A New Proof of Anti-Maximum Principle Via A Bifurcation Approach † ¶

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Abstract

We use a bifurcation approach to prove an abstract version of anti-maximum principle. The proof is different from previous approaches.

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain ($\partial\Omega$ is of class C^2). Let L denote the differential operator:

$$(1.1) \qquad Lu = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au,$$

where $a_{ij} \in C(\overline{\Omega})$, $a_{ij} = a_{ji}$, and $\sum_{i,j=1}^{n} a_{ij}(x)\xi^i\xi^j > 0$ for $x \in \overline{\Omega}$ and $\xi = (\xi^i) \in \mathbf{R}^n \backslash \{0\}$, and $a_i, a \in L^\infty(\Omega)$. We consider a Dirichlet boundary value problem:

$$(1.2) \qquad Lu - \lambda m u = f, \ x \in \Omega, \ \ u = 0, \ x \in \partial\Omega,$$

where $m \in L^\infty(\Omega)$.

Let $p > n$, and let $X = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, and let $Y = L^p(\Omega)$. Let the operator $A : X \to Y$ be defined by $Au = Lu$. Then it is well-known ([7]) that A has a unique principal eigenvalue $\lambda_1(A)$, which is simple and $Au = \lambda_1(A)mu$ has a strict positive eigenfunction φ_1 such that

$$(1.3) \qquad \varphi_1(x) > 0, \ \ x \in \Omega, \ \ \frac{\partial \varphi_1}{\partial n}(x) < 0, \ \ x \in \partial\Omega.$$

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An anti-maximum principle for (1.2) was proved by Clément and Peletier [3] and Hess [6], which can be stated along with classical maximum principle as follows:

Theorem 1.1. *Let A be the elliptic operator defined above and let $\lambda_1(A)$ be its principal eigenvalue. Suppose that $f \in L^p(\Omega)$, $p > n$, such that $f > 0$, and suppose u satisfies the equation*

$$Au - \lambda m u = f \quad \text{in } L^p(\Omega). \tag{1.4}$$

Then there exists $\delta_f > 0$, which depends on f, such that if $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$,

$$u(x) < 0, \;\; x \in \Omega, \;\; \frac{\partial u}{\partial n}(x) > 0, \;\; x \in \partial\Omega; \tag{1.5}$$

and if $\lambda < \lambda_1(A)$,

$$u(x) > 0, \;\; x \in \Omega, \;\; \frac{\partial u}{\partial n}(x) < 0, \;\; x \in \partial\Omega. \tag{1.6}$$

Here the result for $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$ is called anti-maximum principle, and the result for $\lambda < \lambda_1(A)$ is an extended maximum principle. Several extensions and refinements of anti-maximum principles have been proved, see for examples, [1, 2, 4, 13]. In particular, an abstract form of anti-maximum principle was proved by Takáč [13], where he proved it for a strongly positive operator on a cone in a ordered Banach space. His proof is based on a strongly spectral projection and Krein-Rutman Theorem for strongly positive operators.

In this paper we give a new proof of an abstract anti-maximum principle with a bifurcation approach, in which we apply a secondary bifurcation theorem by Crandall and Rabinowitz [5]. The original proofs in [3] and [6] used a Lyapunov-Schmidt reduction, which is also used in the bifurcation theorem in [5]. Thus our proof still has the same essence as the original proof, but with a viewpoint of bifurcation theory. In particular we show the existence of a smooth curve of solutions bifurcating from the trivial solutions, which somehow explains the continuous change of the solutions from positive to negative when it crosses the principal eigenvalue. A similar approach can also be found in Arcoya and Gámez [1] where they used bifurcation from infinity to prove the anti-maximum principle. In fact, here we will prove an abstract version of anti-maximum principle in [2] and [1] for which $f > 0$ is weaken to $\int_\Omega f\varphi_1 dx > 0$. For simplicity, we will use the classical Krein-Rutman theorem to prove the positivity of the solutions. But our approach works for any version of Krein-Rutman theorem proved in many other papers, and our main focus is the bifurcation structure of the problem.

We state and prove our main result in Section 2. We will use $R(T)$ for the range space, and $N(T)$ for the null space of a linear operator T.

2 Main Results and Proof

We first set up the abstract framework of the problem, and we will also recall the classical Krein-Rutman theorem and a secondary bifurcation theorem by Crandall and Rabinowitz [5].

Suppose that Y is an ordered Banach space, *i.e.* there is a cone $K_Y \subset Y$ (a nonempty convex closed subset such that $K_Y \cap (-K_Y) = \{0\}$) and a partial order "$\leq$" such that $x \leq y$ if and only if $x - y \in K_Y$. Let $X \subset Y$ be a Banach space. Then X inherit the partial order from Y, and

$K_X = K_Y \cap X$ is also a cone in X. We assume that the interior $\overset{o}{K}_X$ of K_X is nonempty. We say that $x > 0$ if $x \in K_Y$.

We assume that $A : X \to Y$ is a linear operator such that $T \equiv A^{-1} : Y \to Y$ is a linear compact operator which is strongly positive, *i.e.* for any $x \in K_Y$, $Tx \in \overset{o}{K}_Y$ (which is nonempty since its subset $\overset{o}{K}_X$ is not.) Then the following Krein-Rutman theorem holds (see [9] or [14]):

Theorem 2.1. *Let Y be a real Banach space with an order cone K_Y with nonempty interior. Then a linear, compact, and strongly positive operator $T : Y \to Y$ has the following properties:*

1. *T has exactly one eigenvector x with $x > 0$ and $||x|| = 1$. The corresponding eigenvalue is $r(T)$ (the spectral radius of T) and it is algebraically simple;*
2. *For all $\lambda \in \mathbf{C}$ in the spectrum of T with $\lambda \neq r(T)$, it follows that $|\lambda| < r(T)$;*
3. *The dual operator T^* has $r(T)$ as an algebraically simple eigenvalue with a strictly positive eigenvector x^*.*

To study the anti-maximum principle, we consider the following equation:

$$Au - \lambda u = [\lambda - \lambda_1(A)]^2 f, \tag{2.1}$$

where A is defined above, $\lambda_1(A) = [r(T)]^{-1}$ is the principal eigenvalue of A, and $f \in Y$.

To study (2.1) we recall a theorem of secondary bifurcation by Crandall and Rabinowitz ([5] Theorem 1):

Theorem 2.2. *Let W and Y be Banach spaces, Ω an open subset of W and $G : \Omega \to Y$ be twice differentiable. Let $w : [-1, 1] \to \Omega$ be a simple continuously differentiable arc in Ω such that $G(w(t)) = 0$ for $|t| \leq 1$. Suppose*

1. *$w'(0) \neq 0$,*
2. *dim $N(G'(w(0))) = 2$, codim $R(G'(w(0))) = 1$,*
3. *$N(G'(w(0))) =$ span$\{w'(0), v\}$ for some $v \notin$ span$\{w'(0)\}$;*
4. *$G''(w(0))(w'(0), v) \notin R(G'(w(0)))$.*

Then $w(0)$ is a bifurcation point of $G(w) = 0$ with respect to $C = \{w(t) : t \in [-1, 1]\}$ and in some neighborhood of $w(0)$ the totality of solutions of $G(w) = 0$ form two continuous curves intersecting only at $w(0)$.

Theorem 2.2 can be proved by the following more well-known theorem also due to [5] (Theorem 1.7 in [5]):

Theorem 2.3. *Let X and Y be real Banach spaces, $\lambda_0 \in \mathbf{R}$ and let F be a continuously differentiable mapping of an open neighborhood $V \subset \mathbf{R} \times X$ of $(\lambda_0, 0)$ into Y. Suppose that*

1. *$F(\lambda, 0) = 0$ for $\lambda \in \mathbf{R}$,*
2. *The partial derivative $F_{\lambda u}$ exists and is continuous,*

3. *dim* $N(F_u(\lambda_0,0))$=*codim* $R(F_u(\lambda_0,0)) = 1$,

4. $F_{\lambda u}(\lambda_0,0)w_0 \notin R(F_u(\lambda_0,0))$, *where* $w_0 \in X$ *spans* $N(F_u(\lambda_0,0))$.

Let Z *be any complement of* $span\{w_0\}$ *in* X. *Then there exist an open interval* $I = (-\epsilon, \epsilon)$ *and* C^1 *functions* $\lambda : I \to \mathbf{R}$, $\psi : I \to Z$, *such that* $\lambda(0) = \lambda_0$, $\psi(0) = 0$, *and, if* $u(s) = sw_0 + s\psi(s)$ *for* $s \in I$, *then* $F(\lambda(s), u(s)) = 0$. *Moreover,* $F^{-1}(\{0\})$ *near* $(\lambda_0, 0)$ *consists precisely of the curves* $u = 0$ *and* $(\lambda(s), u(s))$, $s \in I$.

Our main result is the following:

Theorem 2.4. *Let* Y *be an ordered Banach space with an order cone* K_Y, *and let* $X \subset Y$ *be a Banach space with an order cone* $K_X = K_Y \cap X$ *having nonempty interior. Suppose that* $A : X \to Y$ *is a linear operator such that* $A^{-1} : Y \to Y$ *is a linear compact operator which is strongly positive. Let* $\lambda_1(A) = [r(A^{-1})]^{-1}$ *be the principal eigenvalue of* A, *and let* φ_1 *and* φ_1^* *be the normalized principal eigenfuntions of* A^{-1} *and* $(A^{-1})^*$. *Then, for equation*

$$Au - \lambda u = f, \tag{2.2}$$

1. *If* $\lambda < \lambda_1(A)$ *and* $f \in K_Y$ $(f \neq 0)$, *then* $u \in \overset{o}{K}_Y$;

2. *If* $f \in Y$ *and* $\langle \varphi_1^*, f \rangle > 0$, *then there exists* $\delta_f > 0$ *such that when* $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$, $-u \in \overset{o}{K}_Y$, *where* $\langle \cdot, \cdot \rangle$ *is the duality between* Y^* *and* Y.

Corollary 2.5. *Suppose that the conditions in Theorem 2.4 are satisfied. If* $f \in K_Y$ $(f \neq 0)$, *then when* $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$, $-u \in \overset{o}{K}_Y$.

Proof of Theorem 2.4. Let $W = \mathbf{R} \times X$. Define $G : W \to Y$ by $G(\lambda, u) = Au - \lambda u - [\lambda - \lambda_1(A)]^2 f$. Then $G(w) = 0$ has a family of solutions $w(t) = (\lambda_1(A), t\varphi_1)$ for $t \in \mathbf{R}$, where $\varphi_1 > 0$ is the positive eigenvector of $A^{-1}\varphi_1 = r(A^{-1})\varphi_1 = [\lambda_1(A)]^{-1}\varphi_1$ such that $||\varphi_1||_Y = 1$. We verify the conditions in Theorem 2.2. In the following we use $\lambda_1 = \lambda_1(A)$. Obviously G is differentiable as it is linear in u and quadratic in λ. First $w'(0) = (0, \varphi_1) \neq (0,0)$. The derivatives of G are as follows: for $(s,v), (r,z) \in W$,

$$\begin{aligned} G'(\lambda,u)[(s,v)] &= -su - 2s(\lambda - \lambda_1)f + Av - \lambda v, \\ G''(\lambda,u)[(s,v),(r,z)] &= -2srf - sz - rv. \end{aligned} \tag{2.3}$$

Suppose that $(s,v) \in N(G'(w(0)))$, then $Av - \lambda_1 v = 0$, and $s \in \mathbf{R}$, thus $N(G'(w(0))) = span\{(1,0),(0,\varphi_1)\} = span\{(1,0), w'(0)\}$. Suppose that $y \in R(G'(w'(0)))$, then there exists $(s,v) \in W$ such that $Av - \lambda_1 v = y$, thus we have

$$\langle \varphi_1^*, Av \rangle - \lambda_1 \langle \varphi_1^*, v \rangle = \langle \varphi_1^*, y \rangle. \tag{2.4}$$

By using $(A^{-1})^*\varphi_1^* = \lambda_1^{-1}\varphi_1^*$, we obtain $\langle \varphi_1^*, y \rangle = 0$. On the other hand, by the Fredholm theory of compact operators, the equation $(I - \lambda_1 A^{-1})v = A^{-1}y$ is solvable if $\langle \varphi_1^*, y \rangle = 0$. Hence $R(G'(w'(0))) = \{y \in Y : \langle \varphi_1^*, y \rangle = 0\}$ which is codimension one. Finally, $G''(w(0))[(1,0), w'(0)] = -\varphi_1 \notin R(G'(w'(0)))$ since $\langle \varphi_1^*, -\varphi_1 \rangle \neq 0$ since they are both positive. Therefore by Theorem 2.2, the solution set of $G(w) = 0$ near $w(0) = (\lambda_1, 0)$ consists of two intersecting curves.

Let $W = span\{w'(0)\} \oplus Z$, where $Z = \mathbf{R} \times Z_1$, and Z_1 is a compliment of $span\{\varphi_1\}$ in X. We define $F : \mathbf{R} \times Z \to Y$ by $F(t, (\mu, v)) = G(w(t) + (\mu, v))$. Since $\Psi : \mathbf{R} \times Z \to W$ is an isomorphism near $(0,(0,0))$, then the study of $G(w)=0$ near $(\lambda_1,0)$ is equivalent to $F(t,(\mu,v))=0$. We can apply Theorem 2.3 to F defined above, and the set of nontrivial solutions of $F=0$ is $t=\phi(s)$, $(\mu,v)=s(1,0)+s\psi(s)$, where $\phi:(-\delta,\delta)\to\mathbf{R}$ and $\psi:(-\delta,\delta)\to Z$ are continuous, and $\phi(0)=0$, $\psi(0)=(0,0)$. Indeed the partial derivative of F respect to the second argument can be written as

$$F_{(\mu,v)}(0,(0,0))[(\eta,z)]=Az-\lambda_1 z, \tag{2.5}$$

where $\eta\in\mathbf{R}$ and $z\in Z_1$. Thus $N(F_{(\mu,v)}(0,(0,0)))=span\{(1,0)\}$. This implies the nontrivial solutions of $G(w)=0$ can be written as $w=(\lambda(s),u(s))=w(\phi(s))+s(1,0)+s\psi(s)=(\lambda_1+s+s\psi_1(s),\phi(s)\varphi_1+s\psi_1(s))$, where $\psi(s)=(\psi_1(s),\psi_2(s))$, and $s\in(-\delta,\delta)$. We calculate $\phi'(0)$. For $\phi(s)$ in Theorem 2.3, if F_{uu} is also continuous, then (see [11] page 507)

$$\phi'(0)=-\frac{\langle\varphi_1^*,F_{(\mu,v)(\mu,v)}(0,(0,0))[(1,0),(1,0)]\rangle}{2\langle\varphi_1^*,F_{t(\mu,v)}(0,(0,0))[(1,0)]\rangle}. \tag{2.6}$$

For F defined here, $F_{(\mu,v)(\mu,v)}(0,(0,0))[(1,0),(1,0)]=-2f$, $F_{t(\mu,v)}(0,(0,0))[(1,0)]=-\varphi_1$, then from (2.6), we have

$$\phi'(0)=-\frac{\langle\varphi_1^*,f\rangle}{\langle\varphi_1^*,\varphi_1\rangle}. \tag{2.7}$$

If $\langle\varphi_1^*,f\rangle>0$, then $\phi'(0)<0$ and we can assume that for a $\delta_1\in(0,\delta)$, $s\phi(s)<0$ for $|s|\le\delta_1$. Therefore for $s\in(0,\delta)$, $\lambda(s)=\lambda_1+s+s\psi_1(s)>\lambda_1$, and $u(s)=\phi(s)\varphi_1+s\psi_2(s)<0$. Similarly for $s\in(-\delta,0)$, $\lambda(s)<\lambda_1$ and $u(s)>0$. This completes the proof for part 2 (anti-maximum principle.) The part 1 is well-known since u satisfies $(I-\lambda A^{-1})u=A^{-1}f>0$, and u is positive, then $u=(I-\lambda A^{-1})^{-1}A^{-1}f$ is also positive (see [14]). □

Remark 2.6.

1. The classical Krein-Rutman theorem requires that the cone has nonempty interior, which is not satisfied for the L^p setting in the introduction. But one can easily replace Theorem 2.1 by the version of Krein-Rutman theorem proved in [2] or [13], and the bifurcation proof above remains valid without any change.

2. A global bifurcation theorem by Rabinowitz [10] can also be applied to the equation (2.1). Thus the solution curve $(\lambda(s),u(s))$ obtained in Theorem 2.4 is indeed a part of an unbounded branch. But since it is a linear equation, This branch is a curve for at least $\lambda<\lambda_2(A)$, where $[\lambda_2(A)]^{-1}=\sup\{k\in\mathbf{C}:k\in spt(A^{-1})\}$, and $spt(A^{-1})$ is the spectrum of A^{-1}. However the branch for $\lambda_2(A)>\lambda>\lambda_1(A)$ may not be all positive, and it will depend on the function f.

3. In [8], Korman shows that if $f(x)$ is an even front-loaded function (for example, functions f such that $f'(x)\ge 0$ on $(0,\pi/2)$) on $[0,\pi]$, then the anti-maximum principle (for one-dimensional Dirichlet problem $u''+\lambda u=f$) holds for this f and all $\lambda\in(\lambda_1,\lambda_2)$. In general, one does not expect the anti-maximum principle holds for $\lambda>\lambda_2$, and a simple counterexample is $f(x)\equiv 1$. Recently, the author [12] extends the result of Korman to $\lambda\in(\lambda_1,\lambda_2)$ for a more general class of f.

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