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UNIQUENESS AND NONEXISTENCE OF POSITIVE SOLUTIONS TO SEMIPOSITONE PROBLEMS

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Abstract

We consider the uniqueness of the positive solution to a semilinear elliptic equation with Dirichlet boundary condition and the nonlinearity satisfying $f(0) < 0$ and having asymptotic sublinear growth rate. A similar idea is also applied to the nonexistence of a positive solution to a superlinear problem.

1. *Introduction*

We consider the uniqueness of the positive solution to a semilinear elliptic equation

$$-\Delta u = \lambda f(u), \quad x \in D; \qquad u > 0, \quad x \in D; \qquad u(x) = 0, \quad x \in \partial D; \tag{1.1}$$

where D is a smooth bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$, λ is a positive parameter, and $f(u)$ is a smooth function. The nonlinearity $f(u)$ here satisfies $f(0) < 0$; thus some standard comparison principles cannot be applied, and such a nonlinearity is called *semipositone* [**7**]. Semilinear equations with such nolinearities arise from the studies of astrophysics and bioeconomic harvesting problems (see [**20**]). We show in this paper that when f also satisfies a certain sublinear growth rate at infinity, or when f has a zero point at which $f' \leqslant 0$, then the uniqueness of a positive solution for large λ can be proved.

The uniqueness of a positive solution for positive f has been studied in [**11**] (see also [**3**, **10**, **13**, **14**, **24**]). It is proved in [**11**] that if $f(u) > 0$ for $u \in \mathbb{R}^+$ (or $u \in (0, c)$), $f(0) > 0$, or $f(0) = 0$ but $f'(0) > 0$, and f satisfies a certain sublinear growth rate at infinity, or $f(c) = 0$ such that $f' \leqslant 0$ near c, then (1.1) has a unique positive solution when λ is large. Our result here covers the case of $f(0) < 0$ with similar asymptotic behavior. Notice that when $f(0) = f'(0) = 0$, equation (1.1) can have more than one positive solution for all large λ.

The uniqueness of a positive solution for semipositone f and large λ is proved in [**7**], in which f is assumed to be increasing, concave and sublinear, and the domain D is assumed to have convex outer boundary. Our main result improves on that result by not assuming the concavity of f and convexity of D, and we need f to be increasing only for large u (see Theorem 1.1). On the other hand, it is known that when λ is large, (1.1) has a unique positive solution whose maximum value is near a zero point of f where $f' \leqslant 0$ (see [**3**, **10**, **13**, **24**]), and in certain cases, the number of stable positive solutions is the same as the number of zero points

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of f where $f' \leqslant 0$ (see [**13**]). Hence our result also complements these results by showing that there is no other positive solution for large λ. The existence of a positive solution under these conditions is well known. Indeed, one can show that there exists a maximal positive solution for each $\lambda > \lambda_* > 0$ (see [**20**, **23**]).

Our main results in this paper are for general bounded smooth domains. If the domain is a ball in $\mathbb{R}^n$ or an interval in $\mathbb{R}^1$, much better results can be proved (see, for example, [**9**, **21**, **22**]). In particular, it has been proved in [**9**, **22**] that when D is a ball, (1.1) has:

– exactly two positive solutions for $\lambda \in (\lambda_*, \lambda^*]$,
– exactly one positive solution for $\lambda = \lambda_*$ and $\lambda > \lambda^*$, and
– no positive solution for $\lambda < \lambda_*$,

given that f is concave, $f(0) < 0$ and an integral condition on f is satisfied. Hence, in general, the uniqueness of the positive solution does not hold for small λ.

The main ingredients of the proof include the moving plane method, boundary blowup of a solution sequence when $\lambda \to \infty$, information on the half-space solutions and sweeping principles. While all these tools have appeared in earlier works on uniqueness problems, our approach uses some new ideas. For example, we use the moving plane method only at 'good' points on the boundary, where the boundary is convex. Another idea is to consider the monotonicity of a solution not only in the normal direction but also the direction close to normal, which gives extra information on the profile of the limit solution on the half space. These ideas may be useful in tackling other problems.

Throughout the paper, we assume that

$$\begin{aligned}&D \text{ is a bounded connected domain in } \mathbb{R}^n \text{ with } n \geqslant 2,\\ &\text{the boundary } \partial D \text{ is of } C^2 \text{ class, and}\\ &\partial D \text{ satisfies a uniform interior ball condition.}\end{aligned} \tag{1.2}$$

Note that the uniform interior ball condition is satisfied if ∂D is of C^3 class. For our first result, we assume that $f(u)$ satisfies the following conditions.

(f1) $f \in C^{1,\alpha}(\mathbb{R}^+)$ for some $\alpha \in (0,1)$, $f(0) < 0$.
(f2) There exists $b > 0$ such that $f(u) > 0$ for all $u > b$, and $F(u) < 0$ for all $u \in (0, b]$ where $F(u) = \int_0^u f(t)\,dt$.
(f3) $\lim_{u\to\infty} f(u)/u = 0$.
(f4) $\liminf_{u\to\infty} [f(u) - uf'(u)] > 0$.
(f5) There exists $K > 0$ such that $f'(u) \geqslant 0$ for all $u \geqslant K$.

Some examples of nonlinearities f satisfying (f1)–(f5) are $f(u) = \ln(u + 1/2)$ and $f(u) = 2 - 3e^{-u}$, and these functions are concave. But we notice that the conditions (f3)–(f5) are imposed only on large u, and there is no condition imposed on f for $u \leqslant b$ except for (f2), which is a well-known necessary condition so that (1.1) has a positive solution u only with $\max_{x\in D} u(x) > b$ (see [**16**]). Under these conditions, we have the following uniqueness result.

THEOREM 1.1. *Suppose that D satisfies condition* (1.2). *If $f(u)$ satisfies conditions* (f1)–(f5), *then there exists $\lambda_* > 0$ such that when $\lambda > \lambda_*$, equation* (1.1) *has a unique positive solution.*

Another type of nonlinearity that we consider is that f satisfies (f1), and the following conditions hold.

(f6) There exists $c > b > 0$ such that $f(b) = f(c) = 0$, $f(u) > 0$ for $u \in (b, c)$, and $f(u) < 0$ for $u \in (c, \infty)$.
(f7) Let $F(u) = \int_0^u f(t)\,dt$; then $F(u) < 0$ for $u \in (0, b]$ and $F(c) > 0$.
(f8) $f'(u) \leqslant 0$ for $u \in (c - \delta, c)$ with some $\delta > 0$.

THEOREM 1.2. *Suppose that* D *satisfies condition* (1.2). *If* $f(u)$ *satisfies conditions* (f1) *and* (f6)–(f8), *then there exists* $\lambda_* > 0$ *such that when* $\lambda > \lambda_*$, *equation* (1.1) *has a unique positive solution.*

Under essentially the same conditions as in Theorem 1.2 (except that $f(0)$ may not be 0), it is proved in [**10**, **24**] that (1.1) has a unique positive solution whose maximum value is in $(c - \delta, c)$ when λ is large. Thus our result shows that there is no other positive solution for large λ if $f(0) < 0$. An example of f satisfying the conditions in Theorem 1.2 is $f(u) = u - u^p - k$ for some $k > 0$ and $p > 1$, and k is chosen so that (f7) is satisfied. This nonlinearity appears in a diffusive logistic model with a constant harvesting, and the existence of positive solutions has been studied in [**20**, **23**].

Using the same techniques which we develop for sublinear problems, we can also prove a non-existence result for a superlinear problem.

THEOREM 1.3. *Suppose that* D *satisfies condition* (1.2). *If* $f(u)$ *satisfies conditions* (f1), (f2) *and*
(f9) $\liminf_{u\to\infty} f(u)/u > 0$,
then there exists $\lambda_* > 0$ *such that when* $\lambda > \lambda_*$, *equation* (1.1) *has no non-negative solution.*

Notice that we allow the limit in (f9) to be infinity. This result seems to be new, even for the domain being a ball. A typical example here is $f(u) = u^p - k$ for some $p > 1$ and $k > 0$. In [**6**], the nonexistence is proved for a ball domain under more restrictive conditions on f, but including $f(u) = u^p - k$. The existence of a positive solution for $f(u)$ being similar to $u^p - k$ when λ is small has been shown in [**1**, **2**, **8**].

The proofs of these theorems are given in Section 2. We use the notation $B(y; R)$ for the ball $\{x \in \mathbb{R}^n : |x - y| < R\}$, and $x = (x', x_n)$ is the coordinate of $x \in \mathbb{R}^n$ where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

2. *Proof of the main results*

First we recall some maximum principles of elliptic equations. Let

$$Lw = \Delta w + c(x)w \qquad \text{for any } w \in W^{2,p}_{\mathrm{loc}}(D) \cap C(\overline{D}) \text{ and for any } p > n,$$

where $c(x) \in L^\infty(D)$. We say that the *maximum principle* holds for L in D if $Lu \leqslant 0$ in D, $u \geqslant 0$ on ∂D, implies that $u \geqslant 0$ in D. (The strong maximum principle implies that either $u > 0$ in D or $u \equiv 0$ in D.) The following well-known maximum principle on a narrow domain was first observed by Bakelman [**4**].

LEMMA 2.1. *There exists* $\delta = (C \cdot \mathrm{diam}(D) \cdot \|c^+\|_\infty)^{-n}$, *where* C *is a constant depending only on* n *and* $c^+(x) = \max\{c(x), 0\}$, *such that the maximum principle holds for* L *in any subregion* D' *of* D *with* $|D'| < \delta$.

We use the moving plane method to show that for any solution (λ, u) of (1.1), u is increasing, moving from the boundary toward the interior near a boundary point x_0 where D touches a hyperplane. Here we recall the standard setup for the moving plane method. Since D is a bounded domain, it must lie on one side of some hyperplane. Without loss of generality, we assume that $D \subset \{(x', x_n) : x_n > k_0\}$ for some $k_0 \in \mathbb{R}$. Let T_k be the hyperplane defined by $T_k = \{(x', x_n) : x_n = k\}$, and let Σ_k be the half space $H_k = \{(x', x_n) : x_n > k\}$ for any $k \in \mathbb{R}$. We assume that $T_{k_0} \cap \partial D \neq \emptyset$. Define $D_k = H_k \cap D$. We assume that $u(x)$ is extended to $\mathbb{R}^n$ by being zero outside of D.

Lemma 2.2. *Let f be a locally Lipschitz continuous function. Then there exists $k_1 > k_0$ such that for any positive solution (λ, u) of (1.1), we have*

$$u(x', 2k_1 - x_n) > u(x', x_n), \qquad \frac{\partial u}{\partial x_n}(x', x_n) > 0, \qquad x = (x', x_n) \in D_{k_1}. \quad (2.1)$$

Proof. For $x = (x', x_n) \in \mathbb{R}^n$, we define

$$v_k(x', x_n) = u(x', 2k - x_n) \qquad \text{and} \qquad w_k(x', x_n) = v_k(x', x_n) - u(x', x_n). \quad (2.2)$$

Then for $k \leqslant k_0$, we have $w_k(x', x_n) \geqslant 0$. We claim that there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that for $k \in (k_0, k_0 + \varepsilon)$, we have $w_k(x) > 0$ for $x \in D_k$. Indeed, let D_k' be the reflection of D_k with respect to T_k; then for small $\varepsilon > 0$, we have $D_k' \subset D$ for $(k_0, k_0 + \varepsilon)$. Note that w_k satisfies

$$-\Delta w = \lambda c_k(x', x_n) w, \qquad x \in D_k, \quad (2.3)$$

where

$$c_k(x', x_n) = \frac{f(v_k(x', x_n)) - f(u(x', x_n))}{v_k(x', x_n) - u(x', x_n)}.$$

Then $c_k \in L^\infty(D)$ since f is Lipschitz continuous, and $\|c_k\|_{L^\infty(D)}$ is independent of k. Thus there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that when $k \in (k_0, k_0 + \varepsilon)$, we have $|D_k| < \delta$, where D_k is as defined in Lemma 2.1. Then for such k, $w_k(x) > 0$ for $x \in D_k$ since $w_k(x) \geqslant 0$ for $x \in \partial D_k$ and using Lemma 2.1. This also implies that $u(x) > 0$ for $x \in D_k$ when $k \in (k_0, k_0 + \varepsilon)$. Therefore the moving plane process can be initiated. Moreover, since w_k satisfies (2.3), $w_k(x', k) = 0$ and $w_k > 0$ in D_k, then from the Hopf lemma we have

$$-2\frac{\partial u}{\partial x_n}(x', k) = \frac{\partial w_k}{\partial x_n}(x', k) < 0, \qquad \text{if } (x', k) \in \partial D_k. \quad (2.4)$$

Thus we have proved that when $k \in (k_0, k_0 + \varepsilon)$, any positive solution u of equation (1.1) satisfies $w_k(x) > 0$ and $\partial u / \partial x_n(x) > 0$ for $x \in D_k$.

Next we define

$$k_1 = \sup\{k > k_0 : w_k(x) > 0,\ \partial u/\partial x_n(x) > 0,\ \text{for } x \in D_a,\ a \in (k_0, k)\}.$$

Then at $k = k_1$, one of the following happens:

(i) $w_{k_1}(x) \geqslant 0$ for $x \in \overline{D_{k_1}}$, and $w_{k_1}(x) = 0$ at some $x \in D_k$;
(ii) $w_{k_1}(x) > 0$ for $x \in D_{k_1}$, and $\partial u/\partial x_n(x) = 0$ at some $x \in T_{k_1} \cap D$;
(iii) there exists $x \in \partial D \cap \Sigma_{k_1}$ such that the reflection $\overline{x}$ of x about T_k is also on the boundary; or
(iv) T_{k_1} is orthogonal to ∂D.

However, the first case cannot happen because of a maximum principle for $\min u(x) = 0$ (see [**19**]), and the second case cannot occur either, from a boundary point maximum principle (see [**18**]). Hence at $k = k_1$, either case (iii) or case (iv) above can occur. Note that either case depends only on the geometry of D, but not on λ. Hence for any $\lambda > 0$, there exists $k_1 > 0$ such that $w_{k_1}(x) > 0$ and $\partial u/\partial x_n(x) > 0$, for $x \in D_{k_1}$. □

In Lemma 2.1, we show that, for $x \in D_{k_1}$, u is monotonic increasing along the inward normal direction $\tau_0 = (0, 0, \ldots, 1)$ at a 'touching point' $x_0 \in \partial D$. Next we show that the monotonicity of u also holds along any direction close to τ_0 for x in a smaller cap D_{k_3}.

Lemma 2.3. *Suppose that f is a locally Lipschitz continuous function. Then there exist a neighborhood Σ of $\tau_0 = (0, 0, \ldots, 1)$ in S^{n-1} and $k_3 \in (k_0, k_1)$ such that for any positive solution (λ, u) of equation* (1.1) *and any $\tau \in \Sigma$, we have $\nabla u \cdot \tau > 0$ for $x \in D_{k_3}$.*

Proof. Since the boundary ∂D is assumed to be C^2, the normal vector field and the corresponding tangent bundle of ∂D are also C^2. In particular, for a neighborhood Σ_1 of $\tau_0 = (0, 0, \ldots, 1)$ in S^{n-1}, if $\tau \in \Sigma_1$, then the moving plane procedure in the proof of Lemma 2.2 can also be carried out for the direction τ from x_τ near x_0. Moreover, as in the proof of Lemma 2.2, we can move the hyperplane orthogonal to τ into D until the case (iii) or case (iv) listed above occurs. This defines a maximal cap D_{τ,k_1} for each $\tau \in \Sigma_1$ and such D_{τ,k_1} depends only on the geometric properties of D, but not on u, f or λ. For $x \in D_{\tau,k}$, we have $\nabla u \cdot \tau > 0$ following the proof of Lemma 2.2.

We claim that the set $\bigcap_{\tau\in\Sigma} D_{\tau,k_1}$ contains a neighborhood of x_0 for some $\Sigma \subset \Sigma_1$. We choose $k_2 \in (k_0, k_1)$, where k_0 and k_1 are as defined above. Then $D'_{k_2} \subset D$ and T_{k_2} is not orthogonal to ∂D. We fix $x_* \in T_{k_2} \cap D$; then for $\tau \in \Sigma_1$, there exists a unique hyperplane T_{τ,k_2} which contains x_* and is orthogonal to τ. From the smoothness of ∂D, there exists $\Sigma \subset \Sigma_1$ such that for any $\tau \in \Sigma$, T_{τ,k_2} is not orthogonal to ∂D, from the continuity of T_{τ,k_2} on τ. Indeed, for such a choice of k_2, we have $\langle \tau_0, N_x\rangle > 0$, where N_x is the inward normal direction of ∂D at x, and $x \in \partial D \cap \partial D_{k_2}$. Hence by choosing $\Sigma \subset \Sigma_1$, we also have $\langle \tau, N_x\rangle > 0$, where $x \in T_{\tau,k_2} \cap \partial D$. On the other hand, since $\langle \tau, N_x\rangle > 0$, we also conclude that the reflection of D_{τ,k_2} (the cap cut out by T_{τ,k_2} in D) is still inside D. Thus $D_{\tau,k_2} \subset D_{\tau,k_1}$, where D_{τ,k_1} is defined as the maximal cap above. Let $x = (x', x_n) \in T_{\tau,k_2} \cap \partial D$, and let $\tau \in \Sigma$. Then $k_3 \equiv \inf x_n > k_0$, from the continuity of T_{τ,k_2} on τ. Thus

$$\bigcap_{\tau\in\Sigma} D_{\tau,k_1} \supset \bigcap_{\tau\in\Sigma} D_{\tau,k_2} \supset D_{k_3},$$

and the claim is proved. □

Now we use a boundary blowup argument at $x_0 \in T_{k_0} \cap \partial D$. We assume that (λ_m, u_m) is a sequence of solutions to equation (1.1) with $\lambda = \lambda_m$ and $\lambda_m \to \infty$. Let O be a neighborhood of x_0 such that $O \cap D \subset D_{k_3}$, where k_3 is as defined in Lemma 2.3. We choose a C^2 local coordinate system

$$\Psi(x) = (\Psi'(x), \Psi_n(x)) : O \longrightarrow \mathbb{R}^n$$

so that $\Psi(x_0) = 0$, $x \in O \cap D$ if and only if $\Psi_n(x) > 0$.

Define

$$v_m(y) = u_m(\Psi^{-1}(\lambda_m^{-1/2} y))$$

for

$$y \in B^+(\lambda_m^{1/2} R) = \{y = (y', y_n) \in \mathbb{R}^n : y_n > 0,\ |y| < \lambda^{1/2} R\},$$

where R is chosen so that $B^+(R) \subset \Psi(O)$. Then under this transformation, v_m satisfies the following equation:

$$\begin{cases} \Delta v_m + \lambda_m^{-1/2}\left(\displaystyle\sum_{i,j=1}^n a_{ij,m}\frac{\partial^2 v_m}{\partial y_i \partial y_j} + \sum_{i=1}^n b_{i,m}\frac{\partial v_m}{\partial y_i}\right) + f(v_m) = 0, & y \in B^+(\lambda_m^{1/2} R), \\ v_k = 0, & y \in \partial B^+(\lambda_m^{1/2} R), \\ y_n = 0. \end{cases}$$

From well-known arguments, there exists a subsequence (which we still denote by $\{v_m\}$) such that $v_m \to v$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n_+)$, and $v(y)$ is a nonnegative solution of

$$\Delta v + f(v) = 0, \quad y \in \mathbb{R}^n_+; \qquad v(y) = 0, \quad y \in \partial\mathbb{R}^n_+. \tag{2.5}$$

For any $y \in \mathbb{R}^n_+$, we have

$$\begin{aligned} \frac{\partial v_m}{\partial y_n}(y) &= \lambda_m^{-1/2}\sum_{i=1}^n \frac{\partial u_m}{\partial x_i}(\Psi^{-1}(\lambda_m^{-1/2} y)) \cdot \frac{\partial \Psi_i^{-1}}{\partial y_n}(\lambda_m^{-1/2} y) \\ &\to \lambda_m^{-1/2}\frac{\partial u_m}{\partial x_n}(\Psi^{-1}(\lambda_m^{-1/2} y)) + o(\lambda_m^{-1/2}), \end{aligned} \tag{2.6}$$

since

$$\frac{\partial \Psi_i^{-1}}{\partial y_n}(\lambda_m^{-1/2} y) \to \delta_{in} \qquad \text{for } 1 \leqslant i \leqslant n, \text{ as } m \to \infty.$$

In particular, (2.6) implies that $\partial v(y)/\partial y_n \geqslant 0$ for any $y \in \mathbb{R}^n_+$, and thus $v(y) \geqslant 0$ for any $y \in \mathbb{R}^n_+$. From the strong maximum principle, $\partial v/\partial y_n \equiv 0$ or $\partial v(y)/\partial y_n > 0$ for all $y \in \mathbb{R}^n_+$. In the former case, $v \equiv 0$, while $v > 0$ in the latter one. Similarly, if $v > 0$, for any $\tau \in \Sigma$ (defined in Lemma 2.3), we have $\nabla v(y) \cdot \tau > 0$ for any $y \in \mathbb{R}^n_+$ following Lemma 2.3. Indeed, for such a blow-up limit v, we have the following result.

Lemma 2.4. *Suppose that v is a bounded positive solution of equation* (2.5)*, and there exists an open subset Σ of S^{n-1} which contains $\tau_0 = (0, \dots, 0, 1)$ such that $\nabla v(y) \cdot \tau > 0$ for any $y \in \mathbb{R}^n_+$. Then $v(y', y_n) \equiv v_1(y_n)$, where v_1 is a bounded solution of*

$$v'' + f(v) = 0, \quad z \in \mathbb{R}^+; \qquad v(0) = 0; \qquad v'(z) > 0, \quad z \in \mathbb{R}^+, \tag{2.7}$$

and $\lim_{y_n \to \infty} v_1(y_n) = c$, where c satisfies $f(c) = 0$ and $f'(c) \leqslant 0$.

Proof. We first prove that $w(y') = \lim_{y_n \to \infty} v(y', y_n)$ exists, and that w is a positive solution of

$$\Delta w + f(w) = 0, \qquad y' \in \mathbb{R}^{n-1}. \tag{2.8}$$

In fact, v is bounded and increasing with respect to y_n; thus w exists and w is positive. From the arguments in [**15**, p. 8], $v(y', y_n)$ converges to $w(y')$ locally uniformly for y' when $y_n \to \infty$, and w is a classical solution of (2.8).

Let $V(y', y_n) = v(y', y_n) - w(y')$. Then V satisfies

$$-\Delta V = c(y)V, \quad y \in \mathbb{R}^n_+; \qquad V(y) = 0, \quad y \in \partial\mathbb{R}^n_+; \tag{2.9}$$

where $c(y)$ is uniformly bounded for $y \in \mathbb{R}^n_+$ since f is Lipschitz continuous. Since V is the solution of equation (2.9), and $V(y', y_n) \to 0$ locally uniformly for y' when $y_n \to \infty$, then $|\nabla V| \to 0$ for y' in any compact subset of $\mathbb{R}^{n-1}$ as $y_n \to \infty$, from the local sup estimates for the gradient in [**19**, Theorem 8.32]. For any $\tau = (\tau^1, \tau^2, \dots, \tau^{n-1}, \tau^n) \in \Sigma$, define $\tau' = (\tau^1, \tau^2, \dots, \tau^{n-1})$. Then

$$\nabla v \cdot \tau - \nabla w \cdot \tau' = \nabla V \cdot \tau \to 0, \qquad y_n \to \infty, \tag{2.10}$$

uniformly for y' in any compact subset of $\mathbb{R}^{n-1}$. Since $\nabla v \cdot \tau > 0$ for any $y \in \mathbb{R}^n_+$, then $\nabla w \cdot \tau' \geqslant 0$ for any $y' \in \mathbb{R}^{n-1}$. Since Σ is an open neighborhood of τ_0, then the projection of Σ under the mapping $\tau \longmapsto \tau'$ contains an open neighborhood of the origin of $\mathbb{R}^{n-1}$. This implies that $\nabla w \cdot \tau' \geqslant 0$ for any direction τ', and hence $w \equiv c$ for a constant c. Since w is a solution of equation (2.8), it is necessary that $f(c) = 0$. Now from [**5**, Theorem 1.4], we see that $v \equiv v_1(y_n)$. Therefore v_1 must be a solution of equation (2.7), and it is easy to observe that equation (2.7) has no solution if $f'(c) > 0$, so $f'(c) \leqslant 0$. □

REMARK 2.1. We compare Lemma 2.4 with some earlier results of half-space solutions proved in [**12**] and [**5**]. In [**12**], it is proved that if $f(0) \geqslant 0$, then any positive solution v of equation (2.5) is monotonic; that is, $\partial v/\partial y_n > 0$. The question is whether v is symmetric; that is, whether $v \equiv v_1(y_n)$. It is proved in [**5**] that this is true if u is bounded and $f(\sup u) \leqslant 0$. So here we prove the symmetry of v without the condition that $f(\sup u) \leqslant 0$, but with a stronger monotonicity condition (monotonicity for an open set of directions). It is not clear whether the weaker monotonicity ($\partial v/\partial y_n > 0$) implies the stronger one ($\nabla v \cdot \tau > 0$ for $\tau \in \Sigma$ which is an open subset of S^{n-1} containing $\tau_0 = (0, \dots, 0, 1)$). Notice that this is not true if u is unbounded. For example, $v(x_1, x_2) = e^{x_1} x_2$ is an unbounded solution of $\Delta v - v = 0$ in $\mathbb{R}^2_+$ and $v = 0$ on $\partial\mathbb{R}^2_+$, and v is increasing in x_2, but not in any direction close to $\tau_0 = (0, 1)$. However, we conjecture that it is true for bounded v, and from Lemma 2.4 this conjecture would imply the earlier conjecture in [**5**] on the symmetry of v without the extra condition that $f(\sup u) \leqslant 0$. We also mention that the monotonicity of a positive solution along almost normal directions was first proved in [**17**].

From the analysis above, we obtain an important property for the solution u of equation (1.1) when conditions (f1) and (f2) are satisfied.

COROLLARY 2.5. *Suppose that f satisfies conditions* (f1) *and* (f2). *Suppose that v is a nonnegative solution of equation* (2.5), *and that there exists an open subset Σ of S^{n-1} which contains $\tau_0 = (0, \dots, 0, 1)$ such that $\nabla v(y) \cdot \tau > 0$ for any $y \in \mathbb{R}^n_+$. Then v is unbounded. Moreover, for any $M > b$, there exists a ball $B(y_M; R_M) \subset \mathbb{R}^n_+$ such that $v(y) > M$ for $y \in B(y_M; R_M)$.*

Proof. First, v must be positive, for otherwise $v \equiv 0$ but $f(0) < 0$, from (f1). Suppose that v is bounded. From Lemma 2.4, v must be one-dimensional, $v \equiv v_1(y_n)$, and v_1 satisfies (2.7). Let $c = \lim_{y_n \to \infty} v_1(y_n)$. We multiply (2.7) by v_1', and integrate it on $\mathbb{R}^+$; then we obtain

$$-\tfrac{1}{2}[v_1'(0)]^2 + F(c) - F(0) = 0. \tag{2.11}$$

Hence, from (f2), $c > b$. Since $f(u) > 0$ for all $u > b$, this is a contradiction to $f(c) = 0$. Hence v is unbounded. For any $M > b$, there exists $y_M \in \mathbb{R}^n_+$ such that $v(y_M) = M$. Since $\partial v(y)/\partial y_n > 0$, the level set $S_M = \{y \in \mathbb{R}^n_+ : v(y) = M\}$ near y_M is a C^2 surface which separates a neighborhood of y_M into two disjoint sets $\{v(y) > M\}$ and $\{v(y) < M\}$. Hence there exists a ball $B(y_M; R_M) \subset \{v(y) > M\}$ tangent to S_M at y_M. □

To complete the proof of Theorem 1.1, we construct a family of sub-solutions of equation (1.1) when conditions (f1)–(f3) are satisfied. We extend f to $\mathbb{R}$ so that $f(u) < 0$ for $u \in (-a, 0)$, $f(-a) = 0$, $f'(-a) < 0$, and $f \in C^{1,\alpha}(\mathbb{R})$. From (f2), there exists $d > b$ such that

$$F(u) = \int_0^u f(t)\,dt > 0 \qquad \text{for } u > d.$$

We consider the boundary value problem on $\mathbb{R}^n$:

$$\begin{cases} -\Delta w = f(w), & x \in \mathbb{R}^n; \\ \lim_{|x|\to\infty} w(x) = -a. \end{cases} \tag{2.12}$$

Since f satisfies condition (f3), we see from the result of [**22**] that equation (2.12) has a radially symmetric solution $w(x) = w(|x|)$ satisfying $w'(r) < 0$ for $r > 0$. Then there exists $m \in (0, \infty)$ such that $w(m) = 0$. It is easy to verify the following result (see [**10**]).

Lemma 2.6. *Let $w(x)$ be defined as above. Then for any $y \in D$, and $\lambda > \mu m^2[d(y, \partial D)^{-2}]$, we see that*

$$W(x; \lambda, y) = w(\lambda^{1/2}(x - y)), \qquad x \in D, \tag{2.13}$$

is a subsolution of equation (1.1).

Now we show that any positive solution of equation (1.1) must be above one of subsolutions defined in Lemma 2.6.

Lemma 2.7. *There exists $\lambda_* > 0$ such that if $u(x)$ is a positive solution of equation* (1.1) *with $\lambda > \lambda_*$, then there exists $y \in D$ such that $u(x) > W(x; \lambda, y)$ for $x \in D$.*

Proof. We use the same blow-up argument as above, at a boundary point $x_0 \in T_{k_0} \cap \partial D$. Then the limit function v is a positive solution of equation (2.5) satisfying $\partial v/\partial y_n > 0$. From Corollary 2.5, $v(y)$ is unbounded. Let $M = 3w(0) > 0$, where w is the positive radial solution of (2.12). Then from Corollary 2.5, there exists a ball $B(y_M; R_M) \subset \mathbb{R}^n_+$ such that $v(y) > M$ for $y \in B(y_M; R_M)$. When $\lambda > \lambda_a$ for some $\lambda_a > 0$, we can assume that $B(y_M; R_M) \subset B^+(\lambda^{1/2}R)$, where R is chosen so that $B^+(R) \subset \Psi(O)$. Hence there is a ball $B(x_\lambda; \lambda^{-1/2}R_0)$ contained in $\Psi(\lambda^{-1/2}B(y_M; R_M))$. On the other hand, $v_m \to v$ uniformly for $y \in B(y_M; R_M)$, and hence there exists $\lambda_* > \lambda_a$ such that $v_m(y) \geqslant 2w(0)$ when $\lambda_m > \lambda_*$, which implies that $u(x) \geqslant 2w(0) > W(x; \lambda, x_\lambda)$ when $\lambda > \lambda_*$ and $x \in B(x_\lambda; \lambda^{-1/2}R_0)$. □

With Lemma 2.7 and the sweeping principle, we now are able to prove that the solution u is large except near the boundary. Since we assume that D satisfies a uniform interior sphere condition, there exists $\varepsilon > 0$ such that $D = \cup\{B(y;\varepsilon) : y \in D_\varepsilon\}$ where $D_\varepsilon = \{x \in D : d(x, \partial D) > \varepsilon\}$. The following lemma can be proved in the same way as [**24**, Lemma 3.6], if Lemma 2.7 holds.

LEMMA 2.8. *For any positive solution* (λ, u) *of equation* (1.1), *if* $\lambda > m^2\varepsilon^{-2}$, *then there exists* $C_1 > 0$ *such that*

$$u(x) \geqslant \min\{w(0), C\lambda^{1/2}d(x,\partial D)\}, \qquad x \in D. \tag{2.14}$$

Moreover, for any $M > w(0)$ *and* $\lambda > m^2\varepsilon^{-2}$, *there exists* $C_M > 0$ *such that*

$$u(x) > M, \qquad \text{if } d(x,\partial D) > \lambda^{-1/2}[C_1^{-1}w(0) + C_M]. \tag{2.15}$$

Proof. The proof of equation (2.14) is exactly same as that of [**24**, Lemma 3.6]. For equation (2.15), we apply [**11**, Proposition 1] or [**10**, Lemma A.3], and it is easy to check that equation (2.15) holds if $C_M = \tau_M^{-1/2}\lambda_1^{1/2}$. Here, τ_M is the constant such that $f(u) \geqslant \tau_M(u-b)$ for $u \in [b, M]$, and λ_1 is the principal eigenvalue of

$$\Delta\phi + \lambda\phi = 0, \quad y \in B(0;1); \qquad \phi(y) = 0, \quad y \in \partial B(0;1). \tag{2.16}$$

The proof of the lemma is complete. □

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that (λ, u) is a positive solution of (1.1). We consider the linearized eigenvalue problem:

$$-\Delta\psi = \lambda f'(u)\psi + \mu\psi, \quad x \in D; \qquad \psi(x) = 0, \quad x \in \partial D. \tag{2.17}$$

Suppose that (μ_1, ψ_1) is the principal eigen-pair, and that $\psi_1 > 0$ in D. We claim that $\mu_1 > 0$ if λ is sufficiently large. If the claim holds, then the uniqueness follows from a well-known argument using Leray–Schauder degree theory; see for example [**10**, p. 114].

So it remains to prove the claim. Suppose it is not true; then there exists a sequence of positive solutions (λ^n, u^n) such that $\mu^n = \mu_1^n \leqslant 0$. Hence we have

$$-\Delta\psi^n \leqslant \lambda^n f'(u^n)\psi^n, \quad x \in D; \qquad \psi^n(x) = 0, \quad x \in \partial D, \tag{2.18}$$

where $\psi^n = \psi_1^n$ is the principal eigenfunction. We can choose ψ^n so that $\psi^n > 0$ in D and $\max_{x\in\overline{D}} \psi^n(x)/u^n(x) = 1$ since $u^n(x) \geqslant C(\lambda^n)^{1/2}d(x,\partial D)$ for any $x \in D$, by Lemma 2.8. From (f4) and (f5), we can assume that $f(u) - uf'(u) \geqslant k_1 > 0$ and $f'(u) \geqslant 0$ for $u \geqslant M$ for some $k_1, M > 0$. Define

$$D_1 = \{x \in D : u^n(x) > M\}, \qquad D_2 = D\backslash D_1. \tag{2.19}$$

For $x \in D_1$, we have

$$\lambda^n f'(u^n)\psi^n \leqslant \lambda^n f'(u^n)u^n \leqslant \lambda^n[f(u^n) - k_1], \tag{2.20}$$

and for $x \in D_2$,

$$\lambda^n f'(u^n)\psi^n \leqslant \lambda^n|f'(u^n)u^n| \leqslant \lambda^n C_2, \tag{2.21}$$

where $C_2 = \max_{u\leqslant M}|f'(u)u|$. From equations (2.18), (2.20) and (2.21), we obtain

$$-\Delta\psi^n \leqslant \lambda^n[f(u^n) - k_1]\chi_{D_1} + \lambda^n C_2\chi_{D_2}, \qquad x \in D. \tag{2.22}$$

Combining equations (1.1) and (2.22), we have

$$-\Delta(u^n - \psi^n) \geqslant \lambda^n k_1 \chi_{D_1} - \lambda^n (C_2 + C_3)\chi_{D_2}, \tag{2.23}$$

where $C_3 = \max_{u \leqslant M} |f(u)|$. Let ϕ^n be the unique solution of

$$\Delta\phi + [k_1\chi_{D_1} - (C_2 + C_3)\chi_{D_2}] = 0, \quad x \in D; \qquad \phi(x) = 0, \quad x \in \partial D. \tag{2.24}$$

Notice that ϕ^n depends on n since D_1 and D_2 depend on λ^n. From Lemma 2.8,

$$D_1 \supset \{x \in D : d(x, \partial D) \geqslant (\lambda^n)^{1/2}[C_1^{-1}w(0) + C_M]\},$$

where C_1 and C_M are as defined in Lemma 2.8. In particular, the function $k_1\chi_{D_1} - (C_2 + C_3)\chi_{D_2}$ approaches $k_1\chi_D$ as $n \to \infty$. Hence the limit of ϕ^n as $\lambda \to \infty$ is the unique solution ϕ of

$$\Delta\phi + k_1 = 0, \quad x \in D; \qquad \phi(x) = 0, \quad x \in \partial D. \tag{2.25}$$

Since $\phi > 0$ in D and $\partial\phi/\partial n < 0$ on ∂D, for large enough λ, we have $\phi^n > 0$ in D and $\partial\phi^n/\partial n < 0$ on ∂D. Hence there exists $\varepsilon^n > 0$ such that $\phi^n \geqslant \varepsilon^n u^n$. From equation (2.23) and the fact that $(-\Delta)^{-1}$ is an order-preserving operator on $L^p(D)$, we have

$$u^n - \psi^n \geqslant \phi^n \geqslant \varepsilon^n u^n, \tag{2.26}$$

which implies that $(1 - \varepsilon^n)u^n \geqslant \psi^n$, and that is a contradiction of the assumption that $\max_{x \in \overline{D}} \psi^n(x)/u^n(x) = 1$. Thus the claim has been proved. □

Now we indicate how to modify the above proof to prove Theorem 1.2. We claim that for any $\delta_a > 0$, there exists $\lambda_a > 0$ such that if u is a solution of equation (1.1) with $\lambda > \lambda_a$, then $\max_{x \in \overline{D}} u(x) > c - \delta_a$. If we assume that this claim holds, then Theorem 1.2 can be proved by using [**24**, Theorem 1.6], in which it was proved that there exists $\delta_a > 0$ such that for any large λ, there is exactly one solution u_λ of equation (1.1) such that $\max u_\lambda \in (c - \delta_a, c)$. To prove this claim, we use the same moving plane and blowup argument as above, and the limit v of the blowup sequence again satisfies equation (2.5). Since v is bounded, we know from Lemma 2.4 that $v(y', y_n) \equiv v_1(y_n)$ and $v(y_n)$ is a solution of (2.7). From a simple observation of the phase portrait of $v' = w$, $w' = -f(v)$, we see that equation (2.7) has a unique solution which satisfies

$$v'(0) = \left(2\int_0^c f(s)\,ds\right)^{1/2},$$

and $\lim_{z \to \infty} v(z) = c$. Since for large z, $v(z) > c - \delta_a$, and $v_m \to v$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n_+)$, there exists $\widetilde{x}_m \in D$ such that $u_m(\widetilde{x}_m) > c - \delta_a$.

Finally, we show that the methods used above can also be used to prove Theorem 1.3. Suppose that there is a sequence of positive solutions (λ^n, u^n) such that $\lambda^n \to \infty$ as $n \to \infty$. Again, we use the moving plane method and the boundary blowup argument. Then the limit v must be unbounded, by Corollary 2.5, which implies that $\|u^n\|_\infty \to \infty$. Indeed, we can show that u^n is large except near the boundary, following the proof of Lemmas 2.6–2.8, since we can define a function $f_1(u)$ satisfying (f1)–(f3) and $f(u) > f_1(u)$. Thus subsolutions can be constructed from the solutions of $-\Delta w = f_1(w)$. In particular, there exists a ball $B \subset D$ where u^n is large for all large n. From the assumption (f9), we can assume that $f(u) > pu$

when $u > M$ for some $p, M > 0$. We define T_n to be the connected component of $\{x \in D : u^n(x) > M\}$ which contains B. Then the principal eigenvalue $\lambda_1(T_n)$ of

$$-\Delta\phi = \lambda\phi, \quad x \in T_n; \qquad \phi(x) = 0, \quad x \in \partial T_n; \tag{2.27}$$

is bounded above by $\lambda_1(B)$. On the other hand, since $f(u^n) > pu^n$ for $x \in T_n$, we have $-\Delta(u^n - M) = \lambda^n f(u^n) > \lambda^n p(u^n - M)$, and $u^n - M = 0$ on ∂T_n. This implies that $\lambda_1(T_n) > \lambda^n p \to \infty$ as $n \to \infty$, which contradicts $\lambda_1(T_n) \leqslant \lambda_1(B)$. This completes the proof of Theorem 1.3.

REMARK 2.2. The methods of proving Theorem 1.1 can also be used for some related problems. For example, if we replace the assumption that $f(0) < 0$ in Theorem 1.1 by $f(0) > 0$, or $f(0) = 0$ but $f'(0) > 0$, and assume that $b = 0$ in (f2), then the result of Theorem 1.1 remains true. The nonlinearity f in this case is positive, but (f4) is a weaker concavity condition than in previous work [**11**], where $f(u)$ or $u^{-p}f(u)$ (for $p \in (0,1)$) is assumed to tend to a positive constant as $u \to \infty$. However the monotonicity of f is not assumed in [**11**]. Thus neither result covers the other one.

REMARK 2.3. If $f(u) \to C$ as $u \to \infty$ for some positive constant C and $f(0) < 0$, then the uniqueness of the positive solution for D in any dimension can still be proved without the condition on $f'(u)$ in (f5), by using essentially the same argument as in the proof of [**11**, Theorem 1], together with the ideas in this paper.

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