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Imperfect transcritical and pitchfork bifurcations ☆

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Abstract

Imperfect bifurcation phenomena are formulated in framework of analytical bifurcation theory on Banach spaces. In particular the perturbations of transcritical and pitchfork bifurcations at a simple eigenvalue are examined, and two-parameter unfoldings of singularities are rigorously established. Applications include semilinear elliptic equations, imperfect Euler buckling beam problem and perturbed diffusive logistic equation.

Keywords: Imperfect bifurcation; Secondary bifurcation; Bifurcation in Banach spaces

1. Introduction

Nonlinear problems can often be formulated to an abstract equation

$$F(\lambda, u) = 0, \tag{1.1}$$

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where $F:\mathbf{R}\times X\to Y$ is a nonlinear differentiable mapping, and X, Y are Banach spaces. The solutions of nonlinear equation (1.1) and their dependence on the parameter λ have been the subject of extensive studies in the last forty years. Bifurcation could occur at a solution (λ_0,u_0) if it is a *degenerate solution* of (1.1), i.e. the linearized operator $F_u(\lambda_0,u_0)$ is not invertible. Two celebrated theorems of Crandall and Rabinowitz [5,6] are now regarded as foundation of analytical bifurcation theory in infinite-dimensional spaces, and both results are based on implicit function theorem. In both theorems, it is assumed that

(F1) $\dim N(F_u(\lambda_0,u_0)) = \operatorname{codim} R(F_u(\lambda_0,u_0)) = 1$, and $N(F_u(\lambda_0,u_0)) = \operatorname{span}\{w_0\}$,

where $N(F_u)$ and $R(F_u)$ are the null space and the range space of linear operator F_u.

Theorem 1.1. *(Saddle-node bifurcation, [6, Theorem 3.2].) Let $F:\mathbf{R}\times X\to Y$ be continuously differentiable. $F(\lambda_0,u_0)=0$, F satisfies* (F1) *and*

(F2) $F_\lambda(\lambda_0,u_0)\notin R(F_u(\lambda_0,u_0))$.

Then the solutions of (1.1) *near (λ_0,u_0) form a continuously differentiable curve $(\lambda(s),u(s))$, $\lambda(0)=\lambda_0$, $u(0)=u_0$, $\lambda'(0)=0$ and $u'(0)=w_0$. Moreover, if F is k-times continuously differentiable, so are $\lambda(s)$, $u(s)$.*

Theorem 1.2. *(Transcritical and pitchfork bifurcations, [5, Theorem 1.7].) Let $F:\mathbf{R}\times X\to Y$ be continuously differentiable. Suppose that $F(\lambda,u_0)=0$ for $\lambda\in\mathbf{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0,u_0), F satisfies* (F1) *and*

(F3) $F_{\lambda u}(\lambda_0,u_0)[w_0]\notin R(F_u(\lambda_0,u_0))$.

Then the solutions of (1.1) *near (λ_0,u_0) consists precisely of the curves $u=u_0$ and $(\lambda(s),u(s))$, $s\in I=(-\delta,\delta)$, where $(\lambda(s),u(s))$ are continuously differentiable functions such that $\lambda(0)=\lambda_0$, $u(0)=u_0$, $u'(0)=w_0$.*

Applications of Theorems 1.1 and 1.2 can be found in [3,11,17–21,24] and many other books and papers. In [24], the second author studied the perturbations of the bifurcation diagrams appearing in Theorems 1.1 and 1.2 via an investigation of the system consisting (1.1) and $F_u(\lambda,u)[w]=0$. The goal of this paper is to further explore the set of solutions to (1.1) near a bifurcation point (λ_0,u_0) satisfying (F1). Here our focus is the situations when the transversality condition (F2) is violated, that is, the opposite of (F2):

(F2′) $F_\lambda(\lambda_0,u_0)\in R(F_u(\lambda_0,u_0))$.

Our first result (see Theorem 2.1) describes the local solution set of $F(\lambda,u)=0$ near (λ_0,u_0) when (F1) and (F2′) are satisfied, which contrasts to Theorem 1.1 when (F1) and (F2) are satisfied. We show that the solution set is no longer a curve near the degenerate solution (λ_0,u_0) like the case in saddle-node bifurcation, but either an isolated single point, or a pair of transversally intersecting curves, which is similar to the solution set in Theorem 1.2. (See the remark after Theorem 2.1 for a discussion on the connection of Theorems 2.1 and 1.2.) Our proof uses Morse lemma instead of implicit function theorem as in [5], and this idea goes back to Nirenberg

[15,16]. Theorem 2.1 complements Theorem 1.1 better than Theorem 1.2 since (F2) and (F2′) are directly opposite, and we also need a more general further transversality condition than (F3), thus Theorems 2.1 and 1.1 can be regarded as the first two parts of classification of degenerate solutions according to the order of degeneracy.

In the second part of the paper, we continue the work of the second author [24]. We consider the solution set of

$$F(\varepsilon, \lambda, u) = 0, \tag{1.2}$$

which can be considered as a perturbation of (1.1), and the new parameter ε indicates the perturbation. Following [24,25], we investigate the imperfect bifurcation of bifurcation diagram in (λ, u) space under small perturbations. Imperfection of the bifurcation diagrams occur when the small errors or noises destroy the original bifurcation structure, which occur frequently in engineering or other application problems. We prove several new theorems about the symmetry breaking of transcritical and pitchfork bifurcations, see Theorems 3.1–3.3 for the structure of the degenerate solutions, and Theorems 4.1 and 4.3 for the variations of solution set of (1.2). The statements of theorems can be found in respective sections, and here we only sketch the imperfect bifurcation diagrams they represent.

In Figs. 1 and 2, typical symmetry breaking perturbations of transcritical and pitchfork bifurcations are shown. We use analytic bifurcation theory following [5,6] to obtain precise structure of the perturbed local bifurcation diagrams. Another not-so-typical breaking up of the transcritical bifurcation is also shown as an application of the secondary bifurcation theorem which is proved in Section 2, see Figs. 3 and 4. In Section 5, we apply our abstract results to the imperfect bifurcation in classical Euler buckling beam problem and diffusive logistic equation in spatial ecology.

Bifurcation problems concerning the solutions of nonlinear equation (1.1) have been studied in abstract framework since 1970s. One important tool is Lyapunov–Schmidt reduction which

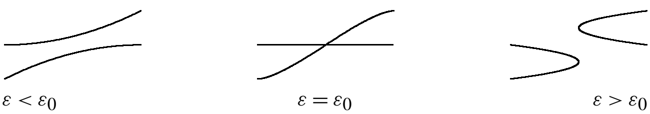

Fig. 1. Typical symmetry breaking of transcritical bifurcation.

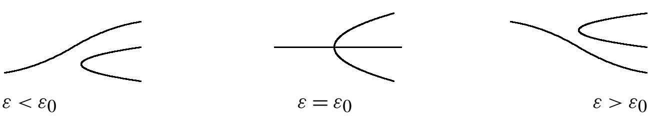

Fig. 2. Typical symmetry breaking of pitchfork bifurcation.

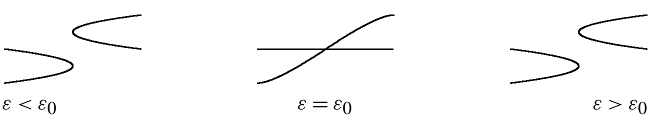

Fig. 3. Non-typical symmetry breaking of transcritical bifurcation (1).

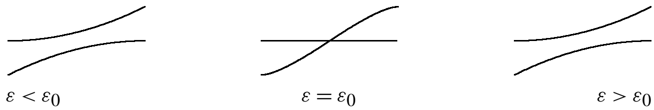

Fig. 4. Non-typical symmetry breaking of transcritical bifurcation (2).

reduces the original problem to a finite-dimensional one, and the theory of singularities of differentiable maps and catastrophe theory are useful in the qualitative studies of such finite-dimensional problems. In particular, the imperfect bifurcation of (Lyapunov–Schmidt) reduced maps are investigated by Golubitsky and Schaeffer [10]. This approach is efficient but may depend on the reductions specific to the application problems. Another approach is to directly deal with infinite-dimensional problems, and the results involve various partial derivatives of the nonlinear maps on Banach spaces but not derivatives of reduced finite-dimensional maps. This could be demonstrated by the work of Crandall and Rabinowitz [5,6], which have been widely adopted in applications. Such results are more convenient and more applicable in real world problems, since in applications one only needs to check certain linearized operators but does not need to perform Lyapunov–Schmidt reduction. On the other hand, some ideas of catastrophe theory have also been generalized to infinite-dimensional setting. The fold and cusp type singularities in Banach spaces have been found in various situations, and a general theory has also been developed, see the survey of Church and Timourian [4]. We also note that the two approaches are not in conflict: for example, when establishing infinite-dimensional bifurcation theorem (see Theorem 2.1), the Lyapunov–Schmidt reduction is used in proof. But the abstract results in the latter approach are free of specific reductions.

In the paper, we use $\|\cdot\|$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range space of L.

2. Crossing curve bifurcation

If we assume F satisfies (F1) at (λ_0, u_0), then we have decompositions of X and Y: $X = N(F_u(\lambda_0, u_0)) \oplus Z$ and $Y = R(F_u(\lambda_0, u_0)) \oplus Y_1$, where Z is a complement of $N(F_u(\lambda_0, u_0))$ in X, and Y_1 is a complement of $R(F_u(\lambda_0, u_0))$. In particular, $F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism. Since $R(F_u(\lambda_0, u_0))$ is codimension one, then there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y\colon \langle l, v\rangle = 0\}$. Thus if F also satisfies (F2′), then the equation

$$F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[v] = 0 \tag{2.1}$$

has a unique solution $v_1 \in Z$. Our main result in this section is the following bifurcation theorem:

Theorem 2.1. *Let $F : \mathbf{R} \times X \to Y$ be a C^2 mapping. Suppose that $F(\lambda_0, u_0) = 0$, F satisfies* (F1) *and* (F2′)*. Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, let $v_1 \in Z$ be the unique solution of* (2.1)*, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y\colon \langle l, v\rangle = 0\}$. We assume that the matrix* (*all derivatives are evaluated at* (λ_0, u_0))

$$H_0 = H_0(\lambda_0, u_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0, w_0]\rangle \end{pmatrix} \tag{2.2}$$

is non-degenerate, i.e., $\det(H_0) \neq 0$.

(1) *If* H_0 *is definite, i.e.* $\det(H_0) > 0$*, then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is* $\{(\lambda_0, u_0)\}$.
(2) *If* H_0 *is indefinite, i.e.* $\det(H_0) < 0$*, then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of two intersecting* C^1 *curves, and the two curves are in form of* $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s y_i(s))$*,* $i = 1, 2$*, where* $s \in (-\delta, \delta)$ *for some* $\delta > 0$*,* (μ_1, η_1) *and* (μ_2, η_2) *are non-zero linear independent solutions of the equation*

$$\begin{aligned}&\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle \mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \eta\mu \\ &\quad + \langle l, F_{uu}[w_0, w_0]\rangle \eta^2 = 0, \qquad (2.3)\end{aligned}$$

where $\theta_i(s)$*,* $y_i(s)$ *are some functions defined on* $s \in (-\delta, \delta)$ *which satisfy* $\theta_i(0) = \theta_i'(0) = 0$*,* $y_i(s) \in Z$*, and* $y_i(0) = y_i'(0) = 0$*,* $i = 1, 2$.

A particularly useful special case is when $F_\lambda(\lambda_0, u_0) = 0$, and immediately we have $v_1 = 0$.

Corollary 2.2. *Let* $F : \mathbf{R} \times X \to Y$ *be a* C^2 *mapping. Suppose that* $F(\lambda_0, u_0) = 0$*,* F *satisfies* (F1) *and* $F_\lambda(\lambda_0, u_0) = 0$*. We assume that the matrix*

$$H_1 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix} \qquad (2.4)$$

is non-degenerate. Then the conclusions of Theorem 2.1 *hold, and Eq.* (2.3) *simplifies to*

$$\langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle \mu^2 + 2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \mu\eta + \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \eta^2 = 0. \qquad (2.5)$$

Corollary 2.3. *Let* $F : \mathbf{R} \times X \to Y$ *be a* C^2 *mapping. Suppose that* $F(\lambda_0, u_0) = 0$*,* F *satisfies* (F1), (F3)*,* $F_\lambda(\lambda_0, u_0) = 0$*,* $F_{\lambda\lambda}(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$*. Then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of two intersecting* C^1 *curves* Γ_i*,* $i = 1, 2$*,* Γ_1 *is tangent to* λ *axis at* (λ_0, u_0) *and it is in form of* $\{(\lambda, u(\lambda)), |\lambda| < \varepsilon\}$*; and* Γ_2 *is the form of* $\{(\lambda_0 + \mu_2 s + s\theta_2(s), u_0 + s w_0 + s y_2(s))\}$*, where*

$$\mu_2 = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}.$$

Remark 2.4. (1) Theorem 2.1 complements Crandall–Rabinowitz saddle-node bifurcation theorem (Theorem 1.1), where (F2) is imposed. Our result is based on the opposite condition (F2′) and a generic second order non-degeneracy condition $\det(H_0) \neq 0$.

(2) Crandall–Rabinowitz theorem of bifurcation from simple eigenvalue (Theorem 1.2) is a special case of Theorem 2.1, Corollaries 2.2 and 2.3. Indeed, in Theorem 1.2, for the curve of constant solutions, any derivative of F in λ is zero, but (F3) is assumed in Theorem 1.2 so the matrix H_0 is indefinite. One curve in Theorem 2.1 part 2 satisfies $\eta_1 = 0$ (and we can assume

$\mu_1 = 1$) and only a higher order projection on w_0, which corresponds to the constant solution branch in Theorem 1.2; the other branch satisfies $\eta_2 = 1$ and

$$\mu_2 = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle} \tag{2.6}$$

from (2.5), which determines the bifurcation direction: if $\mu_2 \neq 0$, then a transcritical bifurcation occurs, and on either side of $\lambda = \lambda_0$, (1.1) has locally exactly two solutions; and if $\mu_2 = 0$ but a higher order non-degeneracy condition is satisfied (see [24]), then a pitchfork bifurcation occurs.

(3) Theorem 2.1 is also a more general result on secondary bifurcations. The well-known version of secondary bifurcation is [5, p. 323, Theorem 1] (see also [7, p. 407, Theorem 29.3]), where a solution curve Γ_1 is given, and it is shown that another curve Γ_2 exists and intersects with Γ_1 transversally. In our result, no any solution curve is given, and we obtain the two curves simultaneously from conditions on F at the bifurcation point. Later we prove that Theorem 2.1 implies the Crandall–Rabinowitz secondary bifurcation theorem (see Theorem 2.7).

(4) Notice that v_1 in (2.1) depends on the choice of the complement subspace Z, so with a different subspace, v_1 can be in form of $v_1 + kw_0$ for some $k \in \mathbf{R}$. However it is easy to check that $\det(H_0)$ is independent of choice of v_1 or Z, and the solutions of (2.3) are also independent of choice of v_1 or Z (up to a constant scale). Note that the solutions of (2.3) are apparently not unique, but there exists two linear independent real-valued solutions since $\det(H_0) < 0$.

To prove Theorem 2.1, we first establish a result in finite-dimensional space, which is of its own interest.

Lemma 2.5. *Suppose that $(x_0, y_0) \in \mathbf{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f: U \to \mathbf{R}$ is a C^p function for $p \geqslant 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian $H = H(x_0, y_0)$ is non-degenerate. Then*

(1) *If H is definite, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);*
(2) *If H is indefinite, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of*

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0. \tag{2.7}$$

Proof. When H is definite, then (x_0, y_0) is either a strict local minimum or a strict maximum point of $f(x, y)$ from calculus. Thus (x_0, y_0) is the unique zero of $f(x, y) = 0$ locally. When H is indefinite, consider the differential equation:

$$x' = \frac{\partial f(x, y)}{\partial y}, \qquad y' = -\frac{\partial f(x, y)}{\partial x}, \qquad \bigl(x(0), y(0)\bigr) \in U. \tag{2.8}$$

Then (2.8) is a Hamiltonian system with potential function $f(x, y)$, and (x_0, y_0) is the only equilibrium point of (2.8) in U (if necessary we can choose smaller U). The Jacobian of (2.8) at (x_0, y_0) is

$$J = \begin{pmatrix} f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \\ -f_{xx}(x_0, y_0) & -f_{xy}(x_0, y_0) \end{pmatrix}. \tag{2.9}$$

Since $\mathrm{Trace}(J) = 0$ and $\mathrm{Det}(J) = \mathrm{Det}(H) < 0$, then (x_0, y_0) is a saddle type equilibrium of (2.8) and J has eigenvalues $\pm k$ for some $k > 0$. From the invariant manifold theory of differential equations, there exists a unique curve $\Gamma_{\mathrm{s}} \subset U$ (the stable manifold) such that Γ_{s} is invariant for (2.8) and for $(x(0), y(0)) \in \Gamma_{\mathrm{s}}$, $(x(t), y(t)) \to (x_0, y_0)$ as $t \to \infty$; and similarly the unstable manifold is another invariant curve Γ_{u} for (2.8) and for $(x(0), y(0)) \in \Gamma_{\mathrm{u}}$, $(x(t), y(t)) \to (x_0, y_0)$ as $t \to -\infty$. Both Γ_{s} and Γ_{u} are C^{p-1} one-dimensional manifold by the stable and unstable manifold theorem [22, p. 107]. $f(x, y) = 0$ for $(x, y) \in \Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{u}}$ since $f(x, y)$ is the Hamiltonian function of the system and $\Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{u}} \cup \{(x_0, y_0)\}$. On the other hand, for any $(x, y) \notin \Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{u}} \cup \{(x_0, y_0)\}$, $f(x, y) \neq 0$. This simply follows from the Morse lemma, the C^{p-1} curves must be identical to $\Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{u}}$.

Finally we consider the tangential direction of Γ_{s} and Γ_{u}. We denote the two curves by $(x_i(t), y_i(t))$, with $i = 1, 2$. Then

$$f\big(x_i(t), y_i(t)\big) = 0. \tag{2.10}$$

Differentiating (2.10) in t twice, we obtain (we omit the subscript i for $x_i(t)$ and $y_i(t)$ in the equation)

$$\begin{aligned} &f_{xx}\big(x(t), y(t)\big)\big(x'(t)\big)^2 + 2f_{xy}\big(x(t), y(t)\big)x'(t)y'(t) + f_{yy}\big(x(t), y(t)\big)\big(y'(t)\big)^2 \\ &\quad + f_x\big(x(t), y(t)\big)x''(t) + f_y\big(x(t), y(t)\big)y''(t) = 0 \end{aligned}$$

evaluating at $t = 0$ and $\nabla f(x_0, y_0) = 0$, we obtain (2.7). □

We remark that Lemma 2.5 can also be deduced from a more general Morse lemma, see Kuiper [13] and Chang [2] (Lemma 4.1 and Theorem 5.1), and a weaker result is proved in Nirenberg [16, Theorem 3.2.1] in which the crossing curves are shown to be C^{p-2}; we give an alternate proof here using invariant manifold theory, and we also remark that C^{p-1} is the optimal regularity, see discussions in Shi and Xie [28].

Next we recall the well-known Lyapunov–Schmidt procedure under the condition (F1). The following version can be found in [16, pp. 36–37 and 40]. We sketch a proof for the completeness of presentation.

Lemma 2.6 *(Lyapunov–Schmidt reduction). Suppose that* $F : \mathbf{R} \times X \to Y$ *is a* C^p *map such that* $F(\lambda_0, u_0) = 0$, *and* F *satisfies* (F1) *at* (λ_0, u_0). *Then* $F(\lambda, u) = 0$ *for* (λ, u) *near* (λ_0, u_0) *can be reduced to* $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$, *where* $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, *where* δ *is a small positive constant,* $l \in Y^*$ *such that* $\langle l, v\rangle = 0$ *if and only if* $v \in R(F_u(\lambda_0, u_0))$, *and* g *is a* C^p *function into* Z *such that* $g(\lambda_0, 0) = 0$ *and* Z *is a complement of* $N(F_u(\lambda_0, u_0))$ *in* X.

Proof. We denote the projection from Y into $R(F_{\mathrm{u}}(\lambda_0, u_0))$ by Q. Then $F(\lambda, u) = 0$ is equivalent to

$$Q \circ F(\lambda, u) = 0, \quad \text{and} \quad (I - Q) \circ F(\lambda, u) = 0. \tag{2.11}$$

We rewrite the first equation in form

$$Q \circ F(\lambda, u_0 + tw_0 + g) = 0 \tag{2.12}$$

where $t \in \mathbf{R}$ and $g \in Z$. Since F satisfies (F1) at (λ_0, u_0), then $g = g(\lambda, t)$ in (2.12) is uniquely solvable from the implicit function theorem for (λ, t) near $(\lambda_0, 0)$, and g is C^p. Hence $u = u_0 + tw_0 + g(\lambda, t)$ is a solution to $F(\lambda, u) = 0$ if and only if $(I - Q) \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0$. Since $R(F_{\mathrm{u}}(\lambda_0, u_0))$ is co-dimensional one, hence it becomes the scalar equation $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$. $\square$

Proof of Theorem 2.1. From the proof of Lemma 2.6, we have

$$f_1(\lambda, t) \equiv Q \circ F\big(\lambda, u_0 + tw_0 + g(\lambda, t)\big) = 0, \tag{2.13}$$

for (λ, t) near $(\lambda_0, 0)$. Differentiating f_1 and evaluating at $(\lambda, t) = (\lambda_0, 0)$, we obtain

$$0 = \nabla f_1 = \big(Q \circ \big(F_\lambda + F_{\mathrm{u}}[g_\lambda]\big), Q \circ F_{\mathrm{u}}[w_0 + g_t]\big). \tag{2.14}$$

Since $F_{\mathrm{u}}[w_0] = 0$ and $g_t \in Z$, and $F_{\mathrm{u}}(\lambda_0, u_0)|_Z$ is an isomorphism, then $g_t(\lambda_0, 0) = 0$. Similarly $g_\lambda \in Z$ and $F_\lambda \in R(F_{\mathrm{u}}(\lambda_0, u_0))$ from (F2′), hence

$$F_\lambda(\lambda_0, u_0) + F_{\mathrm{u}}(\lambda_0, u_0)\big[g_\lambda(\lambda_0, 0)\big] = 0. \tag{2.15}$$

Hence $g_\lambda(\lambda_0, 0) = v_1$, where v_1 is defined as in (2.1).

To prove the statement in Theorem 2.1, we apply Lemma 2.5 to

$$f(\lambda, t) = \big\langle l, F\big(\lambda, u_0 + tw_0 + g(\lambda, t)\big)\big\rangle. \tag{2.16}$$

From Lemma 2.6, $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) is equivalent to $f(\lambda, t) = 0$ for (λ, t) near $(\lambda_0, 0)$. To apply Lemma 2.5, we claim that

$$\nabla f(\lambda_0, 0) = (f_\lambda, f_t) = 0, \quad \text{and} \quad \mathrm{Hess}(f) \text{ is non-degenerate.} \tag{2.17}$$

It is easy to see that

$$\nabla f(\lambda_0, 0) = \big(\big\langle l, F_\lambda(\lambda_0, u_0) + F_{\mathrm{u}}(\lambda_0, u_0)\big[g_\lambda(\lambda_0, 0)\big]\big\rangle, \big\langle l, F_{\mathrm{u}}(\lambda_0, u_0)\big[w_0 + g_t(\lambda_0, 0)\big]\big\rangle\big). \tag{2.18}$$

Thus $\nabla f(\lambda_0, 0) = 0$ from (2.1) and $g_t(\lambda_0, 0) = 0$. For the Hessian matrix, we have

$$\mathrm{Hess}(f) = \begin{pmatrix} f_{\lambda\lambda} & f_{\lambda t} \\ f_{t\lambda} & f_{tt} \end{pmatrix}. \tag{2.19}$$

Here

$$f_{\lambda t}(\lambda_0, 0) = f_{t\lambda}(\lambda_0, 0) = \langle l, F_{\lambda u}[w_0 + g_t] + F_{uu}[w_0 + g_t, g_\lambda] + F_{\mathrm{u}}[g_{\lambda t}]\rangle$$
$$= \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle, \tag{2.20}$$

since $g_t = 0$. Next we have

$$f_{\lambda\lambda}(\lambda_0, 0) = \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[g_\lambda] + F_{uu}[g_\lambda, g_\lambda] + F_{\mathrm{u}}[g_{\lambda\lambda}]\rangle$$
$$= \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle. \tag{2.21}$$

Finally,

$$f_{tt}(\lambda_0, 0) = \langle l, F_{uu}[w_0 + g_t, w_0 + g_t] + F_{\mathrm{u}}[g_{tt}]\rangle = \langle l, F_{uu}[w_0, w_0]\rangle. \tag{2.22}$$

In summary, from our calculation,

$$\mathrm{Hess}(f) = \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0, w_0]\rangle \end{pmatrix}. \tag{2.23}$$

Therefore from Lemma 2.5, we conclude that the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is a pair of intersecting curves if the matrix in (2.23) is indefinite, or is a single point if it is definite.

Now we consider only the former case of two curves. We denote the two curves by $(\lambda_i(s), u_i(s)) = (\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s)))$, with $i = 1, 2$. Then

$$F(\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s))) = 0. \tag{2.24}$$

From Lemma 2.5 the vectors $v_i = (\lambda_i'(0), t_i'(0))$ are the solutions of $v^T H v = 0$, which are the solutions (μ, η) of (2.3). □

Next we give an secondary bifurcation theorem generalizing a well-known one [5, Theorem 1] based on Theorem 2.1. This is not surprising considering that Theorem 2.1 is more general than Theorem 1.2, which implies the secondary bifurcation theorem (see [5]). But here we do not assume the existence of a given solution branch as that in [5].

Theorem 2.7. *Let W and Y be Banach spaces, Ω an open subset of W and $G : \Omega \to Y$ be twice differentiable. Suppose*

(1) $G(w_0) = 0$,
(2) $\dim N(G'(w_0)) = 2$, $\operatorname{codim} R(G'(w_0)) = 1$.

Then

(1) *If for any $\phi(\neq 0) \in N(G'(w_0))$, $G''(w_0)[\phi, \phi] \notin R(G'(w_0))$, then the set of solutions to $G(w) = 0$ near $w = w_0$ is the singleton $\{w_0\}$.*
(2) *If there exists $\phi_1(\neq 0) \in N(G'(w_0))$ such that $G''(w_0)[\phi_1, \phi_1] \in R(G'(w_0))$, and there exists $\phi_2 \in N(G'(w_0))$ such that $G''(w_0)[\phi_1, \phi_2] \notin R(G'(w_0))$, then w_0 is a bifurcation point of $G(w) = 0$ and in some neighborhood of w_0, the totality of solutions of $G(w) = 0$ form*

two continuous curves intersecting only at w_0. *Moreover the solution curves are in form of* $w_0 + s\psi_i + s\theta_i(s)$, $s \in (-\delta, \delta)$, $\theta_i(0) = \theta_i'(0) = 0$, *where* ψ_i $(i = 1, 2)$ *are the two linear independent solutions of the equation* $\langle l, G''(w_0)[\psi, \psi]\rangle = 0$.

Proof. Let $l \in Y^*$ such that $\langle l, y\rangle = 0$ if and only if $y \in R(G'(w_0))$. Then if for any $\phi(\neq 0) \in N(G'(w_0))$, $G''(w_0)[\phi, \phi] \notin R(G'(w_0))$, we must have $\langle l, G''(w_0)[\phi, \phi]\rangle > 0$ (or < 0) for any $\phi(\neq 0) \in N(G'(w_0))$. Without loss of generality, we assume $>$ holds. We assume that $W = \operatorname{span}\{\phi_1\} \oplus X$ is a splitting of W, and we choose $\phi_2 \in X \cap N(G'(w_0))$ so that $\{\phi_1, \phi_2\}$ is a basis of $N(G'(w_0))$. Clearly $X \cap N(G'(w_0)) = \operatorname{span}\{\phi_2\}$. Define $F : I \times X \to Y$ ($I \subset \mathbf{R}$ is an open interval containing 0)

$$F(\lambda, u) = G(w_0 + \lambda\phi_1 + u). \tag{2.25}$$

Then $F \in C^2$ and $F(0, 0) = 0$. We check F satisfies (F1) and (F2′). It is easy to calculate

$$\begin{gathered} F_\lambda(0,0) = G'(w_0)[\phi_1], \qquad F_{\lambda\lambda}(0,0) = G''(w_0)[\phi_1, \phi_1], \\ F_{\mathrm{u}}(0,0)[\psi] = G'(w_0)[\psi], \qquad F_{\lambda u}(0,0)[\psi] = G''(w_0)[\phi_1, \psi], \\ F_{uu}(0,0)[\psi, \theta] = G''(w_0)[\psi, \theta]. \end{gathered} \tag{2.26}$$

Then $N(F_{\mathrm{u}}(0, 0)) = \operatorname{span}\{\phi_2\}$ and $R(F_{\mathrm{u}}(0, 0)) = R(G'(w_0))$, hence (F1) is satisfied. (F2′) is obvious since $F_\lambda(0, 0) = G'(w_0)[\phi_1] = 0$. From above calculation and Corollary 2.2, we have

$$H_1 = \begin{pmatrix} \langle l, G''(w_0)[\phi_1, \phi_1]\rangle & \langle l, G''(w_0)[\phi_1, \phi_2]\rangle \\ \langle l, G''(w_0)[\phi_1, \phi_2]\rangle & \langle l, G''(w_0)[\phi_2, \phi_2]\rangle \end{pmatrix}. \tag{2.27}$$

Since $\langle l, G''(w_0)[\phi, \phi]\rangle > 0$ for any $\phi(\neq 0) \in N(G'(w_0))$, then $\det(H_1) > 0$ since

$$\bigl\langle l, G''(w_0)[k_1\phi_1 + k_2\phi_2, k_1\phi_1 + k_2\phi_2]\bigr\rangle = \mathbf{k}H_1\mathbf{k}^T > 0,$$

which implies that H_1 is positively definite, and here $\mathbf{k} = (k_1, k_2) \in \mathbf{R}^2$ and $\mathbf{k}^T$ is the transpose of $\mathbf{k}$. We apply Theorem 2.1, part 1, then the result follows. For the second part, the calculation above remains true with ϕ_1, ϕ_2 satisfying the conditions in theorem if we choose a complement subspace X to $\operatorname{span}\{\phi_1\}$ so that $\phi_2 \in X$, and $\{\phi_1, \phi_2\}$ makes a basis for $N(G'(w_0))$. But $\det(H_1) < 0$ from the assumptions, hence we can apply part 2 of Theorem 2.1 or Corollary 2.2. For the solutions (μ_i, η_i) of (2.5), $(\mu_1, \eta_1) = (1, 0)$ is one solution since $\langle l, G''(w_0)[\phi_1, \phi_1]\rangle = 0$, and the other solution is given by $\eta_2 = 1$ and $\mu_2 = -\langle l, G''(w_0)[\phi_2, \phi_2]\rangle/(2\langle l, G''(w_0)[\phi_1, \phi_2]\rangle)$. Hence the two solution branches are in form of $w_0 + s\phi_1 + s\theta_1(s)$ and $w_0 + s(\mu_2\phi_1 + \phi_2) + s\theta_2(s)$, and one can verify that $\psi_1 = \phi_1$ and $\psi_2 = \mu_2\phi_1 + \phi_2$ are the two linear independent solutions of $\langle l, G''(w_0)[\psi, \psi]\rangle = 0$. □

The simplest example of Theorem 2.7 is the quadratic map: $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x^2 \pm y^2$. In the $+$ sign case, $(x, y) = (0, 0)$ is the only solution of $f(x, y) = 0$, and in the $-$ sign case, the solutions of $f(x, y) = 0$ are all points on the crossing lines $x = \pm y$. To conclude this section, we illustrate Theorem 2.1 with an example in $X = Y = \mathbf{R}^2$. Another example in infinite-dimensional spaces is shown in Section 5.1.

Example. Define

$$F(\lambda, x_1, x_2) = \begin{pmatrix} 5x_1 + x_1^3 + 2x_1x_2^2 \\ x_2 + x_2^3 + 2x_2x_1^2 \end{pmatrix} - \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.28}$$

where $(x_1, x_2) \in \mathbf{R}^2$ and $\lambda \in \mathbf{R}$. The equation $F(\lambda, x_1, x_2) = 0$ has a branch of zero solutions $\Sigma_0 = \{(\lambda, 0, 0)\colon \lambda \in \mathbf{R}\}$, and it also has two branch of semi-trivial solutions (with one component zero):

$$\Sigma_1 = \bigl\{(\lambda, x_1, 0)\colon x_1 = \pm\sqrt{\lambda - 5}, \lambda > 5\bigr\},$$
$$\Sigma_2 = \bigl\{(\lambda, 0, x_2)\colon x_2 = \pm\sqrt{\lambda - 1}, \lambda > 1\bigr\}. \tag{2.29}$$

We notice that both Σ_1 and Σ_2 are generated through pitchfork bifurcations from Σ_0 at $\lambda = 5$ and $\lambda = 1$, respectively. To further analyze the secondary bifurcations, we need the following calculations:

$$F_\lambda = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad F_{\lambda\lambda} = 0,$$
$$F_x = \begin{pmatrix} 5 + 3x_1^2 + 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 1 + 3x_2^2 + 2x_1^2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_{\lambda x} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$F_{xx} = \left(\begin{pmatrix} 6x_1 & 4x_2 \\ 4x_2 & 4x_1 \end{pmatrix}, \begin{pmatrix} 4x_2 & 4x_1 \\ 4x_1 & 6x_2 \end{pmatrix} \right). \tag{2.30}$$

Along Σ_1, $F_x = \operatorname{diag}(2\lambda - 10, \lambda - 9)$. Thus besides $\lambda = 5$ (primary bifurcation point), $\lambda = 9$ is another degenerate point. We analyze the bifurcation at $(\lambda, x_1, x_2) = (9, 2, 0)$. $N(F_x) = \{(0, a)^T\colon a \in \mathbf{R}\}$, $R(F_x) = \{(a, 0)^T\colon a \in \mathbf{R}\}$, $F_\lambda = (-2, 0)^T \in R(F_x)$. Hence (F1) and (F2′) are satisfied. We choose $Z = R(F_x)$, then the equation $F_\lambda + F_x[v_1] = 0$ has a unique solution $v_1 = (1/4, 0)^T \in Z$. From above calculation, we find that the matrix H_0 in (2.2) to be

$$H_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.31}$$

which is indefinite. Thus we can apply Theorem 2.1 to this equation, and near $(\lambda, x_1, x_2) = (9, 2, 0)$, the solution set is the union of two intersecting curves. Moreover we can choose $(\mu_1, \eta_1) = (1, 0)$ (corresponding to Σ_1) and $(\mu_2, \eta_2) = (0, 1)$ (corresponding to new branch) for the directions of two curves. $\mu_2 = 0$ indicates that the bifurcation is not a linear transcritical one. Indeed it is a pitchfork bifurcation, and the new branch is

$$\Sigma_3 = \bigl\{(\lambda, x_1, x_2)\colon \lambda > 9,\ x_1 = \pm\sqrt{(\lambda + 3)/3},\ x_2 = \pm\sqrt{(\lambda - 9)/3}\bigr\}. \tag{2.32}$$

3. Perturbed problems

We consider an equation

$$F(\varepsilon, \lambda, u) = 0, \tag{3.1}$$

where $F \in C^1(M, Y)$, $M \equiv \mathbf{R} \times \mathbf{R} \times X$, and X, Y are Banach spaces. We define

$$H(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ F_u(\varepsilon, \lambda, u)[w] \end{pmatrix}. \tag{3.2}$$

We consider the solution $(\varepsilon_0, \lambda_0, u_0, w_0)$ of $H(\varepsilon, \lambda, u, w) = 0$. For $(\varepsilon_0, \lambda_0, u_0) \in M$ and $w_0 \in X_1 \equiv \{x \in X\colon \|x\| = 1\}$, By Hahn–Banach theorem (see [24, Lemma 7.1]), there exists a closed subspace X_3 of X with codimension 1 such that $X = L(w_0) \oplus X_3$, where $L(w_0) = \operatorname{span}\{w_0\}$, and $d(w_0, X_3) = \inf\{\|w - x\|\colon x \in X_3\} > 0$. Let $X_2 = w_0 + X_3 = \{w_0 + x\colon x \in X_3\}$. Then X_2 is a closed hyperplane of X with codimension 1. Since X_3 is a closed subspace of X, and X_3 is also a Banach space in the subspace topology. Hence we can regard $M_1 = M \times X_2$ as a Banach space with product topology. Moreover, the tangent space of M_1 is homeomorphic to $M \times X_3$ (see [24] for more on the setting).

In the following we will still use the conditions (Fi) on F defined in previous sections and in [24], but we will use $(\varepsilon_0, \lambda_0, u_0)$ instead of (λ_0, u_0) in all these conditions. In addition to (F1)–(F3) defined above, we also define (following [24])

(F4) $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$;
(F5) $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

We also use the convention that (Fi') means that the condition defined in (Fi) does not hold.

We first prove a refinement of Theorem 2.6 and a generalization of Theorem 2.4 in [24]:

Theorem 3.1. *Let $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $H(T_0) = (0, 0)$. Suppose that the operator F satisfies* (F1), (F2′)*,* (F3)*,* (F4) *and* (F5) *at T_0. Then there exists $\delta > 0$ such that all the solutions of $H(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 form a C^2-curve.*

$$\{T_s = (\varepsilon(s), \lambda(s), u(s), w(s)),\ s \in I = (-\delta, \delta)\}, \tag{3.3}$$

where $\varepsilon(s) = \varepsilon_0 + \tau(s)$, $s \in I$, $\tau(\cdot) \in C^2(I, \mathbf{R})$, $\tau(0) = \tau'(0) = 0$, and

$$\begin{aligned} \lambda(s) &= \lambda_0 + s + z_1(s), \\ u(s) &= u_0 + s(kw_0 + v_1) + z_2(s), \\ w(s) &= w_0 + s\psi_1 + z_3(s), \end{aligned}$$

where $s \in I$, $z_i(\cdot) \in C^2$, $z_i(0) = z_i'(0) = 0$ $(i = 1, 2, 3)$, and $v_1 \in X_3$ is the unique solution of

$$F_u(\varepsilon_0, \lambda_0, u_0)[v_1] + F_\lambda(\varepsilon_0, \lambda_0, u_0) = 0, \tag{3.4}$$

k is the unique number such that

$$\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle + \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0]\rangle + k\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle = 0, \tag{3.5}$$

$\psi_1 \in X_3$ is the unique solution of

$$F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]+kF_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0,w_0] \\ +F_{uu}(\varepsilon_0,\lambda_0,u_0)[v_1,w_0]+F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[\psi]=0. \tag{3.6}$$

Proof. We apply Theorem 1.1 to operator H so we need to verify all the conditions. We define a differential operator $K:\mathbf{R}\times X\times X_3\to Y\times Y$,

$$K[\tau,v,\psi]=H_{(\lambda,u,w)}(\varepsilon_0,\lambda_0,u_0,w_0)[\tau,v,\psi] \\ =\begin{pmatrix}\tau F_\lambda(\varepsilon_0,\lambda_0,u_0)+F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[v]\\ \tau F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]+F_{uu}(\varepsilon_0,\lambda_0,u_0)[v,w_0]+F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[\psi]\end{pmatrix}. \tag{3.7}$$

(1) $\dim N(K)=1$. Suppose that $(\tau,v,\psi)\in N(K)$ and $(\tau,v,\psi)\neq 0$. If $\tau=0$, from $F_u(\varepsilon_0,\lambda_0,u_0)[v]=0$ and (F1), we have $v=kw_0$, and

$$kF_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0,w_0]+F_u(\varepsilon_0,\lambda_0,u_0)[\psi]=0. \tag{3.8}$$

But (F4) is satisfied, thus $k=0$, and $\psi=v=0$. This is a contradiction. Next we consider $\tau\neq 0$. Without loss of generality, we assume that $\tau=1$. Notice that $F_\lambda(\varepsilon_0,\lambda_0,u_0)\in R(F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0))$ from (F2′), then v must be in form of kw_0+v_1, where v_1 is defined in (3.4). Substituting $v=kw_0+v_1$ and applying l to it, we obtain (3.5). From (F4), k can be uniquely determined by (3.5) and ψ_1 is also uniquely determined. Thus $N(K)=\{(1,kw_0+v_1,\psi_1)\}$.

(2) $\operatorname{codim} R(K)=1$. Let $(h,g)\in R(K)$, and let $(\tau,v,\psi)\in\mathbf{R}\times X\times X_3$ satisfy

$$\tau F_\lambda(\varepsilon_0,\lambda_0,u_0)+F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[v]=h, \tag{3.9}$$

$$\tau F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]+F_{uu}(\varepsilon_0,\lambda_0,u_0)[v,w_0]+F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[\psi]=g. \tag{3.10}$$

Applying l to (3.9), we get $\langle l,h\rangle=0$, hence $h\in R(F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0))$, and $R(K)\subset R(F_{\mathrm{u}})\times Y$. Conversely, for any $(h,g)\in R(F_{\mathrm{u}})\times Y$, there exists a unique $v_2\in X_3$ such that $F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0)[v_2]=h$. For any $\tau,k\in\mathbf{R}$, let $v_{\tau,k}=v_2+\tau v_1+kw_0$, where v_1 satisfies (3.4). Then $v_{\tau,k}$ solves (3.9). Substituting $v=v_{\tau,k}$ into (3.10), and applying l, we obtain

$$\tau\big\langle l,F_{\lambda u}[w_0]\big\rangle+\big\langle l,F_{uu}[w_0,v_2]\big\rangle+\tau\big\langle l,F_{uu}[w_0,v_1]\big\rangle+k\big\langle l,F_{uu}[w_0,w_0]\big\rangle=\langle l,g\rangle. \tag{3.11}$$

Then for fixed $\tau\in\mathbf{R}$, from (F4) there exists a unique $k(\tau)$ so that (3.11) holds. With such choice of k, (3.11) holds, then ψ in (3.10) is uniquely solvable in X_3. Therefore $(\tau,v_{\tau,k(\tau)},\psi)$ is a pre-image of (h,g). This implies that $R(K)=R(F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0))\times Y$, and $\operatorname{codim} R(K)=1$.

(3) $H(\varepsilon_0,\lambda_0,u_0,w_0)=0$.

(4) $H_\varepsilon(\varepsilon_0,\lambda_0,u_0,w_0)\notin R(K)$. Since $R(K)=R(F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0))\times Y$, we only need to show that $F_\varepsilon(\varepsilon_0,\lambda_0,u_0,w_0)\notin R(F_{\mathrm{u}}(\varepsilon_0,\lambda_0,u_0))$, but that is exactly assumed in (F5). So the statement of the theorem follows from Theorem 1.1. □

In Theorem 3.1, we have $\varepsilon(0)=\varepsilon_0$, $\varepsilon'(0)=0$, $\lambda(0)=\lambda_0$ and $\lambda'(0)=1$. To completely determine the turning direction of curve of degenerate solutions, we calculate $\varepsilon''(0)$. Let $\{T_{\mathrm{s}}=$

$(\varepsilon(s), \lambda(s), u(s), w(s))$: $s \in (-\delta, \delta)\}$ be a curve of degenerate solutions which we obtain in Theorem 3.1. Differentiating $H(\varepsilon(s), \lambda(s), u(s), w(s)) = 0$ with respect to s, we obtain

$$F_\varepsilon \varepsilon'(s) + F_\lambda \lambda'(s) + F_{\mathrm{u}}\big[u'(s)\big] = 0, \tag{3.12}$$

$$F_{\varepsilon u}\big[w(s)\big]\varepsilon'(s) + F_{\lambda u}\big[w(s)\big]\lambda'(s) + F_{uu}\big[w(s), u'(s)\big] + F_{\mathrm{u}}\big[w'(s)\big] = 0. \tag{3.13}$$

Setting $s = 0$ in (3.12) and (3.13), we get exactly (3.4) and (3.6). We differentiate (3.12) and (3.13) again, and we have

$$\begin{aligned} &F_{\varepsilon\varepsilon}\big[\varepsilon'(s)\big]^2 + F_\varepsilon \varepsilon''(s) + F_{\lambda\lambda}\big[\lambda'(s)\big]^2 + F_\lambda \lambda''(s) + F_{uu}\big[u'(s), u'(s)\big] \\ &\quad + F_{\mathrm{u}}\big[u''(s)\big] + 2F_{\varepsilon\lambda}\varepsilon'(s)\lambda'(s) + 2F_{\varepsilon u}\big[u'(s)\big]\varepsilon'(s) + 2F_{\lambda u}\big[u'(s)\big]\lambda'(s) = 0, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &F_{\varepsilon\varepsilon u}\big[w(s)\big]\big[\varepsilon'(s)\big]^2 + F_{\varepsilon u}\big[w(s)\big]\varepsilon''(s) + F_{\lambda u}\big[w(s)\big]\lambda''(s) \\ &\quad + F_{\lambda\lambda u}\big[w(s)\big]\big[\lambda'(s)\big]^2 + F_{uuu}\big[u'(s), u'(s), w(s)\big] + F_{uu}\big[w(s), u''(s)\big] \\ &\quad + F_{\mathrm{u}}\big[w''(s)\big] + 2F_{\varepsilon\lambda u}\big[w(s)\big]\varepsilon'(s)\lambda'(s) + 2F_{\varepsilon uu}\big[u'(s), w(s)\big]\varepsilon'(s) \\ &\quad + 2F_{\lambda uu}\big[u'(s), w(s)\big]\lambda'(s) + 2F_{\varepsilon u}\big[w'(s)\big]\varepsilon'(s) + 2F_{\lambda u}\big[w'(s)\big]\lambda'(s) \\ &\quad + 2F_{uu}\big[w'(s), u'(s)\big] = 0. \end{aligned} \tag{3.15}$$

Setting $s = 0$ in (3.14) and applying l to it, we obtain

$$\langle l, F_\varepsilon\rangle\varepsilon''(0) + \langle l, F_{\lambda\lambda}\rangle + \big\langle l, F_{uu}[kw_0 + v_1, kw_0 + v_1]\big\rangle + 2\big\langle l, F_{\lambda u}[kw_0 + v_1]\big\rangle = 0, \tag{3.16}$$

where k satisfies (3.5). By (3.5), we have

$$\varepsilon''(0) = -\frac{\langle l, F_{\lambda\lambda}\rangle + \langle l, F_{uu}[v_1, v_1]\rangle + 2\langle l, F_{\lambda u}[v_1]\rangle - k^2\langle l, F_{uu}[w_0, w_0]\rangle}{\langle l, F_\varepsilon\rangle}. \tag{3.17}$$

Our next result is under the assumption (F4′) instead of (F4). In this case an additional transversality condition (3.18) is needed to establish the saddle-node bifurcation of the degenerate solutions.

Theorem 3.2. *Let $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $H(T_0) = (0, 0)$. Suppose that the operator F satisfies* (F1), (F2′), (F3), (F4′) *and* (F5) *at T_0. We also assume that*

$$F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0] \notin R\big(F_{\mathrm{u}}(\varepsilon_0, \lambda_0, u_0)\big), \tag{3.18}$$

where $v_1 \in X_3$ is defined in (3.4)*. Then there exists $\delta > 0$ such that all the solutions of $H(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 form a C^2-curve*:

$$\big\{T_{\mathrm{s}} = \big(\varepsilon(s), \lambda(s), u(s), w(s)\big),\ s \in I = (-\delta, \delta)\big\}, \tag{3.19}$$

where $\varepsilon(s) = \varepsilon_0 + \tau(s)$, $s \in I$, $\tau(\cdot) \in C^2(I, \mathbf{R})$, $\tau(0) = \tau'(0) = 0$, and

$$\lambda(s) = \lambda_0 + z_1(s),$$
$$u(s) = u_0 + s w_0 + z_2(s),$$
$$w(s) = w_0 + s\psi_2 + z_3(s),$$

where $s \in I$, $z_i(\cdot) \in C^2$, $z_i(0) = z_i'(0) = 0$ $(i = 1, 2, 3)$, $\psi_2 \in X_3$ is the unique solution of

$$F_{\mathrm{u}}(\varepsilon_0, \lambda_0, u_0)[\psi_2] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] = 0. \tag{3.20}$$

Proof. Recall K defined in (3.7).

(1) $\dim N(K) = 1$. Let $(\tau, v, \psi) \in N(K)$ and $(\tau, v, \psi) \neq 0$. If $\tau = 0$, from $F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0$ and (F1), we have $v = k w_0$, and we also get (3.8). From (F4′), we may define ψ_2 as in (3.20). Thus $(0, w_0, \psi_2) \in N(K)$. If $\tau \neq 0$, we may assume that $\tau = 1$. From (F2′), The solution v_1 of (3.4) exists. More generally, the solution of (3.4) is in form of $v_1 + k w_0$. Substitute $v_1 + k w_0$ into (3.10) with $g = 0$, the solvability of ψ is equivalent to

$$F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0] \in R\big(F_{\mathrm{u}}(\varepsilon_0, \lambda_0, u_0)\big), \tag{3.21}$$

from (F4′). But we assume (3.18), hence ψ is not solvable, and there is no $(\tau, v, \psi) \in N(K)$ such that $\tau \neq 0$. Hence $N(K) = \operatorname{span}\{(0, w_0, \psi_2)\}$.

(2) $\operatorname{codim} R(K) = 1$. Similar to the proof of Theorem 3.1, we can show that $R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$. The only difference is that now in (3.11), $\langle l, F_{uu}[w_0, w_0]\rangle = 0$. But with (3.18), τ is uniquely determined since $\langle l, F_{\lambda u}[w_0]\rangle + \langle l, F_{uu}[w_0, v_1]\rangle \neq 0$. With (3.11) satisfied, then ψ in (3.10) is also uniquely determined. Thus $R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, and $\operatorname{codim} R(K) = 1$.

(3) $H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) \notin R(H_{(\lambda,u,w)}(\varepsilon_0, \lambda_0, u_0, w_0))$. From (F5), $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$, hence

$$H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) = \begin{pmatrix} F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \\ F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0] \end{pmatrix} \notin R\big(F_u(\varepsilon_0, \lambda_0, u_0)\big) \times Y.$$

(4) $H(\varepsilon_0, \lambda_0, u_0, w_0) = 0$. So the statement of the theorem follows from Theorem 1.1. □

In Theorem 3.2, we have $\varepsilon'(0) = \lambda'(0) = 0$, and we can determine $\varepsilon''(0)$ and $\lambda''(0)$ from (3.14) and (3.15) by setting $s = 0$:

$$F_\varepsilon \varepsilon''(0) + F_\lambda \lambda''(0) + F_{uu}[w_0, w_0] + F_{\mathrm{u}}\big[u''(0)\big] = 0, \tag{3.22}$$

$$\begin{aligned} &F_{\varepsilon u}[w_0]\varepsilon''(0) + F_{\lambda u}[w_0]\lambda''(0) + F_{uuu}[w_0, w_0, w_0] + F_{uu}\big[w_0, u''(0)\big] \\ &\quad + F_{\mathrm{u}}\big[w''(0)\big] + 2F_{uu}[\psi_2, w_0] = 0. \end{aligned} \tag{3.23}$$

Hence we have

$$\varepsilon''(0) = -\frac{\langle l, F_{uu}[w_0, w_0]\rangle}{\langle l, F_\varepsilon\rangle} = 0, \tag{3.24}$$

and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + \langle l, F_{uu}[w_0, u''(0)]\rangle + 2\langle l, F_{uu}[\psi_2, w_0]\rangle}{\langle l, F_{\lambda u}[w_0]\rangle}. \tag{3.25}$$

Since $\varepsilon''(0) = 0$, (3.22) implies that $u''(0) = \lambda''(0)v_1 + \psi_2 + k_1 w_0$ for some $k_1 \in \mathbf{R}$. Substituting this expression of $u''(0)$ into (3.23) and (3.25), we obtain

$$\lambda''(0) = -\frac{\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[\psi_2, w_0]\rangle}{\langle l, F_{\lambda u}[w_0]\rangle + \langle l, F_{uu}[w_0, v_1]\rangle}. \tag{3.26}$$

We differentiate (3.14) again,

$$\begin{aligned}
&F_{\varepsilon\varepsilon\varepsilon}\big[\varepsilon'(s)\big]^3 + F_\varepsilon \varepsilon'''(s) + F_{\lambda\lambda\lambda}\big[\lambda'(s)\big]^3 + F_\lambda \lambda'''(s) + F_{uuu}\big[u'(s), u'(s), u'(s)\big] \\
&\quad + F_{\mathrm{u}}\big[u'''(s)\big] + 3F_{\varepsilon\varepsilon}\varepsilon'(s)\varepsilon''(s) + 3F_{\lambda\lambda}\lambda'(s)\lambda''(s) + 3F_{uu}\big[u''(s), u'(s)\big] \\
&\quad + 3F_{\varepsilon\lambda}\varepsilon''(s)\lambda'(s) + 3F_{\varepsilon\lambda}\varepsilon'(s)\lambda''(s) + 3F_{\varepsilon u}\big[u'(s)\big]\varepsilon''(s) + 3F_{\varepsilon u}\big[u''(s)\big]\varepsilon'(s) \\
&\quad + 3F_{\lambda u}\big[u''(s)\big]\lambda'(s) + 3F_{\lambda u}\big[u'(s)\big]\lambda''(s) + 3F_{\varepsilon\lambda\lambda}\varepsilon'(s)\big(\lambda'(s)\big)^2 \\
&\quad + 3F_{\varepsilon\varepsilon\lambda}\big(\varepsilon'(s)\big)^2\lambda'(s) + 3F_{\varepsilon\varepsilon u}\big[u'(s)\big]\big(\varepsilon'(s)\big)^2 + F_{\varepsilon uu}\big[u'(s), u'(s)\big]\varepsilon'(s) \\
&\quad + 3F_{\lambda\lambda u}\big[u'(s)\big]\big(\lambda'(s)\big)^2 + 3F_{\lambda uu}\big[u'(s), u'(s)\big]\lambda'(s) + 6F_{\varepsilon\lambda u}\big[u'(s)\big]\varepsilon'(s)\lambda'(s) = 0.
\end{aligned} \tag{3.27}$$

Setting $s = 0$ in (3.27) and applying l to it, we obtain

$$\langle l, F_\varepsilon\rangle\varepsilon'''(0) + \big\langle l, F_{uuu}[w_0, w_0, w_0]\big\rangle + 3\big\langle l, F_{uu}\big[u''(0), w_0\big]\big\rangle + 3\big\langle l, F_{\lambda u}[w_0]\big\rangle\lambda''(0) = 0, \tag{3.28}$$

by (3.25), we have

$$\varepsilon'''(0) = 2 \cdot \frac{\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[\psi_2, w_0]\rangle}{\langle l, F_\varepsilon\rangle}. \tag{3.29}$$

We recall from [24] that the bifurcation at $(\varepsilon_0, \lambda_0, u_0)$ is a pitchfork bifurcation if

(F6) $F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0, w_0] + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[\psi_2, w_0] \notin R(F_{\mathrm{u}}(\varepsilon_0, \lambda_0, u_0))$, where $\psi_2 \in X_3$ is the unique solution of (3.20).

Thus we can conclude that if (F6) is satisfied, then $\lambda''(0) \neq 0$ and $\varepsilon'''(0) \neq 0$ in Theorem 3.2.

With (F2′) and (F5′) both satisfied, a crossing curve structure for the degenerate solutions near a transcritical bifurcation point is possible, as we show in the next theorem.

Theorem 3.3. *Let $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $H(T_0) = (0, 0)$. Suppose that the operator F satisfies* (F1), (F2′), (F3), (F4) *and* (F5′) *at T_0. We also assume that* (3.18) *holds, k and v_1 are defined as in* (3.5) *and* (3.4) *respectively, $v_2 \in X_3$ is the unique solution of*

$$F_\varepsilon(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v_2] = 0, \tag{3.30}$$

and the matrix

$$H_1 \equiv \begin{pmatrix} \langle l, D_{11}\rangle & \langle l, D_{12}\rangle \\ \langle l, D_{21}\rangle & \langle l, D_{22}\rangle \end{pmatrix} \tag{3.31}$$

satisfies $\det(H_1) < 0$, *where* D_{ij} *is given by*

$$\begin{aligned} D_{11} &= F_{\varepsilon\varepsilon} + 2\tau_1 F_{\varepsilon\lambda} + 2\tau_1 F_{\varepsilon u}[v_1] + 2F_{\varepsilon u}[v_2] + \tau_1^2 F_{\lambda\lambda} + 2\tau_1^2 F_{\lambda u}[v_1] \\ &\quad + 2\tau_1 F_{\lambda u}[v_2] + \tau_1^2 F_{uu}[v_1, v_1] + 2\tau_1 F_{uu}[v_1, v_2] + F_{uu}[v_2, v_2], \\ D_{12} &= D_{21} = F_{\varepsilon\lambda} + kF_{\varepsilon u}[w_0] + F_{\varepsilon u}[v_1] + \tau_1 F_{\lambda\lambda} + \tau_1 k F_{\lambda u}[w_0] + 2\tau_1 F_{\lambda u}[v_1] \\ &\quad + F_{\lambda u}[v_2] + \tau_1 k F_{uu}[w_0, v_1] + \tau_1 F_{uu}[v_1, v_1] + kF_{uu}[w_0, v_2] + F_{uu}[v_1, v_2], \\ D_{22} &= F_{\lambda\lambda} + 2kF_{\lambda u}[w_0] + 2F_{\lambda u}[v_1] + k^2 F_{uu}[w_0, w_0] + 2kF_{uu}[w_0, v_1] + F_{uu}[v_1, v_1], \end{aligned}$$

here τ_1 *is determined by*

$$\langle l, F_{\varepsilon u}[w_0]\rangle + \tau\langle l, F_{\lambda u}[w_0]\rangle + \tau\langle l, F_{uu}[v_1, w_0]\rangle + \langle l, F_{uu}[v_2, w_0]\rangle = 0. \tag{3.32}$$

Then the solutions set of $H(\varepsilon, \lambda, u, w) = (0, 0)$ *near* T_0 *is the union of two intersecting curves, and the two curves are in form*:

$$\{T_{i,s} = (\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)):\ s \in (-\delta, \delta)\},$$

where $\varepsilon_i(s) = \varepsilon_0 + \mu_i s + s z_{i0}(s)$, $\lambda_i(s) = \lambda_0 + \eta_i s + s z_{i1}(s)$, $u_i(s) = u_0 + \eta_i s(kw_0 + v_1) + s z_{i2}(s)$, $w_i(s) = w_0 + s\eta_i\psi_1 + s z_{i3}(s)$, (μ_1, η_1) *and* (μ_2, η_2) *are non-zero linear independent solutions of the equation*

$$\langle l, D_{11}\rangle\mu^2 + 2\langle l, D_{12}\rangle\eta\mu + \langle l, D_{22}\rangle\eta^2 = 0, \tag{3.33}$$

$z_{ij}(0) = z'_{ij}(0) = 0$, $z_{ij}(s) \in Z$, $i = 1, 2$; $j = 0, 1, 2, 3$. *If* $\det(H_1) > 0$, *then the solution set of* $H(\varepsilon, \lambda, u, w) = (0, 0)$ *near* T_0 *is the singleton* $\{T_0\}$.

Proof. We apply Theorem 2.1. Recall K defined in (3.7). Then the proof of Theorem 3.1 shows that $\dim N(K) = 1$ and $\operatorname{codim} R(K) = 1$ since the same assumptions except (F5′) hold here but (F5′) is not used in this part of proof. In particular we have $N(K) = \operatorname{span}\{W_0 \equiv (1, kw_0 + v_1, \psi_1)\}$, where ψ_1 is defined by (3.6). (F5′) implies that $H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) \in R(H_{(\lambda,u,w)}(\varepsilon_0, \lambda_0, u_0, w_0))$, thus (F2′) is satisfied for H. Equation (2.1) now becomes $H_\varepsilon + H_{(\lambda,u,w)}[\tau, v, \psi] = 0$, which is

$$F_\varepsilon + \tau F_\lambda + F_{\mathrm{u}}[v] = 0, \tag{3.34}$$

$$F_{\varepsilon u}[w_0] + \tau F_{\lambda u}[w_0] + F_{uu}[v, w_0] + F_{\mathrm{u}}[\psi] = 0. \tag{3.35}$$

We look for a solution $(\tau, v, \psi) \in Z_1 \equiv \mathbf{R} \times X_3 \times X_3$. Let v_1 be defined as in (3.4), and let v_2 be defined as in (3.30). Then the solution of (3.34) is given by $v = \tau v_1 + v_2 \in X_3$, and with this form of v, we apply l to (3.35) and we obtain (3.32). Thus $\tau = \tau_1$ can be uniquely determined from (3.18), and subsequently v, ψ can be uniquely determined. We denote this solution to be $V_1 = (\tau_1, \tau_1 v_1 + v_2, \psi_3)$.

Next we calculate the Hessian matrix in (2.2). Since $R(H_{(\lambda,u,w)}(\varepsilon_0, \lambda_0, u_0, w_0)) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, we define $l_1 \in (Y \times Y)^*$ by

$$\langle l_1, (y_1, y_2)\rangle = \langle l, y_1\rangle, \tag{3.36}$$

where $l \in Y^*$ so that $\langle l, y\rangle = 0$ if and only if $y \in R(F_u(\varepsilon_0, \lambda_0, u_0))$. For the simplicity of notations, we also denote $U = (\lambda, u, w)$. Then the matrix H_0 in Theorem 2.1 is in form of

$$\begin{aligned} H_1 &\equiv \begin{pmatrix} \langle l_1, H_{\varepsilon\varepsilon} + 2H_{\varepsilon U}[V_1] + H_{UU}[V_1, V_1]\rangle & \langle l_1, H_{\varepsilon U}[W_0] + H_{UU}[V_1, W_0]\rangle \\ \langle l_1, H_{\varepsilon U}[W_0] + H_{UU}[V_1, W_0]\rangle & \langle l_1, H_{UU}[W_0, W_0]\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle l, F_{\varepsilon\varepsilon} + 2F_{\varepsilon\widetilde{U}}[\widetilde{V}_1] + F_{\widetilde{U}\widetilde{U}}[\widetilde{V}_1, \widetilde{V}_1]\rangle & \langle l, F_{\varepsilon\widetilde{U}}[\widetilde{W}_0] + F_{\widetilde{U}\widetilde{U}}[\widetilde{V}_1, \widetilde{W}_0]\rangle \\ \langle l, F_{\varepsilon\widetilde{U}}[\widetilde{W}_0] + F_{\widetilde{U}\widetilde{U}}[\widetilde{V}_1, \widetilde{W}_0]\rangle & \langle l, F_{\widetilde{U}\widetilde{U}}[\widetilde{W}_0, \widetilde{W}_0]\rangle \end{pmatrix}, \end{aligned} \tag{3.37}$$

where $\widetilde{V}_1 = (\tau_1, \tau_1 v_1 + v_2)$ and $\widetilde{W}_0 = (1, kw_0 + v_1)$ are the projection of V_1 and W_0 from $\mathbf{R} \times X \times X_3$ to $\mathbf{R} \times X$, and $\widetilde{U} = (\lambda, u)$. From direct calculation, we can show that H_1 takes the form in (3.31). Now we can apply Theorem 2.1 to obtain the conclusions of the theorem. □

An important special case of Theorem 3.3 is when we have a constant solution for all λ near λ_0, i.e. we assume that $F(\varepsilon_0, \lambda, u_0) \equiv 0$ for λ near λ_0.

Corollary 3.4. *Let $F \in C^2(M, Y)$, $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $H(T_0) = (0, 0)$. We assume that there exists a neighborhood U of $(\varepsilon_0, \lambda_0)$ in $\mathbf{R}^2$ such that $F(\varepsilon_0, \lambda, u_0) \equiv 0$ for $(\varepsilon_0, \lambda) \in U$ and $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) = 0$. Suppose that at T_0 the operator F satisfies* (F1), (F3), (F4) *and*

(F7) $(\mathbf{v}H_2\mathbf{v}^T)/\langle l, F_{uu}[w_0, w_0]\rangle > 0$, *where* $\mathbf{v} = (\langle l, F_{\lambda u}[w_0]\rangle, \langle l, F_{\varepsilon\lambda}\rangle)$, *and*

$$H_2 \equiv \begin{pmatrix} \langle l, F_{\varepsilon\varepsilon}\rangle & -\langle l, F_{\varepsilon u}[w_0]\rangle \\ -\langle l, F_{\varepsilon u}[w_0]\rangle & \langle l, F_{uu}[w_0, w_0]\rangle \end{pmatrix}, \tag{3.38}$$

then the conclusions of Theorem 3.3 *hold and (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation*

$$\begin{aligned} &\big(\langle l, F_{\varepsilon\varepsilon}\rangle\langle l, F_{\lambda u}[w_0]\rangle - 2\langle l, F_{\varepsilon\lambda}\rangle\langle l, F_{\varepsilon u}[w_0]\rangle\big)\langle l, F_{uu}[w_0, w_0]\rangle\mu^2 \\ &\quad + 2\langle l, F_{\varepsilon\lambda}\rangle\langle l, F_{\lambda u}[w_0]\rangle\langle l, F_{uu}[w_0, w_0]\rangle\eta\mu - \big(\langle l, F_{\lambda u}[w_0]\rangle\big)^3\eta^2 = 0. \end{aligned} \tag{3.39}$$

Proof. We apply Theorem 3.3. (F2′) and (F5′) are satisfied and $v_1 = v_2 = 0$ since $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) = 0$, and $F(\varepsilon_0, \lambda, u_0) \equiv 0$ for $(\varepsilon_0, \lambda) \in U$. (3.18) is also satisfied since $v_1 = 0$ and (F3). Equations (3.5) and (3.32) now become

$$\langle l, F_{\lambda u}[w_0]\rangle + k\langle l, F_{uu}[w_0, w_0]\rangle = 0, \quad \text{and} \quad \langle l, F_{\varepsilon u}[w_0]\rangle + \tau_1\langle l, F_{\lambda u}[w_0]\rangle = 0, \tag{3.40}$$

respectively. For the entries of matrix H_1, we have $\langle l, D_{11}\rangle = \langle l, F_{\varepsilon\varepsilon} + 2\tau_1 F_{\varepsilon\lambda}\rangle$, $\langle l, D_{12}\rangle = \langle l, F_{\varepsilon\lambda}\rangle$, and $\langle l, D_{22}\rangle = k\langle l, F_{\lambda u}[w_0]\rangle$ from (3.32) and Theorem 3.3. With τ_1 and k determined by (3.40), then $\det(H_1) = -\mathbf{v}H_2\mathbf{v}^T/\langle l, F_{uu}[w_0, w_0]\rangle$, where $\mathbf{v}$ and H_2 are defined in (F7). Hence (F7) implies the indefiniteness of H_1. So the statement of the theorem follows from Theorem 3.3 and (3.39) is from calculations. □

Remark 3.5.

(1) We notice that (F7) is satisfied if the matrix H_2 is negatively definite and $\langle l, F_{uu}[w_0, w_0]\rangle < 0$, which only depends on the value of $F(\varepsilon, \lambda, u)$ for $\lambda = \lambda_0$, not λ near λ_0 since no derivatives respect to λ is involves in (F7).
(2) If $(\mathbf{v}H_2\mathbf{v}^T)/\langle l, F_{uu}[w_0, w_0]\rangle < 0$ in (F7), then the set of degenerate solutions near T_0 is the singleton $\{T_0\}$, from Theorem 2.1. This would happen if $\langle l, F_{uu}[w_0, w_0]\rangle < 0$ and H_2 is positively definite (see Fig. 4).

A more special case is when we also assume $F_{\varepsilon u}[w_0] = 0$:

Corollary 3.6. *Assume the conditions in Corollary* 3.4 *are satisfied, and in addition we assume that* $F_{\varepsilon u}[w_0] = 0$, *and*

$$\langle l, F_{\varepsilon\varepsilon}\rangle \cdot \big\langle l, F_{uu}[w_0, w_0]\big\rangle > 0. \tag{3.41}$$

Then the solution set of $H(\varepsilon, \lambda, u, w) = 0$ *near* T_0 *is the union of two curves of form*:

$$\big\{T_{i,\varepsilon} = \big(\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)\big)\colon\ s \in (-\delta, \delta)\big\}, \tag{3.42}$$

where $\varepsilon_i(s) = \varepsilon_0 + \mu_i s + s z_{i0}(s)$, $\lambda_i(s) = \lambda_0 + \eta_i s + s z_{i1}(s)$, $u_i(s) = u_0 + \eta_i s k w_0 + s z_{i2}(s)$, $w_i(s) = w_0 + s\eta_i\psi_1 + s z_{i3}(s)$, (μ_1, η_1) *and* (μ_2, η_2) *are non-zero linear independent solutions of the equation*

$$\langle l, F_{\varepsilon\varepsilon}\rangle \cdot \big\langle l, F_{uu}[w_0, w_0]\big\rangle\mu^2 + 2\langle l, F_{\varepsilon\lambda}\rangle \cdot \big\langle l, F_{uu}[w_0, w_0]\big\rangle\mu\eta - \big(\big\langle l, F_{\lambda u}[w_0]\big\rangle\big)^2\eta^2 = 0. \tag{3.43}$$

Proof. We apply Corollary 3.4. (F7) is satisfied since $\tau_1 = 0$ and the expression in (F7) is now simplified as

$$-\frac{\langle l, F_{\varepsilon\varepsilon}\rangle \cdot (\langle l, F_{\lambda u}[w_0]\rangle)^2}{\langle l, F_{uu}[w_0, w_0]\rangle} - \big(\langle l, F_{\varepsilon\lambda}\rangle\big)^2 < 0,$$

because of (3.41). Therefore we can apply Corollary 3.4 to obtain the results here. Notice that since $F_\varepsilon = 0$ and $F_{\varepsilon u}[w_0] = 0$, then $H_\varepsilon = 0$ thus we can also apply Corollary 2.2. □

4. Perturbation of bifurcation diagrams

In this section we shall apply the results in Section 3 to the original equation $F(\varepsilon, \lambda, u) = 0$, and observe the variations of bifurcation diagrams in (λ, u) space when the parameter ε is perturbed from $\varepsilon = \varepsilon_0$. First Theorem 3.1 is only a generalization of [24, Theorem 2.4], and the variations of the bifurcation diagrams have been studied in [24, Theorem 2.5]. Hence we will not discuss that situation here. The following result illustrates the application of Theorem 3.2.

Theorem 4.1. *Assume the conditions in Theorem 3.2 are satisfied, and $\{T_s\}$ is defined as in Theorem 3.2. In addition we assume that $F \in C^3(M,Y)$, $F(\varepsilon_0,\lambda,u_0)=0$ for $|\lambda-\lambda_0|<\gamma$ for some $\gamma>0$, and F satisfies (F6). Then $\varepsilon'''(0)\neq 0$ and $\lambda''(0)\neq 0$. If $\langle l, F_\varepsilon\rangle<0$, $\langle l, F_{uuu}[w_0,w_0,w_0]\rangle+3\langle l, F_{uu}[w_0,\psi_2]\rangle<0$, where ψ_2 satisfies (3.20), and $\langle l, F_{\lambda u}[w_0]\rangle>0$, then $\varepsilon'''(0)>0$ and $\lambda''(0)>0$, there exist ρ_1, δ_1, $\delta_2>0$ such that for $N=\{(\lambda,u)\in \mathbf{R}\times X\colon |\lambda-\lambda_0|\leqslant\delta_1,\ \|u\|\leqslant\delta_2\}$, we have*

(A) *for $\varepsilon=\varepsilon_0$,*

$$F^{-1}(0)\cap N=\{(\lambda,0)\colon |\lambda-\lambda_0|\leqslant\delta_1\}\cup\Sigma_0,\qquad \Sigma_0=\{(\bar\lambda(t),\bar u(t)),\ |t|\leqslant\eta\},\tag{4.1}$$

where $\bar\lambda(0)=\lambda_0$, $\bar\lambda'(0)=0$, $\bar\lambda''(0)>0$, and $\bar\lambda(\pm\eta)=\lambda_0+\delta_1$;

(B) *for fixed $\varepsilon\in(\varepsilon_0-\rho_1,\varepsilon_0)\cup(\varepsilon_0,\varepsilon_0+\rho_1)$,*

$$F^{-1}(0)\cap N=\Sigma_\varepsilon^+\cup\Sigma_\varepsilon^-,\tag{4.2}$$

where $\Sigma_\varepsilon^+=\{(\bar\lambda_+(t),\bar u_+(t)),\ t\in[-\eta,+\eta]\}$, where $\bar\lambda_+(\pm\eta)=\lambda_0+\delta_1$, $\bar\lambda_+'(0)=0$, $\bar\lambda_+''(0)>0$, and $(\bar\lambda_+(0),\bar u_+(0))$ is the unique degenerate solution on Σ_ε^+; and $\Sigma_\varepsilon^-=\{(\lambda_-,\bar u(\lambda_-))\colon \lambda_-\in[\lambda_0-\delta_1,\lambda_0+\delta_1]\}$ is a monotone curve without degenerate solutions (see Fig. 2).

Proof. From Theorem 3.2 and (3.29), we have $\varepsilon'(0)=0$, $\lambda'(0)=0$, $\varepsilon''(0)=0$, $\varepsilon'''(0)\neq 0$. From the assumptions, (3.26) (with $v_1=0$) and (3.29), $\lambda''(0)>0$ and $\varepsilon'''(0)>0$. When $\varepsilon=\varepsilon_0$, there is a branch of trivial solutions, and a branch of non-trivial ones (which we denote by $\Sigma_0=(\bar\lambda(t),\bar u(t)))$ from Theorem 1.2. The conditions (F3) and (F4′) imply that $\bar\lambda'(0)=0$ and from [24, (4.6)]

$$\bar\lambda''(0)=-\frac{\langle l, F_{uuu}[w_0,w_0,w_0]\rangle+3\langle l, F_{uu}[w_0,\psi_2]\rangle}{3\langle l, F_{\lambda u}[w_0]\rangle}>0.\tag{4.3}$$

Thus when $\varepsilon=\varepsilon_0$, the bifurcation near (λ_0,u_0) is a supercritical pitchfork one, and there is only one degenerate solution on Σ_0 near (λ_0,u_0). Define $N=\{(\lambda,u)\in\mathbf{R}\times X\colon |\lambda-\lambda_0|\leqslant\delta_1$, $\|u\|\leqslant\delta_2\}$, such that (4.1) holds and $\bar\lambda(\pm\eta)=\lambda_0+\delta_1$, for $t\in[-\eta,+\eta]$, $\|\bar u(t)\|\leqslant\delta_2/2$.

We denote by Σ_ε the solution set of (3.1) in N for fixed ε. Since $\lambda''(0)>0$ and $\varepsilon'''(0)>0$, then there exists $\rho_2>0$ such that for $\varepsilon\in(\varepsilon_0,\varepsilon_0+\rho_2)$, (3.1) has a unique degenerate solution $(\bar\lambda_+,\bar u_+)$, where $\bar\lambda_+=\bar\lambda(s_+)>\lambda_0$, $s_+>0$ for s_+ small. Moreover, $\bar u_+$ is a degenerate solution which satisfy the condition of Theorem 1.1. In fact, we only need to check that $F_\lambda(\varepsilon,\bar\lambda_+,\bar u_+)\notin R(F_{\mathrm u}(\varepsilon,\bar\lambda_+,\bar u_+))$. Define $A(s)=\langle l(s), F_\lambda(\varepsilon(s),\lambda(s),u(s))\rangle$, where $l(s)\in Y^*$ satisfying $N(l(s))=R(F_{\mathrm u}(\varepsilon(s),\lambda(s),u(s))$. Then $A'(0)=\langle l, F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]\rangle>0$ and $A(0)=0$, so $A(s_+)>0$. Define $B(s)=\langle l(s), F_{uu}(\varepsilon(s),\lambda(s),u(s))[w(s),w(s)]\rangle$, we have

$$\begin{aligned}B'(0)&=\langle l'(0), F_{uu}[w_0,w_0]\rangle+\langle l, F_{uuu}[w_0,w_0,w_0]+2F_{uu}[\psi_2,w_0]\rangle\\&=\langle l, F_{uuu}[w_0,w_0,w_0]+3F_{uu}[\psi_2,w_0]\rangle<0,\end{aligned}\tag{4.4}$$

since $\varepsilon'(0)=\lambda'(0)=0$, $u'(0)=w_0$, $w'(0)=\psi_2$, $\varepsilon''(0)=0$. In (4.4), we obtain $\langle l'(0), F_{uu}[w_0,w_0]\rangle=\langle l, F_{uu}[\psi_2,w_0]\rangle$ by differentiating $\langle l(s), F_{\mathrm u}(\varepsilon(s),\lambda(s),u(s))\ [w_0]\rangle=0$ twice and using (3.25). In particular $B(s_+)<0$. From Theorem 1.1, near $(\bar\lambda_+,\bar u_+)$, the solutions form a curve $\Sigma_\varepsilon^+ = \{(\bar\lambda_+(t), \bar u_+(t)),\ t \in [-\eta, +\eta]\}$ with $\bar\lambda_+(0) = \bar\lambda_+$, $\bar u_+(0) = \bar u_+$, $\bar\lambda_+''(0) > 0$ and $\bar\lambda_+(\pm\eta) = \lambda_0 + \delta_1$. Here we obtain $\bar\lambda_+''(0) > 0$ by using [24, (4.1)]

$$\bar\lambda_+''(0) = -\frac{\langle \bar l_+, F_{uu}(\bar\lambda_+, \bar u_+)[\overline{w}_+, \overline{w}_+]\rangle}{\langle \bar l_+, F_\lambda(\bar\lambda_+, \bar u_+)\rangle} = -\frac{B(s_+)}{A(s_+)}. \tag{4.5}$$

From the definition of N, there exists $\rho_3 > 0$ such that for $\varepsilon \in [\varepsilon_0, \varepsilon_0 + \rho_3]$, (3.1) has no solution (λ, u) with $\|u\| = \delta_2$ with $|\lambda - \lambda_0| \leqslant \delta_1$. And there exists $\rho_4 > 0$ such that for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_4]$, there is no degenerate solution (λ, u) of (3.1) with $\|u\| \leqslant \delta_2$ and $|\lambda - \lambda_0| \geqslant \delta_1/2$. For $\varepsilon = \varepsilon_0$, (3.1) have two nontrivial solutions $(\lambda_0 + \delta_1, u_\pm^*) \in \partial N$ such that $\|u_\pm^*\| \leqslant \delta_2/2$. Since $u_\pm^*$ is non-degenerate, for fixed $\lambda = \lambda^*$, by implicit function theorem, there exists $\rho_5 > 0$ such that for $\varepsilon \in (\varepsilon_0 - \rho_5, \varepsilon_0 + \rho_5)$, (3.1) has a unique solution $u_\pm^*(\varepsilon)$ near $u_\pm^*$, a unique solution $u_0^*(\varepsilon)$ near the trivial solution u_0 with $\lambda = \lambda_0 + \delta_1$, and a unique solution $u_{*0}(\varepsilon)$ near the trivial solution u_0 with $\lambda = \lambda_0 - \delta_1$. Therefore from the implicit function theorem and the fact that $(\bar\lambda_+, \bar u_+)$ is the only degenerate solution in N, the solution set of (3.1) in N consists of Σ_ε^+ defined above, and another monotone curve which we denote by Σ_ε^-.

We claim that Σ_ε consists of only $\Sigma_\varepsilon^\pm$ for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$ with $\rho_1 = \min\{\rho_2, \rho_3, \rho_4, \rho_5\}$. Suppose that for some $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$, there is another solution (λ, u) which is not on $\Sigma_\varepsilon^\pm$. Then (λ, u) is a non-degenerate solution since the only degenerate solution is $(\bar\lambda_+, \bar u_+)$. So by the implicit function theorem, (λ, u) is on a solution curve $\Sigma_1 = (\tilde\lambda(s), \tilde u(s))$ which can be extended to ∂N. But there is no solution on $\|u\| = \delta_2$, so there is a solution $(\lambda_0 + \delta_1, u^*)$ on Σ_1. On the other hand, Σ_ε has another three solutions with $\lambda = \lambda_0 + \delta_1$, thus there are at least four solutions with $\lambda = \lambda_0 + \delta_1$, which contradicts with the definition of ρ_1 and the arguments in the last paragraph. The proof for $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$ is similar. □

Remark 4.2.

(1) The assumptions on the signs of $\langle l, F_{uuu}[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}[w_0, \psi_2]\rangle$, $\langle l, F_\varepsilon\rangle$ and $\langle l, F_{\lambda u}[w_0]\rangle$ are only for the purpose of fixing an orientation, similar results for other cases can also be established.

(2) From the proof, the solutions $\bar\lambda_+(\pm\eta)$ are two of the three solutions $u_\pm^*$ and u_0^*, but it is not immediately clear which two. However usually with more information, we can show that u_0^* must be one of $\bar\lambda_+(\pm\eta)$, i.e. the perturbation u_0^* of the trivial solution u_0 must be on the curve with turning point, which is consistent with observations in experiments.

Theorem 4.3. *Assume the conditions in Corollary* 3.6 *are satisfied, and in addition we assume* $F_{\varepsilon\lambda}(\varepsilon_0, \lambda_0, u_0) = 0$. *For the purpose of fixing an orientation, we assume that*

$$\big\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\big\rangle > 0, \quad \text{and} \quad \big\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\big\rangle < 0. \tag{4.6}$$

Then there exist $\rho, \delta_1, \delta_2 > 0$ *such that for* $N = \{(\lambda, u) \in \mathbf{R} \times X\colon\ |\lambda - \lambda_0| \leqslant \delta_1,\ \|u\| \leqslant \delta_2\}$,

(A) *for* $\varepsilon = \varepsilon_0$,

$$\begin{aligned} &F^{-1}(0) \cap N = \big\{(\lambda, u_0)\colon\ |\lambda - \lambda_0| \leqslant \delta_1\big\} \cup \Sigma_0, \\ &\Sigma_0 = \big\{\big(\bar\lambda(t), \bar u(t)\big)\colon\ t \in [-\eta, \eta]\big\}, \quad \text{and} \quad \bar\lambda'(t) > 0 \text{ for } t \in [-\eta, \eta]; \end{aligned} \tag{4.7}$$

(B) *for* $\varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0) \cup (\varepsilon_0, \varepsilon + \rho)$,

$$F^{-1}(0) \cap N = \Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-, \quad \Sigma_\varepsilon^\pm = \{(\bar{\lambda}_\pm(t), \bar{u}_\pm(t)) \colon t \in [-\eta, \eta]\}$$

where $\bar{\lambda}'_\pm(0) = 0$, $\bar{\lambda}''_+(0) > 0$, $\bar{\lambda}''_-(0) < 0$ *and there is exactly one turning point on each component* $\Sigma_\varepsilon^\pm$; *moreover* $\bar{\lambda}_+(0) > \lambda_0 > \bar{\lambda}_-(0)$.

Proof. From (4.6) and (3.41), we also have $\langle l, F_{\varepsilon\varepsilon}(\varepsilon_0, \lambda_0, u_0)\rangle < 0$. When $\varepsilon = \varepsilon_0$, by Theorem 1.2, the solution set of $F(\varepsilon_0, \lambda, u) = 0$ near (λ_0, u_0) consists precisely the two curves of form in (4.7), where

$$\bar{\lambda}'(0) = -\frac{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle} > 0.$$

From the conditions in Corollary 3.6, $\mu_i \neq 0$ and $\eta_i \neq 0$ for $i = 1, 2$, thus there are exactly two degenerate solutions for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$ as well as for $\varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0)$. To be more precise, we assume $\mu_i = 1$ for $i = 1, 2$. When $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$, we denote the two degenerate solutions by (λ_+, u_+) and (λ_-, u_-), where $\lambda_+ = \lambda(s_+) = \lambda_0 + \eta_1 s_+ + o(|s_+|)$ and $\lambda_- = \lambda(s_-) = \lambda_0 + \eta_2 s_- + o(|s_-|)$, $s_+ > 0$ and $s_- > 0$ since $\varepsilon = \varepsilon_0 + s_+ + o(|s_+|)$ and $\varepsilon = \varepsilon_0 + s_- + o(|s_-|)$, $u_\pm = u_0 + \eta_i k s_\pm w_0 + o(|s|)$ for $|s|$ small. From (3.43), we have

$$\eta_i = \pm\frac{\sqrt{\langle l, F_{\varepsilon\varepsilon}\rangle \cdot \langle l, F_{uu}[w_0, w_0]\rangle}}{\langle l, F_{\lambda u}[w_0]\rangle}. \tag{4.8}$$

To be definite we assume $\eta_1 > 0$ and $\eta_2 < 0$, then $\lambda_+ > \lambda_0$ and $\lambda_- < \lambda_0$.

Recall the definitions of $A(s)$ and $l(s)$ from the proof of Theorem 4.1, and here first we consider the branch corresponding to (μ_1, η_1). Then A is differentiable, $A(0) = \langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle = 0$, and since $F_{\varepsilon\lambda}(\varepsilon_0, \lambda_0, u_0) = 0$, we have

$$\begin{aligned} A'(0) &= \langle l'(0), F_\lambda\rangle + \mu_1\langle l, F_{\varepsilon\lambda}\rangle + \eta_1\langle l, F_{\lambda\lambda}\rangle + k\eta_1\langle l, F_{\lambda u}[w_0]\rangle \\ &= k\eta_1\langle l, F_{\lambda u}[w_0]\rangle = -\frac{\langle l, F_{\lambda u}[w_0]\rangle}{\langle l, F_{uu}[w_0, w_0]\rangle}\sqrt{\langle l, F_{\varepsilon\varepsilon}\rangle \cdot \langle l, F_{uu}[w_0, w_0]\rangle} > 0. \end{aligned}$$

Thus $A(s_+) > 0$. On the other hand, since

$$\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle < 0,$$

we have

$$\langle l(s), F_{uu}(\varepsilon(s), \lambda(s), u(s))[w(s), w(s)]\rangle < 0,$$

for $|s|$ small. From Theorem 1.1, near (λ_+, u_+) the solution forms a curve $\Sigma_\varepsilon^+ = \{(\bar{\lambda}_+(t), \bar{u}_+(t))\colon t \in [-\eta, \eta]\}$ with $\bar{\lambda}_+(0) = \lambda_+$, $\bar{u}_+(0) = u_+$, $\bar{\lambda}'_+(0) = 0$ and

$$\bar{\lambda}''_+(0) = -\frac{\langle l(s_+), F_{uu}(\varepsilon_+, \lambda_+, u_+)[w(s_+), w(s_+)]\rangle}{\langle l(s_+), F_\lambda(\varepsilon_+, \lambda_+, u_+)\rangle} > 0.$$

Similarly the solutions near (λ_-, u_-) the solution forms a curve $\Sigma_\varepsilon^- = \{(\bar{\lambda}_-(t), \bar{u}_-(t)):\ t \in [-\eta, \eta]\}$ with $\bar{\lambda}_-(0) = \lambda_-$, $\bar{u}_-(0) = u_-$, $\bar{\lambda}'_-(0) = 0$ and $\bar{\lambda}''_-(0) < 0$. The proof for the case $\varepsilon < \varepsilon_0$ is similar. □

It is clear that when $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho) \cup (\varepsilon_0 - \rho, \varepsilon_0)$, there exists an interval $I = (\lambda_0 - \delta_\varepsilon, \lambda_0 + \delta_\varepsilon)$ such that (3.1) has no solutions near (λ_0, u_0) when $\lambda \in I$ (see Fig. 3).

5. Examples

5.1. Euler bucking beam: symmetry break of pitchfork bifurcation

The Euler buckling beam problem was one of the first bifurcation problems to be studied, as early as by Euler, see for example [1,9,10,23] for more historical remarks and details on modeling. Here we follow an example of Reiss [23]:

$$\begin{cases} \phi'' + \lambda \sin\phi = 0, \quad 0 < x < 1,\ \phi'(0) = \phi'(1) = 0, \\ u' = \cos\phi - 1, \quad 0 < x < 1,\ u(0) = 0, \\ w' = \sin\phi, \quad 0 < x < 1,\ w(0) = w(1) = 0. \end{cases} \tag{5.1}$$

Here the length of the elastic column is normalized so that $0 \leqslant x \leqslant 1$, the horizontal and vertical displacements of the buckled axis are denoted by $u(x)$ and $w(x)$ respectively, $\phi(x)$ is the angle between the tangent to the column's axis and the x-axis, and λ is a parameter proportional to the thrust. The boundary conditions imply that the ends of the column are pinned (simply supported). For this classical problem, solutions can be obtained explicitly in terms of elliptic functions, and supercritical pitchfork bifurcations occur along the trivial solutions $\phi = u = w = 0$ from $\lambda = (n\pi)^2$ with $n = 1, 2, \ldots$. The minimal buckling load λ_1 is the Euler buckling load. To account for imperfections in the column, we assume that the initial unstressed axis is a curve instead of a line, and the angle between the tangent of the curve and the horizontal is $\phi_I(x)$. Then the ϕ equation in (5.1) is modified to

$$\phi'' + \lambda \sin\phi = \lambda\varepsilon g(x), \quad 0 < x < 1, \qquad \phi'(0) = \phi'(1) = 0, \tag{5.2}$$

where $g(x) = \phi_I''(x)$ and $\phi_I'(0) = \phi_I'(1) = 0$ (thus $\int_0^1 g(x)\,dx = 0$). In [14,23], singular perturbation methods matching the inner and outer expansions were used to derive the imperfect bifurcation near $(\lambda, u) = (\pi^2, 0)$. Here we will apply Theorems 3.2 and 4.1 to analytically obtain the precise bifurcation diagrams. But first we illustrate how the secondary bifurcation theorem (Theorem 2.7) can be applied to the bifurcation at $\lambda = 0$:

Proposition 5.1. *Suppose that $g(x) \in C^1([0,1], \mathbf{R})$, $\int_0^1 g(x)\,dx = 0$, then for any $\varepsilon \in \mathbf{R}$, the solution set of* (5.2) *near $(\lambda, u) = (0, 0)$ is in form of*

$$\Sigma_\varepsilon = \big\{(0, k):\ k \in \mathbf{R}\big\} \cup \big\{\big(\lambda, \lambda\phi_0(x) + o(|\lambda|)\big):\ \lambda \in (-\delta, \delta)\big\}, \tag{5.3}$$

where $\phi_0(x)$ is the solution of

$$\phi'' = \varepsilon g(x), \quad 0 < x < 1, \qquad \phi'(0) = \phi'(1) = 0. \tag{5.4}$$

Proof. For fixed $\varepsilon \in \mathbf{R}$, define $G(\lambda, \phi) = \phi'' + \lambda \sin\phi - \lambda\varepsilon g$, where $\lambda \in \mathbf{R}$ and $\phi \in X = \{v \in C^2([0,1])\colon v'(0) = v'(1) = 0\}$. We denote $W = \mathbf{R} \times X$ and $Y = C^0([0,1])$. Then $G(0,0) = 0$, and

$$G'(\lambda,\phi)\big[(\tau,\theta)\big] = (\sin\phi - \varepsilon g)\tau + \theta'' + \lambda\cos\phi\theta,$$
$$G''(\lambda,\phi)\big[(\tau_1,\theta_1)(\tau_2,\theta_2)\big] = 0\cdot\tau_1\tau_2 + \cos\phi(\tau_1\theta_2 + \tau_2\theta_1) - \lambda\sin\phi\theta_1\theta_2. \tag{5.5}$$

Thus $G'(0,0)[(\tau,\theta)] = \theta'' - \varepsilon g\tau$, which has a two-dimensional kernel $N(G'(0,0)) = \operatorname{span}\{(0,1), (1,\phi_0)\}$. If $h \in R(G'(0,0))$, then there exists (τ,θ) such that $\theta'' - \varepsilon g\tau = h$. Since $\int_0^1 g(x)\,dx = 0$, then the solvability of (τ,θ) is reduced to $\int_0^1 h(x)\,dx = 0$. Hence $R(G'(0,0)) = \{h \in Y\colon \int_0^1 h(x)\,dx = 0\}$, which is co-dimensional one. Also $G''(0,0)[(0,1)(1,\phi_0)] = 1 \notin R(G'(0,0))$, and $G''(0,0)[(0,1)(0,1)] = 0 \in R(G'(0,0))$. Hence Theorem 2.7 can be applied to obtain results stated since $(0,k)$ is always a solution for any k, and we can reparameterize the nontrivial branch so that λ is the new parameter. $\square$

Next we apply Theorem 4.1 near the first bifurcation point $\lambda_1 = \pi^2$:

Proposition 5.2. *Suppose that $g(x) \in C^1([0,1],\mathbf{R})$, $\int_0^1 g(x)\,dx = 0$, and $\int_0^1 g(x)\cos\pi x\,dx > 0$. Then there exists $\rho, \delta_1, \delta_2 > 0$ such that for $\varepsilon \in (-\rho,\rho)$, $N = \{(\lambda,u) \in \mathbf{R}\times X\colon |\lambda - \pi^2| \leqslant \delta_1, \|u\| \leqslant \delta_2\}$ such that the solution set of (5.2) in N is in form of $\Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-$, where $\Sigma_\varepsilon^+ = \{(\overline{\lambda}_+(t), \overline{u}_+(t)),\ |t| \leqslant \eta\}$ is a $\subset$-shaped curve with $\overline{\lambda}''_+(0) > 0$, and $(\overline{\lambda}_+(0), \overline{u}_+(0))$ is the unique degenerate solution on Σ_ε^+; and $\Sigma_\varepsilon^- = \{(\lambda_-, \overline{u}(\lambda_-))\colon |\lambda_- - \pi^2| \leqslant \delta_1\}$ is a monotone curve without degenerate solutions. Moreover the bifurcation point satisfies*

$$\overline{\lambda}_+(0) = \pi^2 + \frac{3}{2}\pi^2\left(\int_0^1 g(x)\cos(\pi x)\,dx\right)^{2/3}\varepsilon^{2/3} + o\big(\varepsilon^{2/3}\big). \tag{5.6}$$

Proof. We define

$$F(\varepsilon,\lambda,\phi) = \phi'' + \lambda\sin\phi - \lambda\varepsilon g(x), \tag{5.7}$$

where $\varepsilon, \lambda \in \mathbf{R}$ and $\phi \in X = \{v \in C^2([0,1])\colon v'(0) = v'(1) = 0\}$. With

$$F_\phi(\varepsilon,\lambda,\phi)[w] = w'' + \lambda\cos\phi\cdot w \tag{5.8}$$

we can easily check that for $\lambda_1 = \pi^2$ and $w_0 = \cos\pi x$, (F1) and (F2′) are satisfied, and $R(F_\phi(0,\lambda_1,0)) = \{v \in X\colon \int_0^1 v(x)\cos(\pi x)\,dx = 0\}$. Since $F_{\lambda\phi}(0,\lambda_1,0)[w_0] = \cos(0)w_0 = w_0$ and $\int_0^1 w_0(x)\cos(\pi x)\,dx \neq 0$, then (F3) is satisfied; so is (F4′) since $F_{\phi\phi}(0,\lambda_1,0)[w_0,w_0] = -\lambda_1\sin(0)\cdot w_0^2 = 0$. Finally (F5) is satisfied since $F_\varepsilon(0,\lambda_1,0) = -\lambda_1 g(x)$ and we assume that $\int_0^1 g(x)\cos\pi x\,dx \neq 0$. Hence Theorem 3.2 can be applied, with $v_1 = 0$ and $\psi_2 = 0$, and the degenerate solutions are on a curve $\{T_s = (\varepsilon(s),\lambda(s),u(s),w(s))\colon |s| < \delta\}$ such that $\varepsilon'(0) = \varepsilon''(0) = \lambda'(0) = w'(0) = 0$, and $u'(0) = w_0$. Moreover we can verify that $\langle l, F_\varepsilon\rangle < 0$, $\langle l, F_{\phi\phi\phi}[w_0,w_0,w_0]\rangle + 3\langle l, F_{\phi\phi}[w_0,\psi_2]\rangle < 0$, and $\langle l, F_{\lambda\phi}[w_0]\rangle > 0$. Therefore Theorem 4.1

also holds. Equations (3.26) and (3.29) imply that

$$\lambda''(0)=\frac{\lambda_1\int_0^1 w_0^4(x)\,dx}{\int_0^1 w_0^2(x)\,dx}=\frac{3\pi^2}{4}, \tag{5.9}$$

and

$$\varepsilon'''(0)=\frac{2\int_0^1 w_0^4(x)\,dx}{\int_0^1 w_0(x)g(x)\,dx}>0. \tag{5.10}$$

Hence $T_s=((p/6)s^3+o(|s|^3),\pi^2+(3/8)\pi^2s^2+o(s^2),s\cos(\pi x)+o(|s|),\cos(\pi x)+o(|s|))$, where $p=\varepsilon'''(0)>0$. We rescale the parameter s so $\varepsilon(s)=(p/6)s^3$ then $\lambda(s)=\pi^2+(3/8)6^{2/3}\pi^2p^{-2/3}\varepsilon^{2/3}+o(\varepsilon^{2/3})=\pi^2+(3/2)\pi^2(\int_0^1 g(x)\cos(\pi x)\,dx)^{2/3}\varepsilon^{2/3}+o(\varepsilon^{2/3})$, which is the λ-coordinate of the unique degenerate solution for fixed small ε. Thus the turning point in Theorem 4.1 satisfies (5.6). □

Now by using the perturbation analysis of degenerate solutions in Propositions 5.1 and 5.2 and implicit function theorem, we can obtain the following global imperfect bifurcation picture for $0<\lambda<4\pi^2-\delta$ with small ε (the proof is standard, which we omit):

Theorem 5.3. *Suppose that $g(x)\in C^1([0,1],\mathbf{R})$, $\int_0^1 g(x)\,dx=0$, and $\int_0^1 g(x)\cos\pi x\,dx>0$. For any $\delta_1>0$, there exists*

(1) *If $\varepsilon=0$, then (5.2) has the trivial solution $u=0$ for all $\lambda>0$, and has exactly two other solutions $u_+(x)$ and $u_-(x)$ for $\lambda\in(\pi^2,4\pi^2)$ and $u_-(x)=-u_+(x)=u_+(1-x)$;*
(2) *If $\varepsilon\in(-\delta_2,\delta_2)\backslash\{0\}$, then there exists λ_* such that (5.2) has exactly one solution when $\lambda\in(0,\lambda_*)$, has exactly two solutions when $\lambda=\lambda_*$ and has exact three solutions when $\lambda\in(\lambda_*,4\pi^2-\delta_1)$. Moreover $\lambda_*=\lambda_*(\varepsilon)$ is given by (5.6), $\lambda_*>\pi^2$ if $\delta_2>\varepsilon>0$ and $\lambda_*<\pi^2$ if $0>\varepsilon>-\delta_2$.*

Note that we can show that when $\int_0^1 g(x)\cos\pi x\,dx<0$, same results hold but $\lambda_*<\pi^2$ if $\delta_2>\varepsilon>0$ and $\lambda_*>\pi^2$ if $0>\varepsilon>-\delta_2$. We remark that the expansion in (5.6) was also obtained previously in [23], but results we have here are rigorous not just formal perturbation expansion.

5.2. Perturbed diffusive logistic equation: symmetry break of a transcritical bifurcation

Diffusive logistic equation (Fisher equation) is one of most important reaction–diffusion equations with connections to biological invasion of foreign species, propagation of genetic traits [8,12]. The steady state diffusive logistic equation on a bounded domain is in form of

$$\begin{cases}\Delta u+\lambda\big(u-u^2\big)=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{5.11}$$

where λ is a positive parameter, and Ω is a smooth bounded region in $\mathbf{R}^n$ for $n\geqslant 1$. The nonlinearity $f(u)$ can be more general as the one defined in [26,27], but here for the transparency of presentation, we use $f(u)=u-u^2$ as in the classical equation. The perturbed problem of (5.12) arises when the population is harvested:

$$\begin{cases} \Delta u + \lambda\big(u - u^2 - \varepsilon g(x,u)\big) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.12}$$

Perturbation and bifurcation analysis have been performed in [24,26]. Here we demonstrate our new abstract theory with some new results and also rediscovery of old results.

In the following (λ_1, ϕ_1) is the principal eigen-pair of

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.13}$$

and we assume ϕ_1 is normalized and $\phi_1(x) > 0$ in Ω. The bifurcation diagram for unperturbed problem is well known (see [26,27]):

Theorem 5.4. *When $\varepsilon = 0$, (5.12) has no positive solution if $\lambda \leqslant \lambda_1$, and has exactly one positive solution v_λ if $\lambda > \lambda_1$. Moreover, all v_λ's lie on a smooth curve, $\lim_{\lambda\to\lambda_1^-} v_\lambda = 0$ and v_λ is increasing with respect to λ.*

Indeed $(\lambda, u) = (\lambda_1, 0)$ is a bifurcation point where a transcritical bifurcation (see Theorem 1.2) occurs. To show that, we define

$$F^0(\lambda, u) = \Delta u + \lambda\big(u - u^2\big), \tag{5.14}$$

where $\lambda \in \mathbf{R}$ and $u \in X = \{u \in C^{2,\alpha}(\overline{\Omega})\colon\ u = 0 \text{ on } \partial\Omega\}$. Then $u = 0$ is a trivial solution for any λ, $N(F^0_{\mathrm{u}}(\lambda_1, 0)) = \operatorname{span}\{\phi_1\}$, $R(F^0_{\mathrm{u}}(\lambda_1, 0)) = \{v \in C^\alpha(\overline{\Omega})\colon \int_\Omega v\phi_1\,dx = 0\}$, and $F^0_{\lambda u}(\lambda_1, 0)[\phi_1] = \phi_1 \notin R(F^0_{\mathrm{u}}(\lambda_1, 0))$. Hence (F1), (F2′) and (F3) are satisfied, and Theorem 1.2 is applicable. Moreover, $F^0_{uu}(\lambda_1, 0)[\phi_1, \phi_1] = -2\lambda_1\phi_1^2 \notin R(F^0_{\mathrm{u}}(\lambda_1, 0))$ since $\int_\Omega \phi_1^3\,dx \neq 0$, thus (F4) is satisfied, and a transcritical bifurcation occurs at $(\lambda_1, 0)$. Note that all these remain true for a perturbed operator $F(\varepsilon, \lambda, u)$ if $F(0, \lambda, u) \equiv F^0(\lambda, u)$.

First we consider

$$\begin{cases} \Delta u + \lambda\big(u - u^2\big) - \lambda\varepsilon g(x) = 0, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.15}$$

where $g(x) \in C^1(\overline{\Omega})$ and

$$\int_\Omega g(x)\phi_1(x)\,dx \neq 0. \tag{5.16}$$

We define

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda\big(u - u^2\big) - \lambda\varepsilon g(x). \tag{5.17}$$

Then (F1), (F2′), (F3) and (F4) are satisfied at $(\varepsilon, \lambda, u) = (0, \lambda_1, 0)$ since $F(0, \lambda, u) \equiv F^0(\lambda, u)$ defined above. (F5) is also satisfied, as we have $F_\varepsilon(0, \lambda_1, 0) = -\lambda_1 g(x) \notin R(F_{\mathrm{u}}(0, \lambda_1, 0))$ from (5.16). From Theorem 3.1, there exists $\delta > 0$ such that all the degenerate solutions of (5.15) near $T_0 = (0, \lambda_1, 0, \phi_1)$ form a C^2-curve

$$\big\{T_s = \big(\varepsilon(s), \lambda_1 + s + o\big(|s|\big), ks\phi_1 + o\big(|s|\big), \phi_1 + o(1)\big),\quad s \in I = (-\delta, \delta)\big\}, \tag{5.18}$$

where

$$k = \frac{\int_\Omega \phi_1^2(x)\,dx}{2\lambda_1 \int_\Omega \phi_1^3(x)\,dx} > 0, \tag{5.19}$$

$\varepsilon(0) = \varepsilon'(0) = 0$, and from (3.17),

$$\varepsilon''(0) = \frac{k^2\langle l, F_{uu}[w_0, w_0]\rangle}{\langle l, F_\varepsilon\rangle} = \frac{2k^2 \int_\Omega \phi_1^3(x)\,dx}{\int_\Omega g(x)\phi_1(x)\,dx}. \tag{5.20}$$

We assume $\int_\Omega g(x)\phi_1(x)\,dx > 0$ then $\varepsilon''(0) > 0$, and the variation of the bifurcation diagrams near $\varepsilon = 0$ is shown in Fig. 1 (for more detailed proof, see [24, Theorem 2.5 and Section 6.1]).

Next we consider

$$\begin{cases} \Delta u + \lambda\big(u - u^2\big) - \lambda\varepsilon^2 g(x) = 0, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.21}$$

where $g(x) \in C^1(\overline{\Omega})$, and g satisfies

$$\int_\Omega g(x)\phi_1(x)\,dx > 0. \tag{5.22}$$

We define

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda\big(u - u^2\big) - \lambda\varepsilon^2 g(x). \tag{5.23}$$

Again (F1), (F2′), (F3) and (F4) are satisfied at $(\varepsilon, \lambda, u) = (0, \lambda_1, 0)$. But $F_\varepsilon(0, \lambda_1, 0) = 0$ hence (F5′) is satisfied. Moreover we also have $F_{\varepsilon u}(0, \lambda_1, 0)[\phi_1] = 0$, $F_{\varepsilon\varepsilon}(0, \lambda_1, 0) = -2\lambda_1 g(x)$, thus

$$m \equiv \langle l, F_{\varepsilon\varepsilon}\rangle \cdot \big\langle l, F_{uu}[\phi_1, \phi_1]\big\rangle = 4\lambda_1^2 \int_\Omega g(x)\phi_1(x)\,dx \cdot \int_\Omega \phi_1^3(x)\,dx > 0, \tag{5.24}$$

from (5.22). Hence the conclusions of Corollary 3.6 hold and (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$m\mu^2 - q^2\eta^2 = 0, \tag{5.25}$$

where $q = \langle l, F_{\lambda u}(0, \lambda_1, 0)[\phi_1]\rangle = \int_\Omega \phi_1^2\,dx > 0$. We can choose $(\mu_1, \eta_1) = (q, \sqrt{m})$ and $(\mu_2, \eta_2) = (q, -\sqrt{m})$. Hence for $\varepsilon \neq 0$, the transition in Fig. 3 occurs. However, if (5.22) is changed to $\int_\Omega g(x)\phi_1(x)\,dx < 0$, for $\varepsilon \neq 0$, there is no degenerate solutions nearby (see Fig. 4).

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