Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 1285–1292
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Exact multiplicity of boundary blow-up solutions for a bistable problem

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Received 29 April 2006; received in revised form 26 March 2007; accepted 5 April 2007

Abstract

We prove the exact multiplicity of positive boundary blow-up solutions to a semilinear elliptic equation with bistable nonlinearity for the one-dimensional case. We use time-mapping techniques to determine the exact shape of the bifurcation diagram.

Keywords: Boundary blow-up solution; Exact multiplicity; Semilinear elliptic equation; Time-mapping; Bistable

1. Introduction

We study the exact multiplicity of positive solutions $u \le 1/n$ of the problem

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 1/n \in (1, \infty], \end{cases} \tag{1.1}$$

where λ is a positive parameter, $0 \le n < 1$, and

$$f(u) = u(u - \sigma)(1 - u), \quad \sigma \in (1/2, 1).$$

In particular, when $n = 0$, the boundary conditions become

$$u(-1) = u(1) = \infty,$$

and the solution u is a boundary blow-up solution. Our work is motivated by recent works [1,2] on the boundary blow-up solutions of

$$\Delta u + \lambda u(u - \sigma)(1 - u) = 0, \qquad x \in \Omega, \quad u|_{\partial\Omega} = \infty, \tag{1.2}$$

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.04.017

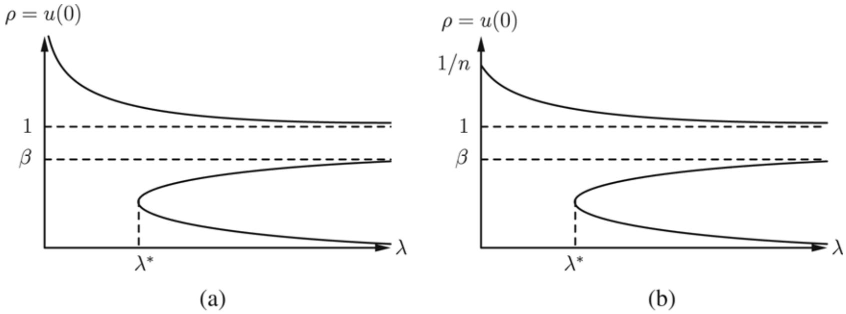

Fig. 1. Bifurcation diagrams for (1.1). (a) $n = 0$; (b) $0 < n < 1$.

where Ω is bounded smooth domain in $\mathbb{R}^N$ with $N \geq 1$. In Aftalion, del Pino and Letelier [1], it was shown that for large $\lambda > 0$, (1.2) has at least three positive solutions $\underline{u}_\lambda < u_\lambda < \overline{u}_\lambda$, and for any compact subset of Ω, $\underline{u}_\lambda \to 0$ and $\overline{u}_\lambda \to 1$ uniformly as $\lambda \to \infty$. In Du and Yan [2], it was shown that the middle solution u_λ is a spike layer solution.

When $n = 0$ (the boundary blow-up case), for the ODE (1.1) with general nonlinearity f, the existence of multiple boundary blow-up solutions was first proved in Anuradha, Brown and Shivaji [3]. We also mention that the multiplicity of boundary blow-up solutions to (1.1) with other nonlinearities was studied by Wang [4] using similar quadrature methods as in this paper. Recently, the exact multiplicity of positive solutions to (1.2) for large λ and ball domains was proved by Guo and Zhou [5].

Our goal in this paper is to determine the exact multiplicity of positive solutions of (1.1) for some parameters (σ, n) for all $\lambda > 0$. We show that there exists $\lambda^* > 0$ such that (1.1) has exactly three solutions when $\lambda > \lambda^*$, exactly two solutions when $\lambda = \lambda^*$, and exactly one solution when $0 < \lambda < \lambda^*$. See Fig. 1. Our proofs are based on the time-mapping method (quadrature method).

For positive solutions u of (1.1), it is well known that the parameter λ and the value $\rho = u(0) = \inf_{-1<x<1} u(x)$ satisfy the following relation

$$\sqrt{2\lambda} = G(\rho) \equiv \int_\rho^{1/n} \frac{1}{\sqrt{F(\rho) - F(u)}} \mathrm{d}u, \qquad \rho \in (0, \beta) \cup (1, 1/n), \tag{1.3}$$

where

$$F(u) = \int_0^u f(t)\mathrm{d}t = -\frac{\sigma}{2}u^2 + \frac{1+\sigma}{3}u^3 - \frac{1}{4}u^4$$

and

$$\beta = \beta(\sigma) = \frac{-1 + 2\sigma + \sqrt{-2 + 2\sigma + 4\sigma^2}}{3} \in (0, 1), \tag{1.4}$$

is the unique point in (0, 1) such that $F(1) - F(\beta) = \int_\beta^1 f(t)\mathrm{d}t = 0$. Thus the bifurcation diagram of (1.1) is determined by the function $G(\rho)$, which is usually called time-mapping. We first have the next proposition:

Proposition 1.1. *Consider* (1.1) *with* $f(u) = u(u - \sigma)(1 - u)$, $\sigma \in (1/2, 1)$. *Then*

$$\lim_{\rho \to 0^+} G(\rho) = \infty, \tag{1.5}$$

$$\lim_{\rho \to \beta^-} G(\rho) = \infty, \tag{1.6}$$

$$\lim_{\rho \to 1^+} G(\rho) = \infty, \tag{1.7}$$

$$\lim_{\rho \to (1/n)^-} G(\rho) = 0. \tag{1.8}$$

In addition,

$G(\rho)$ *is strictly decreasing in* $(1, 1/n)$. (1.9)

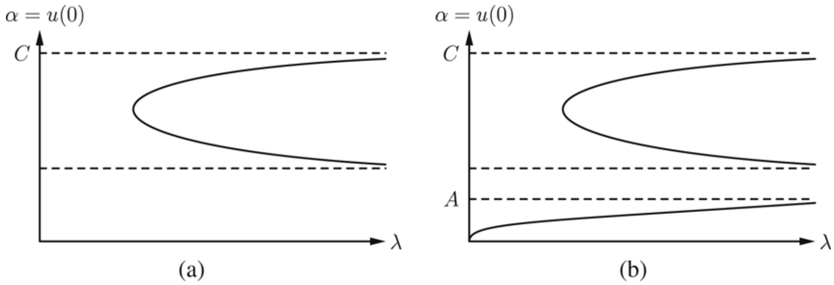

Fig. 2. Bifurcation diagrams for (2.2). (a) $A = 0$; (b) $A > 0$.

Proof. The results $\lim_{\rho\to 0^+} G(\rho) = \lim_{\rho\to\beta^-} G(\rho) = \lim_{\rho\to 1^+} G(\rho) = \infty$ follow by similar arguments used to prove [3, Lemmas 4.2 and 4.3]. In addition, it is easy to see that $\lim_{\rho\to(1/n)^-} G(\rho) = 0$; we omit the proof. Next we show (1.9). For $G(\rho)$ in (1.3), $G'(\rho)$ can be easily computed, cf. e.g. [6, p. 273]. We have

$$G'(\rho) = 2^{-1/2}\int_\rho^{1/n} \frac{\theta(\rho)-\theta(u)}{\rho(\Delta F)^{3/2}}\mathrm{d}u,$$

where $\Delta F = F(\rho) - F(u) > 0$ and $\theta(u) = 2F(u) - uf(u) = \frac{1}{2}u^4 - \frac{1+\sigma}{3}u^3$. We compute that $\theta'(u) = f(u) - uf'(u) = u^2(2u - 1 - \sigma) > 0$ for $u > 1$, since $\sigma < 1$. Thus $G'(\rho) < 0$ for $1 < \rho < 1/n$, and hence (1.9) holds. □

2. Connection with a FitzHugh–Nagumo equation

As pointed out in [2], (1.2) has a connection with another well-known semilinear equation:

$$\Delta u + \lambda(u - A)(u - B)(C - u) = 0, \quad x \in \Omega, u|_{\partial\Omega} = 0, \tag{2.1}$$

where $0 \le A < B < C$ and $2B < C + A$. This equation arises from the studies of dynamics of the FitzHugh–Nagumo equation and population biology. The general results regarding (2.1) can be found in Dancer and Wei [7,8].

The exact multiplicity of positive solutions of one-dimensional version of problem (2.1), *i.e.*

$$u'' + \lambda(u - A)(u - B)(C - u) = 0 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0, \tag{2.2}$$

has been studied extensively. When $A = 0$, Smoller and Wasserman [6] proved that the bifurcation diagram is exactly ⊂-shaped (see Fig. 2(a)). And later, Wang [9] and Korman, Li and Ouyang [10] independently generalized the same result for a general concave–convex nonlinearity $f(u)$ by using the techniques of time-mapping and techniques of bifurcation theory, respectively. The higher dimensional analog for radially symmetric solutions was proved by Korman, Li and Ouyang [11] for a two-dimensional ball, and ball in all dimensions by Ouyang and Shi [12,13]. For the case of $A > 0$, the exact multiplicity result as in Fig. 2(b) was proved independently by Wang [9,14] and Korman, Li and Ouyang [10,15] but all of them need some extra conditions on nonlinearity $f(u)$.

To be more consistent with the previous results in [10,15,16], we consider the following rescaled version of (2.2):

$$u'' + \lambda(u - a)(u - b)(1 - u) = 0 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0, \tag{2.3}$$

where $0 < a < b < 1$. In the following, we denote $h(u) = (u - a)(u - b)(1 - u)$, and $H(u) = \int_0^u h(t)\mathrm{d}t$. It is well known that the necessary condition for the existence of more than one positive solutions for (2.3) is $\int_a^1 h(u)\mathrm{d}u > 0$, or equivalently, $2b < 1 + a$. Hence the valid parameter range of (a, b) for bifurcation diagrams consisting two components is given by $0 < 2a < 2b < 1 + a$, which is a triangular region Δ in (a, b)-plane. The conjecture is that for any $(a, b) \in \Delta$, the bifurcation diagram of (2.3) is precisely like Fig. 2(b). See Theorem 2.1 for a precise statement.

So far, to our knowledge, under different assumptions, there are five known papers [6,9,10,14,16] in which the exact multiplicity of positive solutions for (2.3) is proved, *i.e.* the conjecture above holds. The proof in [6] holds when

$$b > 8a - 1, \tag{2.4}$$

see [9, pg. 50]. In [14], the assumption is

$$\theta(\gamma) < 0, \tag{2.5}$$

where

$$\theta(x) = 2H(x) - xh(x) = abx - \frac{1+a+b}{3}x^3 + \frac{1}{2}x^4,$$

and $b < \gamma < 1$ satisfying $\int_a^\gamma h(x)\mathrm{d}x = 0$, or $H(\gamma) = H(a)$. Note that γ can be calculated as

$$\gamma = \gamma(a,b) = \frac{2 + 2b - a - \sqrt{4 + 2a - 10b - 2a^2 + b^2 - ab}}{3}. \tag{2.6}$$

Thus $\theta(\gamma)$ can also be calculated in terms of a and b with the help of computer algebra systems such as `Maple` or `Mathematica`. The expression of $\theta(\gamma)$ is too long to write here, but we denote it by $g_1(a,b) \equiv \theta(\gamma)$; hence (2.5) is equivalent to

$$g_1(a,b) < 0.$$

In [9], another condition is given by

$$\varphi(\gamma) < 0, \tag{2.7}$$

where

$$\varphi(x) = 3\theta(x) - x\theta'(x) = 6H(x) - 4xh(x) + x^2h'(x) = 2abx - \frac{1}{2}x^4,$$

and γ is given by (2.6) again. Similarly, (2.7) is equivalent to

$$g_2(a,b) \equiv \varphi(\gamma) < 0,$$

where $g_2(a,b)$ can be calculated by computer algebra systems.

In [10], two new conditions are derived for the exact multiplicity results:

$$g_3(a,b) \equiv 24a^2(b-1)^2 - 8a(1+b)^3 + (1+b)^4 \geq 0, \tag{2.8}$$

or

$$g_4(a,b) \equiv \int_a^\xi h(u)\mathrm{d}u \leq 0,$$

where

$$\xi = \frac{1 + a + b + \sqrt{1 - a - b + a^2 + b^2 - ab}}{3}.$$

Note that (2.4) implies (2.8). Summarizing these results in [6,9,10,14], we conclude that the exact multiplicity has been obtained if (a,b) belongs to the following set:

$$\Delta_1 \equiv \{(a,b) \in \Delta : g_1(a,b) < 0, \text{ or } g_2(a,b) < 0, \text{ or } g_3(a,b) \geq 0 \text{ or } g_4(a,b) \leq 0\}. \tag{2.9}$$

We notice that all these four inequalities do not imply each other. In fact, with the help of a computer algebra system, we could draw the diagram of Δ_1 by drawing $g_i(a,b) = 0$ for $1 \leq i \leq 4$ (see Fig. 3). The area of Δ_1 is 44.2% of that of Δ. Here the area of Δ_1 is calculated by using `Mathematica 5.0` and a command similar to that in [16], where they calculated the area of

$$\widetilde{\Delta_1} \equiv \{(a,b) \in \Delta : g_3(a,b) \geq 0 \text{ or } g_4(a,b) \leq 0\}$$

to be 41.5%. The uncovered region $\Delta_2 \equiv \Delta \setminus \Delta_1$ is the lower right part of Δ (the component including $(1,1)$), see Fig. 3 for illustration. It is easy to see that $\partial\Delta_1 \bigcap \{g_i(a,b) = 0\} \neq \emptyset$ for each $i = 1,2,3,4$. Recently, using a

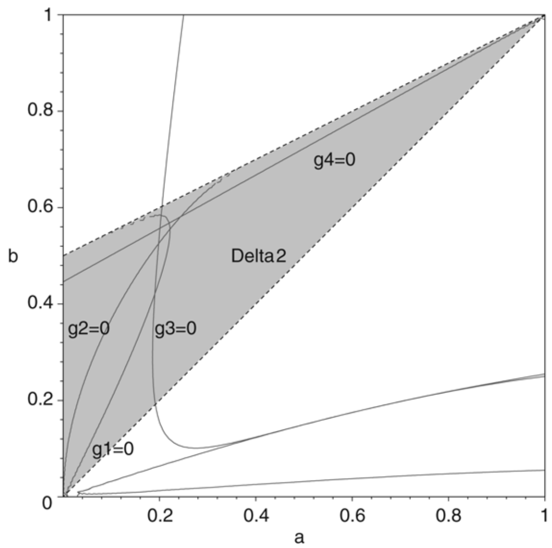

Fig. 3. Region Δ_1 on (a,b)-plane. Shaded area is Δ, and $\Delta_2 = \Delta \setminus \Delta_1$ is the connected component in the right lower part of Δ.

computer-assisted proof, Korman, Li and Ouyang [16] proved the exact multiplicity result for (2.3) as in Fig. 2(b) for all $(a,b) \in \Delta$. But a complete analytical proof is still beyond the reach for all $(a,b) \in \Delta_2$.

For positive solutions u of (2.3), it is well known that the parameter λ and the value $\alpha = u(0) = \max_{x\in[-1,1]} u(x)$ satisfy the following relation

$$\sqrt{2\lambda} = T(\alpha) \equiv \int_0^\alpha \frac{1}{\sqrt{H(\alpha)-H(u)}}\mathrm{d}u, \quad \alpha \in (0,a)\cup(\eta,1),$$

where

$$H(u) = \int_0^u h(t)\mathrm{d}t = abu - \frac{a+b+ab}{2}u^2 + \frac{1+a+b}{3}u^3 - \frac{1}{4}u^4$$

and

$$\eta = \frac{2-a+2b-\sqrt{2(1+a-2b)(2-a-b)}}{3}$$

is the unique point in $(b,1)$ such that $H(\eta) - H(a) = \int_a^\eta f(t)\mathrm{d}t = 0$; see e.g. [9].

We have the following theorem.

Theorem 2.1 (*See Figs.* 2*(b) and* 3). *Consider* (2.3). *Suppose* $(a,b) = (a_0,b_0) \in \Delta_1$ *defined in* (2.9). *Then* $T(\alpha)$ *satisfies*

(i) $\lim_{\alpha\to 0^+} T(\alpha) = 0$, $\lim_{\alpha\to a^-} T(\alpha) = \lim_{\alpha\to\eta^+} T(\alpha) = \lim_{\alpha\to 1^+} T(\alpha) = \infty$;
(ii) $T(\alpha)$ *is strictly increasing in* $(0,a)$;
(iii) $T(\alpha)$ *has exactly one critical point, a minimum, in* $(\gamma,1)$.

We then make several changes of variables to (2.3). Let $v = 1-u$. Then v satisfies

$$v'' + \lambda v(v-m)(n-v) = 0 \quad \text{in } (-1,1), \qquad v(-1) = v(1) = 1,$$

where $m = 1-b$ and $n = 1-a$. Notice that $1 > n > m > 0$, and since $1+a > 2b$, then $m > n/2$. Next we let $w = v/n$. Then w satisfies

$$w'' + \bar{\lambda} w(w-\sigma)(1-w) = 0 \quad \text{in } (-1,1), \qquad w(-1) = w(1) = 1/n, \tag{2.10}$$

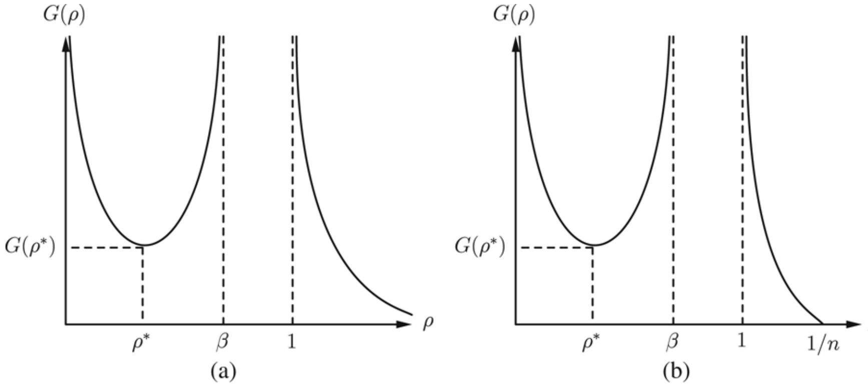

Fig. 4. Graphs for $G(\rho)(=\sqrt{\lambda})$ for (1.3). (a) $n = 0$; (b) $0 < n < 1$.

where $\overline{\lambda} = \lambda n^2$, and

$$\sigma = \frac{m}{n} = \frac{1-b}{1-a} \in (1/2, 1).$$

So we derive Problem (1.1). The changes of variables above do not alter the shapes of bifurcation diagrams. Indeed if the bifurcation diagram of (2.3) is given by $\lambda(\alpha) = (1/2)[G_1(\alpha)]^2$, then the one for (2.10) is simply $\overline{\lambda}(\rho) = (1/2)n^2[G_1(1 - n\rho)]^2$, where $\rho = w(0)$ in (2.10) and $\alpha = u(0)$ in (2.3). In particular, the two bifurcation diagrams have the same number of turning points, and the upper branch for (2.3) corresponds to the lower branch for (2.10), and the lower branch for (2.3) corresponds to the upper branch for (2.10). So by above arguments and Proposition 1.1, we obtain and state the main results in the next section.

3. Main results

The main results in this paper are the next Theorem 3.1 and Corollaries 3.2–3.4. In particular, Theorem 3.1 follows from Theorem 2.1. In the (σ, n)-plane, we define rectangular region

$$\square \equiv \{(\sigma, n) : 1/2 < \sigma < 1, 0 < n < 1\},$$

and region

$$\square_1 \equiv \{(\sigma, n) \in \square : g_1(\sigma, n) < 0, \text{ or } g_2(\sigma, n) < 0, \text{ or } g_3(\sigma, n) \geq 0 \text{ or } g_4(\sigma, n) \leq 0\},$$

Here $g_i(\sigma, n) \equiv g_i(a(\sigma, n), b(\sigma, n))$ from the discussions at the end of Section 2, and $g_i(a, b)$ ($i = 1, 2, 3, 4$) are defined in Section 2. Note that, in this section, for $G(\rho)$ in (1.3), to make it more clear for the dependence on the nonlinearity f, we write $G_f(\rho)$ instead of $G(\rho)$.

Theorem 3.1 (*See Figs. 3, 4(b) and 5*). *Consider (1.1) with*

$$0 < n = n_0 < 1 \quad \textit{and} \quad f = f_0(u) = u(u - \sigma_0)(1 - u), \quad \sigma_0 \in (1/2, 1).$$

Suppose $(a, b) = (a_0, b_0) \equiv (1 - n_0, 1 - \sigma_0 n_0) \in \Delta_1$ defined in (2.9). Then $(\sigma, n) = (\sigma_0, n_0) \in \square_1$. So, in addition to (1.5)–(1.9), $G_{f_0}(\rho)$ has exactly one critical point, a minimum at some ρ^*, in $(0, \beta)$, where β is defined in (1.4).

By Fig. 3 and some easy computations to

$$g_3(a, b) = 24a^2(b - 1)^2 - 8a(1 + b)^3 + (1 + b)^4,$$

we obtain the next corollary.

Corollary 3.2 (*See Figs. 3, 4(b) and 5*). *Consider (1.1) with*

$$0 < n = n_0 < 1 \quad \textit{and} \quad f = u(u - \sigma)(1 - u), \quad \sigma \in (1/2, 1).$$

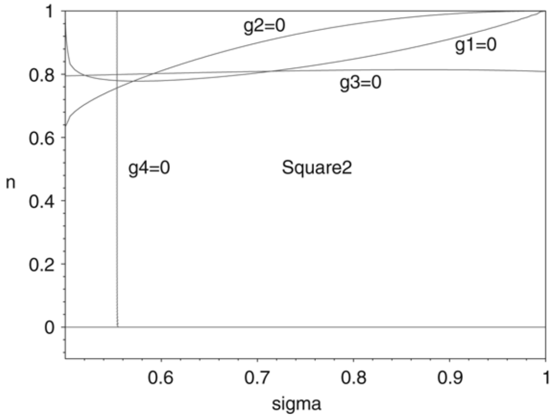

Fig. 5. Region $\square_1$ on (σ, n)-plane. $\square_2 = \square \setminus \square_1$ is the connected component in the right lower part of $\square = (1/2, 1) \times (0, 1)$.

Suppose $n_0 > 1 - a^* \approx 0.814$ *where* $(a^*, b^*) \approx (0.186, 0.289)$ *satisfies* $g_3(a^*, b^*) = 0$ *and* $(\partial g_3/\partial b)(a^*, b^*) = 0$. *Then, in addition to* (1.5)–(1.9), $G_f(\rho)$ *has exactly one critical point, a minimum at some* ρ^*, *in* $(0, \beta)$.

Corollary 3.3 (*See, Figs.* 4(*b*) *and* 5). *Consider* (1.1) *with*

$$0 < n = n_0 < 1 \quad \text{and} \quad f = f_0(u) = u(u - \sigma_0)(1 - u).$$

For any parameter $(\sigma, n) = (\sigma_0, n_0) \in \square$, $(\sigma_0, n_0) \in \square_1$ *if* $\sigma_0 \leq 0.553$. *So, in addition to* (1.5)–(1.9), $G_{f_0}(\rho)$ *has exactly one critical point, a minimum at some* ρ^*, *in* $(0, \beta)$.

Proof. Setting $(\sigma, n) = (0.553, n_0) \in \square$ in $g_4(\sigma, n)$, we obtain

$$g_4(0.553, n_0) < 0 \quad \text{for all } 0 < n = n_0 < 1.$$

(The expression of $g_4(0.553, n_0)$ is too big to write here.) So the result follows. $\square$

Corollary 3.3 implies the next corollary for (1.1) with boundary blow-up conditions; *i.e.*, $n = 0$. This follows from the convergence of G, G' and G'' as $n \to \infty$. Indeed, let ρ^* be a limit point of the unique minimum $\rho^*(n)$ in Corollary 3.3; then we can show that $G'_{f_0}(\rho) > 0$ for $\rho \in (0, \rho^*)$ and $G'_{f_0}(\rho) < 0$ for $\rho \in (\rho^*, \beta)$ by limiting arguments. Hence we obtain

Corollary 3.4 (*See Figs.* 4(*a*) *and* 5). *Consider the boundary blow-up problem*

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = \infty, \end{cases}$$

where

$$f = f_0(u) = u(u - \sigma_0)(1 - u), \quad \sigma_0 \in (1/2, 1).$$

If $\sigma_0 < 0.553$, *then* $G_{f_0}(\rho)$ *satisfies*

(i) $\lim_{\rho \to 0^+} G_{f_0}(\rho) = \lim_{\alpha \to \beta^-} G_{f_0}(\rho) = \lim_{\alpha \to 1^+} G_{f_0}(\rho) = \infty$, $\lim_{\alpha \to \infty} G_{f_0}(\rho) = 0$;
(ii) $G_{f_0}(\rho)$ *is strictly increasing in* $(1, \infty)$;
(iii) $G_{f_0}(\rho)$ *has exactly one critical point, a minimum at some* ρ^*, *in* $(0, \beta)$.

We finally note that Korman, Li and Ouyang's [16] and our work imply a computer assisted proof of Corollary 3.4 in the general case $1/2 < \sigma_0 < 1$ (since G cannot be constant on an interval).

Acknowledgements

The authors thank Prof. Yihong Du for introducing this problem to them and for helpful discussions, and they also thank an anonymous referee for helpful suggestions. J.S. is partially supported by United States NSF grants DMS-0314736 and EF-0436318, College of William and Mary junior research leave, and overseas scholar grant from Department of Education of Heilongjiang Province, China. S.W. is partially supported by National Science Council, ROC.

References

[1] A. Aftalion, M. del Pino, R. Letelier, Multiple boundary blow-up solutions for nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 225–235.
[2] Y. Du, S.S. Yan, Boundary blow-up solutions with a spike layer, J. Differential Equations 205 (2004) 156–184.
[3] V. Anuradha, C. Brown, R. Shivaji, Explosive nonnegative solutions to two point boundary value problems, Nonlinear Anal. 26 (1996) 613–630.
[4] S.-H. Wang, Existence and multiplicity of boundary blow-up nonnegative solutions to two-point boundary value problems, Nonlinear Anal. 42 (2000) 139–162.
[5] Z.M. Guo, F. Zhou, Exact multiplicity for boundary blow-up solutions, J. Differential Equations 228 (2006) 486–506.
[6] J. Smoller, A. Wasserman, Global bifurcation of steady-state solutions, J. Differential Equations 39 (1981) 269–290.
[7] E.N. Dancer, J.C. Wei, On the profile of solutions with two sharp layers to a singularly perturbed semilinear Dirichlet problem, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 691–701.
[8] E.N. Dancer, J.C. Wei, On the location of spikes of solutions with two sharp layers for a singularly perturbed semilinear Dirichlet problem, J. Differential Equations 157 (1999) 82–101.
[9] S.-H. Wang, On the time map of a nonlinear two-point boundary value problem, Differential Integral Equations 7 (1994) 49–55.
[10] P. Korman, Y. Li, T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalising cubic, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 599–616.
[11] P. Korman, Y. Li, T. Ouyang, An exact multiplicity result for a class of semilinear equations, Comm. Partial Differential Equations 22 (1997) 661–684.
[12] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, J. Differential Equations 146 (1998) 121–156.
[13] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem: II, J. Differential Equations 158 (1999) 94–151.
[14] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, J. Differential Equations 77 (1989) 199–202.
[15] P. Korman, Y. Li, T. Ouyang, Perturbation of global solution curves for semilinear problems, Adv. Nonlinear Stud. 3 (2003) 289–299.
[16] P. Korman, Y. Li, T. Ouyang, Computing the location and the direction of bifurcation, Int. Math. Res. Lett. 12 (2005) 933–944.