, \text{ and } b(x,t) = b_0 \geq 0, \quad x \in \partial\Omega. \tag{3}$$

With transformation $t' = kt$, $D'_A = D_A/k$, $D'_B = D_B/k$, and omitting the primes for typographical simplicity, we can convert (2) and (3) into

$$\begin{cases} \dfrac{\partial a}{\partial t} = D_A \Delta a - ab^p, \ \ \dfrac{\partial b}{\partial t} = D_B \Delta b + ab^p, & t > 0, \ \ x \in \Omega, \\ a(x,t) = a_0 > 0, \text{ and } b(x,t) = b_0 \geq 0, & t > 0, \ \ x \in \partial\Omega, \\ a(x,0) = A_0(x) \geq 0, \ \ b(x,0) = B_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{4}$$

Here we also add the initial concentrations of A and B as part of equations.

Many studies have been on the same equation but with different boundary conditions:

$$\begin{cases} \dfrac{\partial a}{\partial t} = D_A \Delta a - ab^p, \ \ \dfrac{\partial b}{\partial t} = D_B \Delta b + ab^p, & t > 0, \ \ x \in \Omega, \\ \alpha_1(x)\dfrac{\partial a(x,t)}{\partial n} + (1-\alpha_1(x))a(x,t) = 0, & t > 0, \ \ x \in \partial\Omega, \\ \alpha_2(x)\dfrac{\partial b(x,t)}{\partial n} + (1-\alpha_2(x))b(x,t) = 0, & t > 0, \ \ x \in \partial\Omega, \\ a(x,0) = A_0(x) \geq 0, \ \ b(x,0) = B_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{5}$$

where $\alpha_i(x)$ $(i = 1,2)$ is a non-negative C^2-function on $\partial\Omega$ such that $0 \leq \alpha_i(x) \leq 1$, and $\partial/\partial n$ denotes the outer normal derivative. Alikakos [1] obtained L^∞ bounds of global solutions of (5) when $\alpha_i(x) \equiv 1$ (homogeneous Neumann boundary condition) and $1 \leq p \leq (n+2)/n$; Masuda [14] showed the same result for arbitrary $p > 0$; Haraux and Youkana [10] gave a shorter proof of Masuda's result based on a Lyapunov functional argument, and their result also held for the case b^p is replaced by e^{cb} for some $c > 0$. Moreover in [14, 10], it was shown that any solution $(a(x,t), b(x,t))$ converges to a constant steady state solution (c_1, c_2) such that $c_1 \cdot c_2 = 0$. Indeed in the Neumann case, we can further obtain $c_1 = 0$ and $c_2 = \int_\Omega [A_0(x) + B_0(x)]dx > 0$, while when $0 \leq \alpha_i(x) < 1$ we must have $c_1 = c_2 = 0$. Other related results about this model can be found in [6, 13, 15, 21, 23].

From the viewpoint of chemical reactions, the results above are not surprising, since the outer flux of the reactant A is outward or zero along the boundary, and the total amount of A and $A + B$ are both decreasing, *i.e.*

$$\frac{\partial a(x,t)}{\partial n} \leq 0, \ \ \frac{\partial b(x,t)}{\partial n} \leq 0, \ \ \frac{d}{dt}\int_\Omega a(x,t)dx < 0, \quad \text{and } \frac{d}{dt}\int_\Omega [a(x,t)+b(x,t)]dx < 0,$$

which can be easily shown from the boundary conditions and integration of the equations. Hence sooner or later all reactant A will be consumed inside Ω or leak to the exterior without reenforcement, and the amount of the catalyst/product B may initially increase due to reaction but eventually drop to zero because the exhaustion of A and the leakage, except in the closed system (Neumann boundary condition) $c_2 > 0$ because of no leakage. Hence the completely studied system (5) does not describe a sustainable chemical reaction. A source for the reactant is needed for continuous reaction inside the reactor.

The purpose of the current paper is to investigate (4) in which the source of A is the reservoir exterior to the reactor. We shall mainly study the asymptotic behavior of solutions for (4) in which the external reservoir is without any catalyst

B, that is, $b_0 = 0$. Our main assumption is equal diffusion coefficients ($D_A = D_B$) and reactor Ω is spherical, *i.e.* $\Omega = B^n = \{x \in \mathbf{R}^n : |x| < 1\}$, the unit ball in $\mathbf{R}^n$. Spherical geometry for the reactor is typical: it represents spherical ($n = 3$, and $\Omega = B^3$), cylindrical ($n = 2$, and $\Omega = B^2$), and linear ($n = 1$, and $\Omega = (-1, 1)$).

It will be shown that either (a) the trivial solution $(a_0, 0)$ is globally asymptotically stable in the nonnegative cone; or (b) there is a codimension one separating manifold M_2 passing through the unique positive steady state (a_2, b_2) which is degenerate such that all solutions for (4) with $b_0 = 0$ is convergent to either $(a_0, 0)$ or (a_2, b_2); or (c) there are exactly two positive steady states (a_2, b_2) and (a_1, b_1) both of which are hyperbolic such that the global stable manifold for (a_2, b_2) is a codimension one C^1 manifold which separates the positive cone into two parts: one is the basin of attraction for $(a_0, 0)$, the other is the basin of attraction for (a_1, b_1). Since the first and third alternatives are hyperbolic and structurally stable, for $b_0 > 0$ sufficiently small and/or $D_A/D_B \approx 1$, the same conclusions hold. The main tools used in this paper are the results on existence and exact multiplicity of positive solutions of scalar elliptic equation with Dirichlet boundary condition, the theory of monotone dynamical systems and the theory of asymptotically autonomous dynamical systems. The organization of the present paper is as follows. In Section 2 we recall results on the steady state equations. Section 3 presents the results on scalar parabolic equation which correspond to that in Section 2. Section 4 gives the classification results stated above. We conclude the paper with some more remarks on the application to the chemical reaction model in Section 5.

2. **Steady state solutions.** Here we collect some results concerning the steady state solutions of (4). Most of them are previously known, thus we will omit the proofs unless necessary. The steady states of (4) satisfy

$$\begin{cases} D_A \Delta a - ab^p = 0, \;\; D_B \Delta b + ab^p = 0, & x \in \Omega, \\ a(x) = a_0 > 0, \;\; b(x) = b_0 \geq 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

By adding the two equations, we obtain

$$\begin{cases} \Delta(D_A a + D_B b) = 0, & x \in \Omega, \\ D_A a + D_b b = D_A a_0 + D_B b_0, & x \in \partial\Omega. \end{cases} \tag{7}$$

From the uniqueness of solution to Laplace equation, we have $D_A a(x) + D_B b(x) \equiv D_A a_0 + D_B b_0$. Define

$$k = \frac{D_B b_0}{D_A a_0 + D_B b_0}, \quad v(x) = \frac{b(x)}{D_A D_B^{-1} a_0 + b_0} - k, \quad \text{and } \lambda = \frac{(D_A D_B^{-1} a_0 + b_0)^p}{D_A}. \tag{8}$$

Then $v(x)$ satisfies

$$\begin{cases} \Delta v + \lambda (v + k)^p (1 - v - k) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

Here $\lambda > 0$, $k \in [0, 1)$ and $p > 1$. From the maximum principle, if $v(x)$ is a nonnegative solution of (9), then either $v(x) \equiv 0$ or $0 < v(x) < 1 - k$ for $x \in \Omega$. To state the result, we recall the definition of stability of a solution. Consider the eigenvalue problem:

$$\begin{cases} \Delta \omega + \lambda f'(v) \omega = \mu \omega, & x \in \Omega, \\ \omega = 0, & x \in \partial\Omega, \end{cases} \tag{10}$$

where (λ, v) is a solution to (9), $f(v) = (v+k)^p(1-v-k)$. It is well-known that the eigenvalue problem has a sequence of real eigenvalues $\mu_1 > \mu_2 \geq \mu_3 \cdots \geq \mu_n \geq \cdots \to -\infty$, and μ_1 is the principal eigenvalue with a positive eigenfunction $\psi_1 > 0$. If $\mu_1(v) < 0$, then we say that (λ, v) is stable; if $\mu_1(v) > 0$, it is unstable; and when $\mu_1(v) = 0$, it is neutrally stable. If (λ, v) is unstable, then the number of positive eigenvalues of the eigenvalue problem (counting the multiplicity) is the Morse index $M(v)$ of (λ, v).

First we consider the case when $k = 0$ which corresponds to the case $b_0 = 0$ in the original problem, *i.e.*, the external reservoir is without any catalyst B. Then (9) becomes

$$\begin{cases} \Delta v + \lambda v^p(1-v) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{11}$$

Here $\lambda > 0$, and $p > 1$. The existence, and exact multiplicity and uniqueness of positive solutions to (11) when $\Omega = B^n$, the unit ball in $\mathbf{R}^n$, are summarized in the following:

Theorem 2.1. *Assume that $p > 1$, and $\Omega = B^n = \{x \in \mathbf{R}^n : |x| < 1\}$ for $n \geq 1$.*

1. *There exists $\lambda_* > 0$ such that when $0 < \lambda < \lambda_*$, the only non-negative solution to (11) is $v = 0$; when $\lambda > \lambda_*$, (11) has exactly two positive solutions $v_{\lambda,1}(x) > v_{\lambda,2}(x) > 0$; and when $\lambda = \lambda_*$, (11) has a unique positive solution $v_{\lambda,1}(x)$.*
2. *All positive solutions of (11) are radially symmetric, and strictly decreasing along the radial direction.*
3. *All positive solutions of (11) lie on a single smooth solution curve in the space $\mathbf{R}_+ \times C^2(\overline{B^n})$, which consists of two branches $v_{\lambda,1}(x) > v_{\lambda,2}(x)$ for $\lambda > \lambda_*$; the mapping $\lambda \mapsto v_{\lambda,1}(x)$ is continuous and increasing, $\lim_{\lambda\to\infty} v_{\lambda,1}(0) = 1$; the mapping $\lambda \mapsto v_{\lambda,2}(x)$ is continuous and decreasing and $\lim_{\lambda\to\infty} v_{\lambda,2}(0) = \theta \geq 0$; $\theta = 0$ if $n \leq 2$, or $n \geq 3$ and $p \leq (n+2)/(n-2)$, and $\theta > 0$ if $n \geq 3$ and $p > (n+2)/(n-2)$; for $\lambda > \lambda_*$, $v_{\lambda,1}$ is stable, and $v_{\lambda,2}$ is unstable with Morse index 1 (see Figure 1.)*

This result is included in [17] Theorem 3, and a detailed proof can be found in [25] Theorem 1.1.

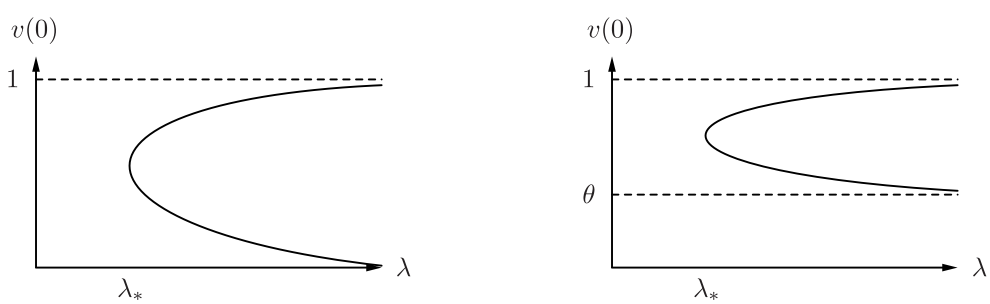

FIGURE 1. Bifurcation diagrams for (11) when $\Omega = B^n$, Left: $p \leq \dfrac{n+2}{n-2}$ or $n \leq 2$; Right: $p > \dfrac{n+2}{n-2}$.

When $k > 0$ in (9) (or $b_0 > 0$ in (6)), $v = 0$ is no longer a solution to (11). The result is proved in [25] Theorem 1.2 for small $k > 0$:

Theorem 2.2. *Suppose that $\Omega = B^n$, and n and p satisfy one of the followings:*

$$n = 1 \text{ or } n = 2, \text{ and } 1 < p < \infty, \text{ or}$$
$$n \geq 3, \text{ and } 1 < p \leq \frac{n+2}{n-2}.$$

1. *There exists $k_0 > 0$ such that when $k \in (0, k_0)$, the bifurcation diagram is exactly S-shaped. More precisely, there exist $0 < \lambda_* < \lambda^* < \infty$ such that (9) has exactly three positive solutions if $\lambda^* > \lambda > \lambda_*$, has exactly one positive solution if $\lambda > \lambda_*$ or $\lambda < \lambda^*$, and has exactly two positive solutions if $\lambda = \lambda_*$ or $\lambda = \lambda^*$.*
2. *All positive solutions of (11) are radially symmetric, and strictly decreasing along the radial direction.*
3. *All positive solutions of (9) lie on a single smooth solution curve in the space $\mathbf{R}_+ \times C^2(\overline{B^n})$, which consists of three branches*

$$\Gamma_* = \{(\lambda, v_*(x,\lambda)) : 0 < \lambda \leq \lambda^*\}$$
$$\Gamma_m = \{(\lambda, v_m(x,\lambda)) : \lambda_* \leq \lambda \leq \lambda^*\}$$
$$\text{and} \quad \Gamma^* = \{(\lambda, v^*(x,\lambda) : \lambda_* \leq \lambda < \infty\};$$

$\lim_{\lambda\to 0^+} v_*(x,\lambda) = 0$, $\lim_{\lambda\to\infty} v^*(0,\lambda) = 1-k$; for $\lambda_* < \lambda < \lambda^*$, $v_*(x,\lambda) < v_m(x,\lambda) < v^*(x,\lambda)$; the mappings $\lambda \mapsto v^*(x,\lambda)$ and $\lambda \mapsto v_*(x,\lambda)$ are continuous and increasing; $v^*(x,\lambda)$ and $v_*(x,\lambda)$ are stable, and $v_m(x,\lambda)$ is unstable with Morse index 1. (see Figure 2.)

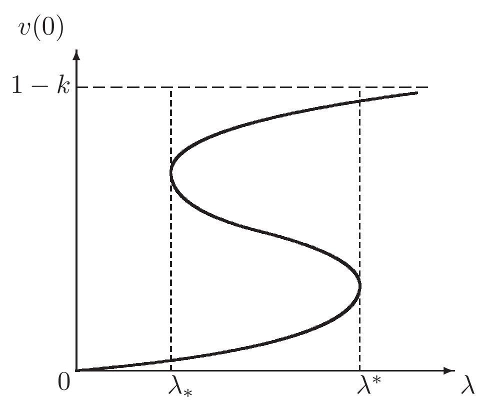

FIGURE 2. S-shaped bifurcation diagram for (9) with small $k > 0$ and $\Omega = B^n$.

3. **Dynamics of scalar equation.** In this section we consider the dynamics of

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \lambda(1-v)v^p, & t > 0, \ x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \\ v(x,t) = 0, & t > 0, \ x \in \partial\Omega. \end{cases} \tag{12}$$

Notice that (12) is the corresponding parabolic equation to (11), but it is not derived from original problem (4). We will make a connection between the dynamics of (12) and (4) in the next section.

First we briefly review a dynamical system setting in [19] sections 2 and 3 (unless otherwise specified), see also for example [11, 22]. Let $X = L^q(\Omega)$ with $n < q < \infty$, and let A be the Laplace operator Δ defined on $W_0^{2,q}(\Omega)$. Then A is the generator of an analytic semigroup on X, and we define X^α, $0 \le \alpha < 1$, to be the fractional power spaces associated with A. If we choose $(n+q)/2q < \alpha < 1$, then X^α is continuously imbedded in $C^{1,\mu}(\overline{\Omega})$ where $0 < \mu < \alpha - (n+q)/2q$. Since we assume $p > 1$, then $f(v) = v^p(1-v)$ is locally Lipschitz continuous. Hence (12) defines a compact C^1 semiflow $\Phi_t v_0 = v(\cdot, t, v_0)$ on X^α for $t > 0$. From the maximum principle, $v(x,t,v_0) > 0$ for $t > 0$ and $x \in \Omega$ if $v_0(x) \ge (\not\equiv) 0$. Moreover since $f(v) < 0$ for $v > 1$, then the solution $v(x,t,v_0)$ of (12) is global in X^α. Define $X_+ = \{u \in X^\alpha : u(x) \ge 0,\ x \in \overline{\Omega}\}$. Then the interior of X_+ is nonempty, and the semiflow Φ_t defined above is strongly monotone, and strongly order-preserving.

Here we recall a theorem of saddle-point property of Jiang, Liang and Zhao [12]:

Theorem 3.1. *Let (X, X_+) be an ordered Banach space with positive cone X_+ having nonempty interior, and let the strongly order preserving semiflow Φ be C^1 on X_+, and satisfy*

(A1) There is a positive number τ such that the mapping Φ_τ is a strict α-contraction, that is, there is a positive number $k < 1$ such that $\alpha(\Phi_\tau(B)) \le k\alpha(B)$ for any bounded subset $B \subset X_+$, where $\alpha(\cdot)$ is the Kuratowski-measure of noncompactness; and

(A2) The semiflow Φ is uniformly bounded in the sense that $O(B) = \bigcup_{t \ge 0} \Phi_t(B)$ is bounded whenever B is a bounded subset of X_+,

with Φ_τ being strongly monotone. Suppose that Φ has exactly two locally stable equilibria $a < b$, and for any other possible equilibrium c; $D_x(\Phi_\tau c)$ is strongly positive and the spectral radius $r(D_x(\Phi_\tau c)) > 1$. Then $M = X_+ \backslash (B_a \cup B_b)$ is an unordered and positively invariant Lipschiz submanifold with codimension one in the order norm $|\cdot|_v$, where B_a and B_b are the basins of attraction of a and b respectively. Furthermore, such an M is a C^1-submanifold if Φ_τ is compact.

Theorem 3.1 is one of the major tools for our main result in this section, and we also recall another result regarding the convergence of solutions to (12) when $\Omega = B^n$:

Theorem 3.2. *Consider*

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + f(v), & t > 0, \quad x \in B^n, \\ v(x,0) = v_0(x) \ge 0, & x \in B^n, \\ v(x,t) = 0, & t > 0, \quad x \in \partial B^n, \end{cases} \tag{13}$$

where $n \ge 1$ and $f \in C^1(\mathbf{R}^+)$. Let $v(x,t)$ be a bounded global solution of (13) with $v(x,t) \ge 0$. Then the ω-limit set of $v(x,t)$ is a single steady state $v^*(x) > 0$ or $v^*(x) \equiv 0$; if v^* is positive, then v^* is radially symmetric and strictly decreasing along the radial direction.

Theorem 3.2 was proved in [9]. In the following we denote by $v(x,t,v_0)$ the solution of (13). Now we are able to prove our main result in this section:

Theorem 3.3. *Let X, X^α and X_+ be defined as above, and let Φ be the semiflow generated by (12) in X_+. Assume that $\Omega = B^n$, and λ_* is as defined in Theorem 2.1.*

1. *If $0 < \lambda < \lambda_*$, then for any $v_0 \in X_+$, $||v(x,t,v_0)||_{X^\alpha} \to 0$ as $t \to \infty$;*

2. *Let* $S = \{u \in X_+ : ||u||_{X^\alpha} = 1\}$. *If* $\lambda > \lambda_*$, *for each* $u_0 \in S$, *there exists a unique* $\beta = \beta(u_0) \in (0, \infty]$ *such that*

$$\begin{aligned} &||v(x,t,\beta u_0) - v_{\lambda,1}||_{X^\alpha} \to 0, \quad \text{if } \beta > \beta(u_0); \\ &||v(x,t,\beta u_0) - v_{\lambda,2}||_{X^\alpha} \to 0, \quad \text{if } \beta = \beta(u_0); \text{ and} \\ &||v(x,t,\beta u_0)||_{X^\alpha} \to 0, \quad \text{if } 0 < \beta < \beta(u_0), \end{aligned} \tag{14}$$

and $M = \{\beta(u_0)u_0 : u_0 \in S, \beta(u_0) < \infty\}$ *is a co-dimension one Lipschitz continuous subminifold of* X_+ *(possibly with boundary). And* $\mathrm{Int}(X_+) \cap M = \{\beta(u_0)u_0 : u_0 \in \mathrm{Int}(X_+) \cap S\}$ *is non-empty with* $0 < \beta(u_0) < \infty$ *for any* $u_0 \in \mathrm{Int}(X_+) \cap S$.

3. *If* $\lambda = \lambda_*$, *then the same conclusions in part 2 hold with* $v_{\lambda,1} \equiv v_{\lambda,2}$.

Proof. When $\lambda < \lambda_*$, $v = 0$ is the only steady state solution, and (12) is a gradient system so that the ω-limit is consisted of the steady states. Hence the only possible ω-limit set is the singleton $\{0\}$. If $\lambda > \lambda_*$, from Theorem 2.1, (12) has exactly three non-negative steady states, with two stable ones 0 and $v_{\lambda,1}$, and $v_{\lambda,2}$ is unstable with Morse index 1. From the setup above, it is straightforward to apply Theorem 3.1. Condition (A1) is automatically fulfilled since the semiflow here is compact, and (A2) is also satisfied since $f(u) < 0$ for $u > 1$, hence the bounded subset $D = \{v \in X^\alpha : 0 \le v(x) \le 1\}$ is positively invariant and globally attracting. To show that $r(D_x(\Phi_\tau c)) > 1$ for any other steady state c, we notice that the only other steady state is $v_{\lambda,2}$ with Morse index 1, hence (10) has a positive eigenvalue $\mu_1 > 0$ and eigenvector $\psi_1 > 0$ with $v = v_{\lambda,2}$. From definition $D_x(\Phi_\tau c)[\phi]$ is the value $W(\tau, \cdot)$, the solution of

$$\begin{cases} W_t = \Delta W + f'(c)W & x \in B^n, \\ W(x,0) = \phi(x) \ge 0, & x \in B^n, \\ W(x,t) = 0, & x \in \partial B^n. \end{cases}$$

Then $D_x(\Phi_\tau v_{\lambda,2})[\psi_1] = e^{\mu_1 \tau}\psi_1$, which implies $r(D_x(\Phi_\tau v_{\lambda,2})) > 1$ as $\mu_1 > 0$. Therefore the existence of the separatrix submanifold M, and the partition of X_+ into $X_+ = M \cup B(0) \cup B(v_{\lambda,1})$, follow from Theorem 3.1, where $B(e)$ is the basin of attraction of the equilibrium e, with $e = 0$ or $v_{\lambda,1}$. We claim that for any $u_0 \in S$, there exists at most one β such that $\beta u_0 \in M$. In fact, from the result of [24], M is totally unordered. Thus if there exist $\beta_1 > \beta_2 > 0$ such that $\beta_i u_0 \in M$ for $i = 1, 2$, then $\beta_1 u_0 \ge \beta_2 u_0$ which is a contradiction. This implies if $u_0 \ge v_0$ but $u_0 \not\equiv v_0$, then there is at most one of u_0 and v_0 to be in M.

Fix any $u_0 \in \mathrm{Int}(X_+) \cap S$, then $u_0(x) > 0$ for $x \in B^n$, and $\partial u_0(x)/\partial n < 0$ for $x \in \partial B^n$. We choose any $v_* \in M$, then there exists a $\beta_1 > 0$ such that $\beta_1 u_0(x) > v_*(x)$ for all $x \in B^n$. Since $v(\cdot, t, v_*) \in M$ for any $t > 0$, $\lim_{t\to\infty} v(x,t,v_*) = v_{\lambda,2}$. So $\beta u_0 \notin B(0)$, the basin of attraction of 0. Also $\beta u_0 \notin M$ since $\beta u_0 > v_*$. From Theorem 3.1, $\beta u_0 \in B(v_{\lambda,1})$, the basin of attraction of $v_{\lambda,1}$. On the other hand, it is clear that for small $\beta_2 > 0$, $\beta_2 u_0 \in B(0)$ since 0 is locally stable. Let $G_1 = \{\beta > 0 : \beta u_0 \in B(v_{\lambda,1})\}$ and $G_0 = \{\beta > 0 : \beta u_0 \in B(0)\}$. Then G_1 and G_0 are both open and for any $\beta_1 \in G_1$, and $\beta_0 \in G_0$, $\beta_1 > \beta_0$, thus there exists $\beta_* > 0$ neither in G_1 nor G_0, and such a β_* is unique from arguments above. Clearly $\beta_* u_0 \in M$. If $v_0 \in M$, from Theorem 3.2, the ω-limit set of $\{v(\cdot, t, v_0)\}$ is a single steady state, which can only be $v_{\lambda,2}$ as $v_0 \notin G_0 \cup G_1$.

When $\lambda = \lambda_*$, from Takác[24], the upper boundary M of $B(0)$ is a Lipschitz submanifold whose codimension is one. Since the system is gradient-like, the basin of attraction $B(v_{\lambda,1})$ is $X_+ \setminus B(0)$. Then the result in part 3 follows. $\square$

Remark 1.
1. Since X^α we choose here satisfies $X^\alpha \subset C^{1,\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$, the convergence in X^α-norm in Theorem 3.3 implies convergence in $C^{1,\mu}$-norm. Such a choice also guarantees the nonemptiness of $\mathrm{Int}(X_+)$.
2. If $u_0 \in S\backslash\mathrm{Int}(X_+)$, then it is not clear whether $\beta(u_0)$ in Theorem 3.3 is finite any more. When it is finite, it becomes a boundary point of the manifold M.
3. In Theorem 3.3, M is identical to the stable manifold of steady state solution $v_{\lambda,2}$.

4. **Dynamics of system.** In this section we consider (4) in a ball domain with $b_0 = 0$:

$$\begin{cases} a_t = D_A\Delta a - ab^p, \;\; b_t = D_B\Delta b + ab^p, & t>0,\; x\in B^n, \\ a(x,0) = A_0(x) \ge 0, \;\; b(x,0) = B_0(x) \ge 0, & x \in B^n, \\ a(x,t) = a_0, \text{ and } b(x,t) = 0, & t>0,\; x\in \partial B^n. \end{cases} \tag{15}$$

From the arguments in Section 2, the steady state solutions of (15) can be associated with the solutions of (11). Here we fix the values of D_A and D_B to be arbitrary positive numbers, but use a_0 as a bifurcation parameter. From (8), λ is proportional to a_0^p. The following assertion is clear from Theorem 2.1:

Proposition 4.1. *Let $v_{\lambda,1}$, $v_{\lambda,2}$ and λ_* be defined in Theorem 2.1. Assume $D_A, D_B > 0$ are fixed, and define $a_0^* = \lambda_*^{1/p} D_A^{(1-p)/p} D_B$. Then when $0 < a_0 < a_0^*$, the only non-negative steady state of (15) is $(a_0, 0)$; when $a_0 > a_0^*$, there are exactly three non-negative steady states: $(a_0, 0)$, $(a_1, b_1) \equiv (a_0(1-v_{\lambda,1}), D_A D_B^{-1} a_0 v_{\lambda,1})$, and $(a_2, b_2) \equiv (a_0(1-v_{\lambda,2}), D_A D_B^{-1} a_0 v_{\lambda,2})$, where $\lambda = D_A^{p-1} D_B^{-p} a_0^p$; and when $a_0 = a_0^*$, there are exactly two solutions $(a_0, 0)$ and (a_1, b_1) defined as in the case of $a_0 > a_0^*$.*

Next we consider the case of equal diffusion coefficients $D_A = D_B \equiv D$. Define $h(x,t) = a(x,t) + b(x,t)$. Then $h(x,t)$ satisfies $h_t = D\Delta h$, and $h(x,t) = a_0$ on ∂B^n. Then from the standard theory of heat equations, $||h(x,t) - a_0||_{C^2(\overline{\Omega})} \to 0$ as $t \to \infty$. It is also easy to show that in this case, the stability of steady states in Proposition 4.1 is the same as that of corresponding solutions of (11). Indeed the linearized eigenvalue problem at a steady state solution (a_*, b_*) of (15) is

$$\begin{cases} D\Delta\phi - b_*^p\phi - pb_*^{p-1}(a_0 - b_*)\psi = \eta\phi, & x \in B^n, \\ D\Delta\psi + b_*^p\phi + pb_*^{p-1}(a_0 - b_*)\psi = \eta\psi, & x\in B^n, \\ \phi(x) = \psi(x) = 0, & x \in \partial B^n. \end{cases} \tag{16}$$

Then $\phi + \psi$ satisfies

$$D\Delta(\phi+\psi) = \eta(\phi+\psi), \;\; x \in B^n, \quad \phi(x) + \psi(x) = 0, \;\; x \in \partial B^n. \tag{17}$$

If $\phi + \psi \not\equiv 0$, then $\eta < 0$ must be an eigenvalue of Laplace operator with zero boundary condition. If $\phi + \psi \equiv 0$, then the equation of ψ is reduced to (10). Hence the number of non-negative eigenvalues of (16) is same as that of (10).

Notice that when $D_A = D_B \equiv D$, the hyperplane $T = \{(a,b) \in X^\alpha \times X^\alpha : a(x) + b(x) = a_0, a(x) \ge 0, b(x) \ge 0, \forall x \in \overline{\Omega}\}$ is invariant for (15). Let $\Phi_t(v_0)$ and $\Psi_t(A_0, B_0)$ be the solution semiflows for (12) and (15) respectively. Then we claim that (X_+, Φ_t) and (T, Ψ_t) are topologically equivalent. In fact, if we define

$H : X_+ \to T$ by $v_0 \to (a_0(1 - v_0), a_0 v_0)$, then H is a homeomorphism and it is easy to check that $H(\Phi_t(v_0)) = \Psi_{t/D}(H(v_0))$. When we restrict the solution semiflow for (15) on T, a direct consequence of Theorem 3.3 and Proposition 4.1 is that for $a_0 > a_0^*$, there exists a submanifold $M_1 \subset T$ which separates the basins of attraction on T of two locally stable steady solutions $(a_0, 0)$ and (a_1, b_1), and $M_1 = \{(a, b) \in T : a_0^{-1} b \in M\}$, where M is the submanifold defined in Theorem 3.3.

The dynamics on T is only part of the whole dynamic picture of (15), with T being homeomorphic to X_+. The property of $h(x, t)$ implies that the hyperplane T is asymptotically attracting, and (15) can be regarded as an asymptotically autonomous system. We recall the definitions from [16] and [3]: let (X, d) be a metric space, and let $\Phi : K \times X \to X$ be a mapping with $K = \{(t, s) : t_0 \le s \le t < \infty\}$. Φ is called *nonautonomous semiflow* if it is continuous and satisfies

(i) $\Phi(s, s, x) = x$, $s \ge t_0$, and
(ii) $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$, $t \ge s \ge r \ge t_0$.

And the semiflow is called *autonomous*, if, in addition,

(iii) $\Phi(t + r, s + r, x) = \Phi(t, s, x)$.

Define $\Theta(t, x) = \Phi(t + t_0, t_0, x)$. Then Θ is an autonomous semiflow. Finally a nonautonomous semiflow Φ on X is called *asymptotically autonomous* with limit semiflow Θ, if Θ is an autonomous semiflow on X, and $\Phi(t_j + s_j, s_j, x_j) \to \Theta(t, x)$, $j \to \infty$, for any sequences $t_j \to t$, $s_j \to \infty$, $x_j \to x$, $j \to \infty$ with $x, x_j \in X$, $0 \le t, t_j < \infty$ and $s_j \ge t_0$. In the following, $\mathcal{O}_\Phi(s, x)$ denotes the forward orbit $\mathcal{O}_\Phi(s, x) = \{\Phi(t, s, x) : t \ge s\}$, and if $\mathcal{O}_\Phi(s, x)$ has compact closure in X, then the ω-limit set of (s, x) is defined by $\omega_\Phi(s, x) = \bigcap_{\tau \ge s} \overline{\{\Phi(t, s, x) : t \ge \tau\}}$. The ω-limit set of x for Θ can be defined similarly. We shall apply the following result of Mischaikow, Smith and Thieme ([16] Theorem 1.8):

Theorem 4.2. *Let Φ be an asymptotically autonomous semiflow with limit semiflow Θ, and let $\mathcal{O}_\Phi(s, x)$ have compact closure in X. Then $\omega = \omega_\Phi(s, x)$ has the following properties:*

1. *ω is nonempty, compact, and connected.*
2. *ω is invariant for the semiflow Θ: $\Theta(t, \omega) = \omega$ for each $t \ge 0$.*
3. *ω attracts $\Phi(t, s, x)$: $dist_X(\Phi(t, s, x), \omega) \to 0$, $t \to \infty$.*
4. *ω is chain recurrent for Θ.*

This result will be sufficient for our application here, and a more general result was proved in Chen and Poláčik [3] Lemma 7.5 as it also applies to discrete systems. [3] also contains interesting additional information and a discussion of the relation between chain recurrent points and equilibria if the limit system has a Lyapunov functional.

To apply Theorem 4.2, we notice that (15) can be reduced to

$$\begin{cases} b_t = D\Delta b + (h(x, t) - b)b^p, & t > 0,\ x \in B^n, \\ b(x, 0) = B_0(x) \ge 0, & x \in B^n, \\ b(x, t) = 0, & t > 0,\ x \in \partial B^n, \end{cases} \tag{18}$$

where $h(x, t)$ is uniquely determined by A_0 and B_0. The theory in [11, 19] can be applied to (18), so it generates a nonautonomous semiflow Φ. On the other hand, we consider

$$\begin{cases} v_t = D\Delta v + (a_0 - v)v^p, & t > 0,\ x \in B^n, \\ v(x,0) = B_0(x) \geq 0, & x \in B^n, \\ v(x,t) = 0, & t > 0,\ x \in \partial B^n. \end{cases} \tag{19}$$

It is clear that (19) is a mere rescaling of (12), and we denote the semiflow generated by (19) by Θ, which is autonomous. We claim that Φ is an asymptotically autonomous semiflow with limit autonomous semiflow Θ. Indeed $g_1(t,b) \equiv [h(\cdot,t) - b]b^p \to (a_0 - b)b^p$ uniformly for b in bounded subsets of X^α as $t \to \infty$, since $h(x,t) \to a_0$ uniformly for $x \in \overline{\Omega}$ and $||b||_\infty$ is bounded as $X^\alpha \hookrightarrow C^{1,\mu}(\overline{\Omega})$. Therefore from the arguments in [16] page 1673, the claim follows from [2]. Now we are in the position to apply Theorem 4.2. Let $(A_0, B_0) \in X_+ \times X_+$ (in fact we only need $B_0 \in X_+$ and A_0 is only needed for the definition of function $h(x,t)$). From Theorem 4.2, $\omega = \omega_\Phi(s, B_0)$ is nonempty, compact and connected; ω is invariant and chain-recurrent for Θ.

From Theorem 3.3, each orbit of Θ converges to a steady state, and there are at most three steady states in X_+. Hence a compact invariant subset of Θ can only be either a steady state or a set consisting of two steady states and a connecting orbit. However ω is also chain recurrent for Θ, hence the latter one is impossible. Summarizing the above arguments, we have proved:

Theorem 4.3. *Suppose that $D_A = D_B \equiv D$, and $\Omega = B^n$. Let X, X^α and X_+ be defined as in Section 3, and let a_0^* be defined as in Proposition 4.1.*

1. *If $0 < a_0 < a_0^*$, then for any $(A_0, B_0) \in X_+ \times X_+$, $||a(x,t)||_{X^\alpha} \to 0$ and $||b(x,t)||_{X^\alpha} \to 0$ as $t \to \infty$;*
2. *If $a_0 \geq a_0^*$, then for any $(A_0, B_0) \in X_+ \times X_+$, there exists a steady state $(a^*(x), b^*(x))$ of (15) such that $||a(x,t) - a^*(x)||_{X^\alpha} \to 0$ and $||b(x,t) - b^*(x)||_{X^\alpha} \to 0$ as $t \to \infty$, and (a^*, b^*) must be one of three (or two) solutions described in Proposition 4.1.*

Note that part 1 in Theorem 4.3 also holds for a general bounded smooth domain Ω. On the other hand, with the convergence to steady state solution established now, we can use invariant manifold theory to conclude the existence of a separatrix manifold of codimension one in the system case:

Theorem 4.4. *Under the condition of Theorem 4.3, $M_2 = (X_+ \times X_+) \backslash (B((a_0, 0)) \cup B((a_1, b_1)))$ is a C^1 injectively immersed manifold of codimension one in $X_+ \times X_+$, and M_2 is the global stable manifold of the unstable steady state (a_2, b_2). Here we denote the basin of attraction of a steady state e by $B(e)$.*

Proof. From previous arguments, we know that (a_2, b_2) is unstable with Morse index 1 (here we extend the stability of a scalar equation to a system since they have the same stability). Thus (a_2, b_2) has a codimension one stable manifold locally defined near (a_2, b_2), and from the smoothness of nonlinearity the local stable manifold is C^1. On the other hand, $(a_0, 0)$ and (a_1, b_2) are both locally stable, thus Theorem 4.3 implies that $M_2 = (X_+ \times X_+) \backslash (B((a_0, 0)) \cup B((a_1, b_1)))$ is the set of all points which converge to (a_2, b_2), that is, M_2 is the global stable manifold $W^s((a_2, b_2))$ for hyperbolic steady state (a_2, b_2). From invariant manifold theory (see for example, [11] Theorem 6.1.9), M_2 contains the local stable manifold and M_2 is a C^1 injectively immersed manifold of codimension one in $X_+ \times X_+$. $\square$

In Theorem 4.4 we show the existence of of codimension one manifold which separates the initial value space into two parts: one is the basin of attraction for $(a_0, 0)$, the other is the basin attraction for (a_1, b_1). The existence of such a threshold manifold is interesting since the system (15) is not a monotone dynamical system, but a predator-prey type system without ordering structure. With a comparison argument, we can better characterize the basin of attraction of the two stable steady states. We recall that $M_1 = \{(a,b) \in T : a_0^{-1}b \in M\}$, where M is the submanifold described in Theorem 3.3, and b_i $(i = 1, 2)$ are the steady states of (19) which are rescaling of $v_{\lambda,i}$ defined in Theorem 2.1.

Corollary 4.5. *Assume the conditions of Theorem 4.3 are satisfied, and $a \geq a_0^*$.*

1. *If $B_0 \in B(0)$ and $A_0 + B_0 \leq a_0$, or $B_0 \in M_1$ and $A_0 + B_0 \leq (\not\equiv) a_0$, then $(A_0, B_0) \in B((a_0, 0))$.*
2. *If $B_0 \in B(b_1)$ and $A_0 + B_0 \geq a_0$, or $B_0 \in M_1$ and $A_0 + B_0 \geq (\not\equiv) a_0$, then $(A_0, B_0) \in B((a_1, b_1))$.*
3. *If $B_0 \in M_1$ and $A_0 + B_0 \equiv a_0$, then $(A_0, B_0) \in B((a_2, b_2))$.*

Proof. The results when $A_0 + B_0 \equiv a_0$ follow from Theorem 3.3, see discussions in earlier this section. Suppose that $B_0 \in B(0)$ and $A_0 + B_0 \leq (\not\equiv) a_0$. Then $A_0(x) + B_0(x) \leq h(x,t) \leq a_0$ for any $t \geq 0$ from the maximum principle of elliptic equations. Hence $b(x,t)$ satisfies

$$b_t = D\Delta b + ab^p = D\Delta b + (h - b)b^p \leq D\Delta b + (a_0 - b)b^p, \quad t > 0, \quad x \in B^n, \qquad (20)$$

and strict inequality holds for some $x \in B^n$. Therefore $b(x,t)$ is a strict lower solution of (19), and from the maximum principle of parabolic equations, $b(x,t) \leq v(x,t)$, the solution of (19) with $v(x,0) = B_0(x)$. Since $\lim_{t\to\infty} v(x,t) = 0$, and from Theorem 4.3, $\lim_{t\to\infty} b(x,t)$ exists, we must have $\lim_{t\to\infty} b(x,t) = 0$ since $0 < b_2(x) < b_1(x)$.

If $B_0 \in M_1$ and $A_0 + B_0 \leq (\not\equiv) a_0$, we still use the proof above except now $\lim_{t\to\infty} v(x,t) = b_2$ since $B_0 \in M_1$. Then the limit of $b(x,t)$ is either 0 or b_2. Since $b(x,t) \leq (\not\equiv) v(x,t)$, then $b(x,1) \leq v(x,1)$ and $b(x,1) < v(x,1)$ for some $x \in B^n$. Let $v_1(x,t)$ be the solution of (19) with $v_1(x,1) = b(x,1)$. Then $b(x,t) \leq v_1(x,t)$ for $t > 1$ and $x \in B^n$, and $v_1(x,1) \leq (\not\equiv) v(x,1)$, thus $v_1(x,1) \in B(0)$ from Theorem 3.3, which implies $0 \leq \lim_{t\to\infty} b(x,t) \leq \lim_{t\to\infty} v_1(x,t) = 0$. The proof of part 2 is similar. □

Remark 2. 1. Results similar to those in Corollary 4.5 are obtained in [23] for (4) with $\Omega = \mathbf{R}^n$. But the results in [23] for B_0 are only compared with steady states, and here the comparison is made with the threshold manifold containing the steady state. It is not clear whether a threshold manifold exists for $\mathbf{R}^n$ case. Recently Polácik [20] proved the existence of such a threshold manifold exists for a scalar equation in $\mathbf{R}^n$ with condition $f'(0) < 0$ (notice that in the model here $f(u) = u^p(1-u)$ satisfying $f'(0) = 0$.)

2. The separatrix manifold M_2 in Theorem 4.4 is only an immersed manifold not an imbedded submanifold, thus the separation of basins of attraction in Theorem 4.4 is weaker than the scalar case. In Theorem 3.3, the manifold M is an imbedded submanifold from Theorem 6.1.10 of [11] since a Lyapunov function exists for the scalar equation case. Indeed by using invariant manifold theory as in the proof of Theorem 4.4, we can show the separatrix manifold M in Theorem 3.3 is smooth, and it is the global imbedded stable manifold of X_+.

3. Suppose that $a_0 \neq a_0^*$. Then it follows from Theorem 4.3 that either $(a_0, 0)$ is globally asymptotically stable or there are exactly three steady states $(a_0, 0)$, (a_2, b_2), (a_1, b_1) which are hyperbolic, the global stable manifold $W^s((a_2, b_2))$ separates the state space into two parts: one is the basin of attraction for $(a_0, 0)$, the other is the basin attraction for (a_1, b_1). In these two alternatives, the systems are structurally stable. Thus if $a_0 \neq a_0^*$, we perturb parameters b_0, D_A and D_B in (4) so that $b_0 > 0$ is sufficiently small and D_A, D_B are both in a neighborhood of $D > 0$, then Theorem 4.3 still holds in these two alternatives because of structural stability. On the other hand, the exact multiplicity result for the steady state solutions with small $b_0 > 0$ is shown in Theorem 2.2. Thus Theorem 4.3 with small $b_0 > 0$ and equal diffusion coefficients can also be proved directly with similar proofs.
4. It is possible to extend the results here for a unit ball to a general smooth domain Ω with small diffusion coefficient D. The key would be establishing corresponding exact multiplicity of nonnegative steady states. Ideas in [4, 5] can be adapted to achieve that, but the details will appear elsewhere.

5. **Concluding remarks.** The main goal of the current paper is to show the dynamical bistability of a prototypical chemical reaction (1) in a precise sense. When the spatial variables in the model is ignored, the dynamics of chemical reaction is simple, the total mass of the two chemicals remain constant, and all reactant A eventually convert to B. The reaction stops when all reactant A is depleted. Because of mass conservation, the kinetic equation of the reaction is equivalent to scalar one $v' = v^p(1 - v)$, and the steady state $v = 1$ is globally asymptotically stable. A more realistic model is the reaction-diffusion system (2). As we discuss in the introduction, the dynamics are still similar to kinetic ones if there is no-flux boundary condition, thus the system (4) with continuous feeding of reactant A (and also possible feeding of catalyst B) is more reasonable.

Diffusion is a stablizing force which makes the neutrally stable steady state $v = 0$ a stable one in the scalar reaction-diffusion model $v_t = D\Delta v + v^p(1-v)$. Note that $v = 0$ represents the failure of the reaction, while $v = 1$ indicates the success of the reaction. Thus the scalar reaction-diffusion model has a bistable dynamics when $a_0 > 0$ is large and $b_0 = 0$, which is the exterior reservoir has a high concentration in the reactant A and zero concentration of the autocatalyst B. When the diffusion coefficients of the two chemicals are equal, then the bistable picture remains true even for the full system. In this bistable dynamics, the success/failure of the reaction depends on the initial distribution of reactant and catalyst inside the reactor. Roughly speaking, the reaction will succeed if there is enough catalyst and reactant inside the reactor, then the conversion from A to B will occur until the concentration of B in the rector reaches a high level (positive steady state); on the other hand, if there is not enough catalyst and reactant inside the reactor, the concentration of B will eventually drop to zero (trivial steady state). More precisely we the show border line between the two different asymptotic behavior is hair-trigging: a global codimension one manifold separates the success/failure of the reaction.

We shall notice that when $b_0 > 0$ is small (a small feeding of autocatalyst), the bistable structures remains but only for a_0 in an intermediate range (see Figure 2). Here when a_0 is sufficiently large, the dynamics is not bistable no matter how small the initial amount of autocatalyst is inside the reactor, since diffusion will bring exterior catalyst in for sustainable reaction. Hence the reaction is always success and B can reach a high concentration at the end. The bistable range (λ_*, λ^*) for this case depends on the smallness of b_0, and as $b_0 \to 0^+$, $\lambda^* \to \infty$.

Acknowledgements. The authors would like to thank Peter Polácik and the anonymous referee for some helpful comments and discussions.

REFERENCES

[1] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations, **4** (1979), 827–868.

[2] J. M. Ball, *On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations*, J. Differential Equations, **27** (1978), 224–265.

[3] X.-Y. Chen and P. Poláčik, *Gradient-like structure and Morse decompositions for time-periodic one-dimensional parabolic equations*, J. Dynam. Differential Equations, **7** (1995), 73–107.

[4] E. N. Dancer, *Stable and finite Morse index solutions on $\mathbf{R}^n$ or on bounded domains with small diffusion*, Trans. Amer. Math. Soc., **357** (2005), 1225–1243.

[5] E. N. Dancer and J. Shi, *Uniqueness of positive solution to sublinear semipositone problem*, Bull. London Math. Soc, **38** (2006), 1033–1044.

[6] W. B. Fitzgibbon, S. L. Hollis and J. J. Morgan, *Stability and Lyapunov functions for reaction-diffusion systems*, SIAM J. Math. Anal., **28** (1997), 595–610.

[7] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243.

[8] P. Gray and S. K. Scott, "Chemical Oscillations and Instabilies: Nonlinear Chemical Kinectics," Clarendon Press, Oxford, 1990.

[9] A. Haraux and P. Polácik, *Convergence to a positive equilibrium for some nonlinear evolution equations in a ball*, Acta Math. Univ. Comenian. (N.S.), **61** (1992), 129–141.

[10] A. Haraux and A. Youkana, *On a result of K. Masuda concerning reaction-diffusion equations*, Tohoku Math. J. (2), **40** (1988), 159–163.

[11] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, 840, Springer-Verlag, Berlin-New York, 1981.

[12] J. Jiang, X. Liang and X. Zhao, *Saddle-point behavior for monotone semiflows and reaction-diffusion models*, J. Differential Equations, **203** (2004), 313–330.

[13] Y. Li and Y. W. Qi, *The global dynamics of isothermal chemical systems with critical nonlinearity*, Nonlinearity, **16** (2003), 1057–1074.

[14] Masuda, Kyûya, *On the global existence and asymptotic behavior of solutions of reaction-diffusion equations*, Hokkaido Math. J., **12** (1983), 360–370.

[15] R. H. Martin and M. Pierre, *Nonlinear reaction-diffusion systems*, In "Nonlinear Equations in the Applied Sciences," 363–398, Math. Sci. Engrg., **185**, Academic Press, Boston, MA, 1992.

[16] K. Mischaikow, H. Smith and H. R. Thieme, *Asymptotically autonomous semiflows: chain recurrence and Lyapunov functions*, Trans. Amer. Math. Soc., **347** (1995), 1669–1685.

[17] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problem*, J. Differential Equations, **146** (1998), 121–156.

[18] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problem: II*, J. Differential Equations, **158** (1999), 94–151.

[19] P. Polácik, *Parabolic equations: asymptotic behavior and dynamics on invariant manifolds*, In "Handbook of Dynamical Systems," Vol. 2, 835–883, North-Holland, Amsterdam, 2002.

[20] P. Polácik, *On a threshold behaviour in parabolic equations on $\mathbf{R}^N$*, Preprint.

[21] Y. W. Qi, *Dynamics and universality of an isothermal combustion problem in 2D*, Rev. Math. Phys., **18** (2006), 285–310.

[22] H. L. Smith, "Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems," Mathematical Surveys and Monographs, 41, American Mathematical Society, Providence, RI, 1995.

[23] J. Shi and X. Wang, *Hair-triggered instability of radial steady states, spread and extinction in semilinear heat equations*, J. Differential Equations, **231** (2006), 235–251.

[24] P. Takác, *Domains of attraction of generic ω-limit sets for strongly monotone semiflows*, Z. Anal. Anwendungen, **10** (1991), 275–317.

[25] Y. Zhao, Y. Wang and J. Shi, *Exact multiplicity of solutions and s-shaped bifurcation curve for a class of semi-linear elliptic equations from a chemical reaction model*, Jour. Math. Anal. Appl., **331** (2007), 263–278.

Received January 2007; Final version May 2007.

E-mail address: jiangjf@mail.tongji.edu.cn
E-mail address: shij@math.wm.edu