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Exact multiplicity of positive solutions to a superlinear problem *

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Abstract

We generalize previous uniqueness results on a semilinear elliptic equation with zero Dirichlet boundary condition and superlinear, subcritical nonlinearity. Our proof is based on a bifurcation approach and a Pohozaev type integral identity, which greatly simplifies the previous arguments.

1 Introduction

We consider the exact multiplicity of the solutions to the semilinear elliptic equation

$$\begin{gathered} \Delta u + \lambda f(u) = 0 \quad \text{in } B^n, \\ u > 0 \quad \text{in } B^n, \\ u = 0 \quad \text{on } \partial B^n, \end{gathered} \tag{1.1}$$

where B^n is the unit ball in $\mathbb{R}^n$ with $n \geq 3$, and λ is a positive parameter. The uniqueness and exact multiplicity of the positive solutions to (1.1) have been extensively studied in the past two decades, and in particular a systematic approach has been developed in [12] and [13]. (More references can be found therein.)

In this paper we assume that f satisfies

(D1) $f \in C^1(\overline{\mathbb{R}^+})$, $f(0) = 0$, $f(u) > 0$, $f'(u) > 0$ for $u > 0$;

(D2) There exists $p, q > 0$ such that for all $u > 0$,

$$1 \leq q \leq K_f(u) \leq p < \frac{n+2}{n-2}, \quad \text{where } K_f(u) = \frac{uf'(u)}{f(u)}; \tag{1.2}$$

(D3) Let

$$A_f(u) = (p-1)\big[nF(u) - \frac{n-2}{2} f(u)u\big] + [f'(u)u - pf(u)]u, \tag{1.3}$$

where $F(u) = \int_0^u f(t)dt$. Then $A_f(u) \geq 0$ for $u \geq 0$.

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From (D2), $uf'(u) \geq f(u)$ for all $u > 0$, thus the function $f(u)/u$ is increasing for $u > 0$. We define

$$\lambda_0 = \frac{\lambda_1}{f'(0)}, \quad \text{and } \lambda_\infty = \frac{\lambda_1}{f'(\infty)}, \tag{1.4}$$

where $f'(\infty) = \lim_{u\to\infty} f(u)/u$ and λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(B^n)$. When $f'(0) = 0$, we understand that $\lambda_0 = \infty$ and when $f'(\infty) = \infty$, $\lambda_\infty = 0$. Then our main result is as follows.

Theorem 1.1 *Suppose that f satisfies (D1), (D2), and (D3). Then* (1.1) *has no solution for $0 < \lambda \leq \lambda_\infty$ and $\lambda \geq \lambda_0$, and has exactly one solution for $\lambda_\infty < \lambda < \lambda_0$. Moreover all solutions lie on a single smooth solution curve in (λ, u) space, which starts from $(\lambda_0, 0)$ and continues to the left up to (λ_∞, ∞), and there is no any turning point on the curve. (see Figures. 1 and 2.)*

In particular, for the special nonlinearity $f(u) = u^p + u^q$, Theorem 1.1 implies that

Corollary 1.2 *Let $f(u) = u^p + u^q$, and $p > q$.*

1. *If $q = 1$ and $p < \frac{n+2}{n-2}$, then $\lambda_0 = \lambda_1$ and* (1.1) *has no solution for $0 < \lambda \leq \lambda_\infty$ and $\lambda \geq \lambda_1$, and has exactly one solution for $\lambda_* < \lambda < \lambda_1$; (see Figure 1)*
2. *If $q > 1$, $p < \frac{n+2}{n-2}$ and*
$$\frac{n(p-1)}{2(q+1)} \leq 1, \tag{1.5}$$
then (1.1) *has exactly one solution for $0 < \lambda < \infty$. (see Figure 2)*

Our result is a generalization of previous results by Kwong and Li [11], Srikanth [16], Yadava [17], Zhang [18] where (1) of Corollary 1.2 was proved by different methods, and Yadava [17], Zhang [19] where (2) of Corollary 1.2 was proved. All these previous proofs seem to be complicated and lengthy, and our proof is much simpler than all of them. On the other hand, Erbe and Tang [6] prove the results in Corollary 1.2 even without (1.5), but their result can not imply Theorem 1.1, and the methods are quite different.

Our method also works for the case of $f(u) = u^q + u^p$ with $p = (n+2)/(n-2)$, see Section 3 for details. In this case, Brezis and Nirenberg [1] first showed the existence of a solution.

We use a bifurcation approach similar to that in [12] and [13], and some techniques in [13] are also used here. But the difference is that instead of showing that the degenerate solution is neutrally stable (Morse index is 0), we show that the Morse index of the degenerate solution is very high (≥ 2), thus turning points can not occur in a branch of solutions (which have Morse index 1) obtained from the Mountain Pass Lemma. Here the function $A_f(u)$ introduced in (D3) provides a Pohozaev type identity, which is the key of the proof. We introduce some preliminaries in Section 2, and the main results are proved in Section 3.

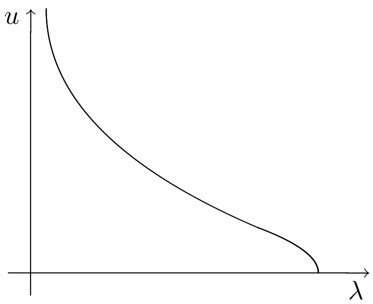

Fig. 1: Bifurcation diagram for $f'(0) > 0$

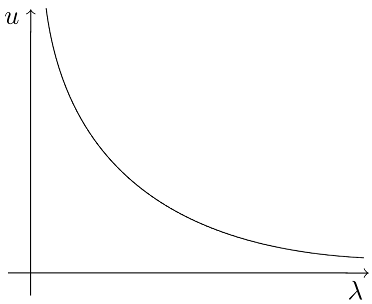

Fig. 2: Bifurcation diagram for $f'(0) = 0$

2 Preliminaries

A framework of using the bifurcation method to prove the exact multiplicity of solutions of (1.1) was established in Ouyang and Shi [12], [13]. (see also [9], [10], [8].) Here we briefly recall the approach in [13] without the proof since all proofs can be found in [13]. One remarkable result regarding (1.1) was proved by Gidas, Ni and Nirenberg [7] in 1979. They showed that if f is locally Lipschitz continuous in $[0,\infty)$, then all positive solutions of (1.1) are radially symmetric. This result sets the foundation of our analysis of positive solutions to (1.1). We summarize some basic facts on (1.1).

Lemma 2.1 *1. If f is locally Lipschitz continuous in $[0,\infty)$, then all positive solutions of* (1.1) *are radially symmetric, and satisfy*

$$\begin{gathered}(r^{n-1}u')' + \lambda r^{n-1} f(u) = 0, \quad r \in (0,1), \\ u'(0) = u(1) = 0;\end{gathered} \tag{2.1}$$

2. If u is a positive solution to (1.1)*, and w is a solution of the linearized problem (if it exists):*

$$\begin{gathered}\Delta w + \lambda f'(u) w = 0 \quad \text{in } B^n, \\ w = 0 \quad \text{on } \partial B^n.\end{gathered} \tag{2.2}$$

then w is also radially symmetric and satisfies

$$\begin{gathered}(r^{n-1}w')' + \lambda r^{n-1} f'(u) w = 0, \quad r \in (0,1), \\ w'(0) = w(1) = 0;\end{gathered} \tag{2.3}$$

3. For any $d > 0$, there is at most one $\lambda_d > 0$ such that (1.1) *has a positive solution $u(\cdot)$ with $\lambda = \lambda_d$ and $u(0) = d$. Let $T = \{d > 0 :$ (1.1) has a positive solution with $u(0) = d\}$, then T is open; $\lambda(d) = \lambda_d$ is a well-defined continuous function from T to $\mathbb{R}^+$.*

Because of (3), we call $\mathbb{R}^+ \times \mathbb{R}^+ = \{(\lambda, d) | \lambda > 0, d > 0\}$ the *phase space*, and $\Sigma = \{(\lambda(d), d) : d \in T\}$ the *bifurcation diagram.* A solution (λ, u) of (1.1) or (2.1) is a *degenerate* solution if (2.2) or (2.3) has a non-trivial solution. At a degenerate solution $(\lambda(d), u(d))$, $\lambda'(d) = 0$, and it is referred as a *turning point* of Σ if $\lambda''(0) \neq 0$. We define the *Morse index* $M(u)$ of a solution (λ, u) to be the number of negative eigenvalues of the following eigenvalue problem

$$\begin{gathered} (r^{n-1}\phi')' + \lambda f'(u)\phi = -\mu\phi, \quad r \in (0,1), \\ \phi'(0) = \phi(1) = 0. \end{gathered} \tag{2.4}$$

It is well-known that the eigenvalues $\mu_1, \mu_2, \dots$ of (2.4) are all simple, and the eigenfunction ϕ_i corresponding to μ_i has exactly $i-1$ simple zeros in $(0,1)$ for $i \in \mathbf{N}$. We also call a solution (λ, u) *stable* if $\mu_1(u) > 0$, otherwise it is *unstable.* One of our main tools is the Sturm comparison lemma, which we include for the sake of completeness.

Lemma 2.2 *Let $Lu(t) = [(p(t)u'(t)]' + q(t)u(t)$, where $p(t)$ and $q(t)$ are continuous in $[a,b]$ and $p(t) \geq 0$, $t \in [a,b]$. Suppose $Lw(t) = 0$, $w \not\equiv 0$.*

1. *If there exists $v \in C^2[a,b]$ such that $Lv(t) \cdot v(t) \leq (\not\equiv)0$, then w has at most one zero in $[a,b]$. If in addition, $w'(a) = 0$ or $p(a) = 0$, then w does not have any zero in $[a,b]$.*
2. *If there exists $v \in C^2[a,b]$ such that $Lv(t) \cdot v(t) \geq (\not\equiv)0$, and $v(a) = v(b) = 0$, then w has at least one zero in (a,b). If $w'(a) = 0$ or $p(a) = 0$, then w has at least one zero in $[a,b]$ even if $v(a) \neq 0$.*

The proof is standard, and we refer to [12]. In the following, we will always use the notation $Lw(r) = (r^{n-1}w')' + \lambda r^{n-1} f'(u)w$, where u is a solution to (2.1). We will say that we apply the *integral procedure* to two equations: $Lu = g_1(r)$ and $Lv = g_2(r)$, which means we multiply the first equation by v and multiply the second equation by u, integrate both over $[0,1]$ and subtract, so we obtain $\int_0^1 (vLu - uLv)dr + \int_0^1 (vg_1 - ug_2)dr = 0$. The first term can be simplified via the integration by parts and boundary conditions of u and v. The following are some calculation which will be used in the proofs.

Lemma 2.3 *Let u and w be the solutions of (2.1) and (2.3) respectively, and let $F(u) = \int_0^u f(t)dt$. Then*

$$Lu = \lambda r^{n-1}[f'(u)u - f(u)], \tag{2.5}$$

$$Lw = 0, \tag{2.6}$$

$$L(ru_r) = -2\lambda r^{n-1} f(u), \tag{2.7}$$

$$\int_0^1 r^{n-1} f(u)w dr = \int_0^1 r^{n-1} f'(u)uw dr = \frac{1}{2\lambda} u_r(1) w_r(1), \tag{2.8}$$

$$\int_0^1 r^{n-1} \big[nF(u) - \frac{n-2}{2} f(u)u\big] dr = \frac{1}{2\lambda} u_r^2(1), \tag{2.9}$$

$$\int_0^1 r^{n-1}[2nF(u) - nf(u)u]dr - \int_0^1 r^{n-1}[f_u(u)u - f(u)]ru_r(r)dr = 0. \tag{2.10}$$

Proof (2.5)-(2.7) are by direct calculations. The first part of (2.8) is obtained by applying integral procedure to (2.5) and (2.6), and the second equality in (2.8) is obtained by applying the integral procedure to (2.6) and (2.7). (see also [12] for a more general identity.) (2.9) is the well-known Pohozaev's identity, and it is obtained by integrating ru_rLu. Finally, (2.10) is obtained by applying the integral procedure to (2.5) and (2.7), and combining with (2.9). □

3 Proof of Main Results

Note that (D1) and (D2) imply that for $u \geq 0$,

$$f'(u)u - pf(u) \leq 0 \quad \text{and} \quad f'(u)u - qf(u) \geq 0. \tag{3.1}$$

Lemma 3.1 *Suppose that f satisfies (D1) and (D2), and u is a degenerate solution of* (2.1)*. Let w be a solution of* (2.3)*. Then w must change sign in* $(0,1)$*.*

Proof By (2.8), we have $\int_0^1 r^{n-1}[f'(u)u - f(u)]wdr = 0$. Since $q \geq 1$ and (3.1), then w must change sign in $(0,1)$. □

The following lemma is the key to our method.

Lemma 3.2 *Suppose that f satisfies (D1), (D2) and (D3), and u is a degenerate solution of* (2.1)*. Let w be a solution of* (2.3)*. Then w has at least two zeros in* $(0,1)$*.*

Proof We use a test function $v(r) = w(r) - u(r)$, where w is a solution of (2.3). It is easy to see that $Lv = -Lu = -\lambda r^{n-1}[f'(u)u - f(u)] \leq 0$. Note that the solutions of (2.3) is a one parameter family which can be parameterized by $w_r(1)$, and we will specify $w_r(1)$ later. By (3.1) and $u > 0$, we have $\int_0^1 r^{n-1}[f'(u)u - pf(u)]udr < 0$. On the other hand, by (2.8), $\int_0^1 r^{n-1}[f'(u)u - pf(u)]wdr = (2\lambda)^{-1}(1-p)u_r(1)w_r(1)$. Since $f(u) > 0$, then $u_r(1) < 0$ and $w_r(1) \neq 0$. therefore we can choose $w_r(1)$ such that

$$\int_0^1 r^{n-1}[f'(u)u - pf(u)]udr = \int_0^1 r^{n-1}[f'(u)u - pf(u)]wdr. \tag{3.2}$$

And by this choice, $w_r(1) < 0$. Therefore, using (2.9), we obtain

$$\begin{aligned}
&\frac{1-p}{2\lambda}u_r(1)v_r(1) \\
&= \frac{1-p}{2\lambda}u_r(1)w_r(1) - \frac{1-p}{2\lambda}u_r^2(1) \\
&= \int_0^1 r^{n-1}[f'(u)u - pf(u)]wdr + (p-1)\int_0^1 r^{n-1}\big[nF(u) - \frac{n-2}{2}f(u)u\big]dr \\
&= \int_0^1 r^{n-1}A_f(u)dr > 0.
\end{aligned}$$

Thus $v_r(1) > 0$. By (3.2), $\int_0^1 r^{n-1}[f'(u)u - pf(u)]vdr = 0$, and $f'(u)u - pf(u) \leq 0$ for $u \geq 0$. Hence v must change sign in $(0,1)$.

Let r_1 be the first zero of v left of 1. Then $v_r(1) > 0$ implies $v(r) < 0$ in $(r_1, 1)$. Since $Lv \leq 0$ in $(0,1)$, then by Lemma 2.2, w has at least one zero in $(r_1, 1)$. Let $r_2(> r_1)$ be the first zero of w left of 1. Then $w_r(1) < 0$ implies $w(r) > 0$ in $(r_2, 1)$, and $w(r) < 0$ in $(r_2 - \delta, r_2)$ for a small $\delta > 0$. But $w(r_1) = v(r_1) + u(r_1) = u(r_1) > 0$, so w has another zero in (r_1, r_2). Therefore w has at least two zeros in $(0,1)$. □

Corollary 3.3 *Suppose that f satisfies (D1), (D2) and (D3), and u is a degenerate solution of* (2.1). *Then the Morse index $M(u) \geq 2$, and $0 = \mu_i(u)$ for some $i \geq 3$.*

Proof Since w has at least two zeros in $(0,1)$, then $0 = \mu_i(u)$ for some $i \geq 3$. □

Note that in the proof of Lemma 3.2, the condition $p < (n+2)/(n-2)$ is *not* needed. This fact is useful when discussing the case of critical exponent.

Proof of Theorem 1.1 We first prove the case when $f'(0) > 0$. In this case, $\lambda_0 = \lambda_1/f'(0)$ is a bifurcation point where a bifurcation from the trivial solutions occurs. From a theorem of Crandall and Rabinowitz [4] (or see Theorem 3.1 (2) in [13]), the local structure of the solution set of (1.1) near $(\lambda, u) = (\lambda_0, 0)$ consists of two parts: $\Sigma_0 = \{(\lambda, 0) : \lambda > 0\}$ and $\Sigma_1 = \{(\lambda(s), u(s)) : |s| \leq \delta\}$, where $\lambda(0) = \lambda_0$, $u(s) = s\phi_1 + o(|s|)$, and ϕ_1 is the positive eigenfunction corresponding to λ_1. Moreover, from Proposition 3.4 (1) in [13], the bifurcation is subcritical, so $\lambda'(s) \leq 0$ for $s \in [0, \delta]$. On the other hand, by Theorem 1.16 in [5], $\mu_1(s) \leq 0$ where $\mu_1(s)$ is the principal eigenvalue of (2.4) with $u = u(s)$. If $\mu_1(s) = 0$ for some $s \in (0, \delta)$, then $u(s)$ is a degenerate solution of (1.1), that contradicts with Corollary 3.3. Thus $\mu_1(s) < 0$ and $\mu_2(s) > 0$ for $s \in (0, \delta)$ with some small $\delta > 0$ by the continuity of the eigenvalues with respect to s. Thus $u(s)$ is a non-degenerate solution with Morse index 1, and in that case we can apply the implicit function theorem to extend Σ_1 further. Suppose $s_0 = sup\{s > 0 : \mu_1(s) < 0 \text{ and } \mu_2(s) > 0\}$. If $s_0 < \infty$, then at $s = s_0$, $u(s)$ is still well-defined, which is the solution of initial value problem $(r^{n-1}u')' + \lambda(s_0)r^{n-1}f(u) = 0$, $u'(0) = 0$ and $u(0) = s_0$. So either $\mu_1(s_0) = 0$ or $\mu_2(s_0) = 0$ by the continuity, and the Morse index of $u(s_0)$ is either 0 or 1, which again reaches a contradiction with Corollary 3.3. Therefore $s_0 = \infty$, and $\lambda'(s) < 0$ for all $s > 0$. When $f'(\infty) < \infty$, then $\lim_{s\to\infty} \lambda(s) = \lambda_\infty$. When $f'(\infty) = \infty$, then $\lim_{s\to\infty} \lambda(s) = 0$. (see [15] for the proofs).

Next we prove the case of $f'(0) = 0$. In this case, the proof is similar as long as we can show that for some (λ, s) there exists a solution (1.1) such that $u(s, 0) = s$, $\mu_1(s) < 0$ and $\mu_2(s) > 0$. This can be obtained by the well-known Mountain Pass Lemma. We verify that Theorem 2.15 in Rabinowitz [14] can be applied here. (For the convenience of the readers, we include the statement of the theorem after the proof.) Let $p(x, \xi) = f(\xi)$, and we would relate conditions (D1-D3) to (p1-p4) in Theorem 3.4. Obviously, (D1) implies (p1) and we can assume (p3) since we only consider the case of $f'(0)=0$. Also if $p<(n+2)/(n-2)$ in (D2), then (p2) is true, since

$$\Big[\frac{f(u)}{u^p}\Big]' = \frac{f'(u)u - pf(u)}{u^{p+1}} \le 0, \tag{3.3}$$

for all $u\ge 0$. Finally, we notice that in (D2), if $q>1$, then (p4) is also satisfied. So if $q>1$, from the result of Rabinowitz (see Theorem 3.4 below), for each $\lambda>0$, (1.1) has a positive solution u. If $q=1$, we notice that in the proof of the result of Rabinowitz, (p4) is only used in proving that there is a function u such that $I(u)=\int_{B^n}[(1/2)|\nabla u|^2-\lambda P(u)]dx\le 0$, but that can also be achieved if we let λ be sufficiently large when $q=1$. So in the case of $q=1$, (1.1) has a positive solution u for sufficiently large λ. (Indeed (1.1) may not have a solution if $f'(\infty)<\infty$).

Thus in any case of $f'(0)=0$, we obtain a solution (λ,u) of (1.1) by the Mountain Pass Lemma. On the other hand, from Theorem 1.6 and Corollary 3.1 in Chapter II of Chang [3], the Morse index of (λ,u) is 1 if it is non-degenerate, and is 0 if it is degenerate. But from Corollary 3.3, the latter case can not happen, so (λ,u) must satisfy $\mu_1(s)<0$ and $\mu_2(s)>0$. Thus the continuation arguments in the proof of the case $f'(0)>0$ can also be carried over to here. Finally, from Proposition 6.6 in [13], since $p<(n+2)/(n-2)$, the domain of the function $\lambda(s)$ should be all $(0,\infty)$, and $\lim_{s\to 0^+}\lambda(s)=\infty$. Similar to the case of $f'(0)>0$, when $f'(\infty)<\infty$, then $\lim_{s\to\infty}\lambda(s)=\lambda_\infty$. When $f'(\infty)=\infty$, then $\lim_{s\to\infty}\lambda(s)=0$. (see [15] for the proofs). □

The following is Theorem 2.15 and Corollary 2.23 in Rabinowitz [14].

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Consider the equation

$$\begin{gathered}\Delta u + \lambda p(x,u) = 0 \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{3.4}$$

Assume that

(p1) $p(x,\xi)$ is locally Lipschitz continuous in $\overline{\Omega}\times\mathbb{R}$,

(p2) there exists $a_1,a_2\ge 0$, such that $|p(x,\xi)|\le a_1+a_2|\xi|^s$, where $0\le s<(n+2)/(n-2)$ if $n>2$,

(p3) $p(x,\xi)=o(|\xi|)$ as $\xi\to 0$, and

(p4) there exists constants $\mu>2$ and $r\ge 0$ such that for $|\xi|\ge r$, $0<\mu P(x,\xi)\le \xi p(x,\xi)$.

Theorem 3.4 *Under assumptions (p1)–(p4), equation* (3.4) *possesses a positive classical solution.*

Finally we discuss the critical exponent case. In fact, Lemma 3.2 is even true when $p>(n+2)/(n-2)$, but in that case the existence of the solution is not clear in general. When $p=(n+2)/(n-2)$ in (D2), and $f(u)=u^p+u^q$, (D3) is also satisfied if (1.5) is also satisfied. So again if we can show the existence of a solution with Morse index 1, then the uniqueness part is implied by Lemma 3.2 and the continuity argument in the proof of Theorem 1.1. In the case of $q = 1$, this can be done by the bifurcation result which we used in the proof of Theorem 1.1, but $\lim_{s\to\infty} \lambda(s)$ may not be 0 as shown in [1]. In the case of $q > 1$, (1.5) can only be satisfied for $n \geq 4$, and in that case, it is proved by Brezis and Nirenberg that (1.1) always has a positive solution via a modified Mountain Pass Lemma, so we can still prove that the Morse index of the solution is 1 in that case. So summarizing these discussion, we have

Theorem 3.5 *Consider*

$$\begin{gathered} \Delta u + \lambda(u^p + u^q) = 0 \quad \text{in } B^n, \\ u > 0 \quad \text{in } B^n, \\ u = 0 \quad \text{on } \partial B^n, \end{gathered} \tag{3.5}$$

where $p = (n+2)/(n-2)$. *Then*

1. *If* $q = 1$, *then* (3.5) *has no solution for* $0 < \lambda \leq \lambda_*$ *and* $\lambda \geq \lambda_0$, *and has exactly one solution for* $\lambda_* < \lambda < \lambda_0$, *where* $\lambda_* = 0$ *when* $n \geq 4$ *and* $\lambda_* = \lambda_1/4$ *when* $n = 3$;
2. *If* $q > 1$, q *satisfies* (1.5) *and* $n \geq 4$, *then* (3.5) *has no solution for* $\lambda \geq \lambda_0$, *and has exactly one solution for* $0 < \lambda < \lambda_0$.

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