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Exact multiplicity of solutions to superlinear and sublinear problems

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1. Introduction

We consider a semilinear elliptic equation

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^n$, $n \geqslant 1$, and λ is a positive parameter. Throughout the paper, we assume that f satisfies

$(f1)$ $f \in C^1(\mathbf{R}, \mathbf{R})$, $f(0) = 0$, $f'(u) > 0$ *for* $u \in \mathbf{R}$;

$(f2)$ $\lim_{u\to\infty} \frac{f(u)}{u} = f_+ > 0$, $\lim_{u\to-\infty} \frac{f(u)}{u} = f_- > 0$.

For the definiteness, we assume $f_+ \geqslant f_-$, and when $f_+ = f_-$, we use $f_\pm$ to represent it. We will consider f being either superlinear or sublinear. f is said to be *superlinear* if $f(u)/u$ is decreasing in $(0, \infty)$ and is increasing in $(-\infty, 0)$; and f is said to be *sublinear* if $f(u)/u$ is increasing in $(0, \infty)$ and is decreasing in $(-\infty, 0)$.

The semilinear equation (1.1) with f satisfying (f1), (f2) has been studied extensively since early 1970s. Several different approaches, like variational methods (Morse

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theory), Leray–Schauder degree theory, Lyapunov–Schmidt reduction method, fixed-point index theory, have been successfully applied to show the existence of one solution or multiple solutions to (1.1) under various additional assumptions. In these works, usually a nonparameterized version of (1.1) is studied

$$\Delta u + f(u) = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

For the results in the study of this problem we refer the readers to [1–3,5–11,13,18,19, 21,22,29] and the references therein.

In this paper, and also in an earlier paper [31], we use a bifurcation approach combining with Leray–Schauder degree theory to study (1.1) which contains a positive parameter λ. Also contrast to most previous works, we focus on the *exact number* of all nontrivial solutions for λ in certain parameter range. This is also the reason we need to add the condition of superlinearity or sublinearity, and it is usually not needed if we only consider the existence. To state our results, we introduce some notations. We denote by λ_k the kth eigenvalue of

$$\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

In this paper, we assume all λ_ks are simple eigenvalues. If f is superlinear, we define

$$\lambda_k^0 = \frac{\lambda_k}{f'(0)}, \quad \lambda_{k,+}^{\infty} = \frac{\lambda_k}{f_+}, \quad \lambda_{k,-}^{\infty} = \frac{\lambda_k}{f_-}, \quad \text{and} \quad \lambda_k^M = \frac{\lambda_k}{\sup_{u\in\mathbf{R}}} f'(u). \tag{1.4}$$

For superlinear f, we have $f'(u) \geqslant f(u)/u$ for $u \in \mathbf{R}$. Therefore, for $k \in \mathbf{N}$, f satisfying (f1), (f2) and being superlinear, $\lambda_k^M \leqslant \lambda_{k,+}^{\infty} \leqslant \lambda_{k,-}^{\infty} < \lambda_k^0$. If in addition f satisfies $uf''(u) \geqslant 0$ and $f_+ = f_-$, then $\lambda_k^M = \lambda_{k,+}^{\infty} = \lambda_{k,-}^{\infty}$. In the case of $f_+ = f_-$, we use $\lambda_k^{\infty} = \lambda_k/f_{\pm}$. For $k \in \mathbf{N}$, we define two open intervals $I_k = (\lambda_{k,+}^{\infty}, \lambda_k^0)$, and $\widetilde{I}_k = (\lambda_k^M, \lambda_k^0)$. We define the *Morse index* $M(u)$ of a solution u to (1.1) to be the number of negative eigenvalues of the following problem:

$$\Delta\phi + \lambda f'(u)\phi = -\mu\phi \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

If u is a solution to (1.1), and 0 is not an eigenvalue of (1.5), then u is a *nondegenerate* solution, otherwise it is *degenerate*.

When $f_+ = f_-$, our main results can be summarized as follows (Here we assume that f satisfies (f1), (f2) and f is superlinear, and for simplicity, we assume that $\lambda_k^M = \lambda_{k,\pm}^{\infty}$, thus $I_k = \widetilde{I}_k$. The more general results will be found in the later part of the paper.):

(A) If $\lambda \notin \bigcup_{j\in\mathbf{N}} I_j$, then (1.1) has only the trivial solution $u = 0$;

(B) If $\lambda \in I_k \setminus \bigcup_{j\neq k} I_j$, then (1.1) has exactly two nontrivial solutions which are nondegenerate and with Morse index $M(u) = k$;

(C) If $\lambda \in (I_k \cap I_{k+1}) \setminus \bigcup_{j \neq k,k+1} I_j$, then there exists $\varepsilon > 0$ such that (1.1) has exactly four nontrivial solutions for $\lambda \in (\lambda_{k+1}^{\infty}, \lambda_{k+1}^{\infty} + \varepsilon) \cup (\lambda_k^0 - \varepsilon, \lambda_k^0)$ (which is: near the boundary of $I_k \cap I_{k+1}$), and all of them are nondegenerate with two of them having Morse index $M(u) = k$, the other two $M(u) = k + 1$.

Part (A) and a special case of Part (B) was also proved in [31]. In fact, in [31], we prove that, if for $k \in \mathbf{N}$, I_k is separated from all other I_js ($j \neq k$), then (1.1) has exactly two nontrivial solutions for $\lambda \in I_k$ and these nontrivial solutions lie on two smooth curves $\Sigma_k^{\pm} = \{(\lambda, u_k^{\pm}(\lambda, \cdot)) \colon \lambda \in I_k\}$, Σ_k^+ and Σ_k^- join at $(\lambda_k^0, 0)$, and $\lim_{\lambda \to (\lambda_k^{\infty})^+} ||u_k^{\pm}(\lambda, \cdot)||_{L^2(\Omega)} = \infty$. Part (B) is also a generalization of a result by Castro and Lazer [11] and Ambrosetti and Mancini [2]. Both their results are for the nonparameter equation (1.2). In this paper we give a simple proof based on our bifurcation approach and eigenvalue comparison argument.

Part (B) gives the exact number of the solutions of (1.1) for λ belonging to only one of I_js but not in the overlap of more than one I_js. A general question is: how many solutions does (1.1) have for λ belonging to the overlap of exactly k intervals I_js? Or what is the lower bound of the number of the solutions? A conjecture by Castro and Lazer [11] in our context is: (1.1) has at least $2k$ nontrivial solutions if λ belongs to the overlap of exactly $k I_j$s. If f is an odd function, this can be proved using Lusternik–Schnirelman theory (see [11] Theorem C). Parts (B) and (C) show that $2k$ can be the exact number of the nontrivial solutions for $k = 1, 2$. However, even for $k = 2$, the upper bound of the number of the nontrivial solution is not necessarily $2k$. In fact, using the domain perturbation theory in Dancer [16], we show that (Proposition 4.2) for $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{k \geqslant 3} I_k$, (1.1) can have as many as 8 nontrivial solutions for $\lambda \in (\lambda_2^{\infty} + \varepsilon, \lambda_1^0 - \varepsilon)$ (which is: in the interior part of $I_1 \cap I_2$), where $\varepsilon > 0$ is a small positive constant. That also shows the result in (C) in some sense is optimal. But, in Section 7, we also show that when $\Omega = (0, \pi)$ (the spatial dimension is 1), then (1.1) has exactly $2k$ solutions if λ belongs to exactly $k I_j$'s. (See Theorem 7.1.)

In the context of the bifurcation problem, the conjecture above can be in another form: whether the solution curve bifurcating from $(\lambda_k^0, 0)$ and the one from $(\lambda_k^{\infty}, \infty)$ are connected? If this is true, then certainly the conjecture above will be true. A lot of effect has been devoted to improve the lower bound of the number of the solutions, see for examples [6,5,9,10,18,19] where the existence of three or four nontrivial solutions is shown under various conditions for λ in the overlap of I_k's. Usually, one of overlapping I_k's is I_1 and two of such solutions are of one sign. When all I_k's are not necessarily I_1 (or equivalently, f crosses higher eigenvalues,) Amann and Zehnder [3] proved there exists at least one solution if λ is in an overlap, and recently Li and Willem [22] proved there exist at least two solutions under some additional conditions.

We also consider the solution set of (1.1) or (1.2) when $f_+ \neq f_-$. In this case, f is called a *jumping nonlinearity*, and the Fučík spectrum of Δ is important in determining the number of the solutions of (1.1) or (1.2). Consider

$$\Delta\phi + a\phi^+ - b\phi^- = 0 \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

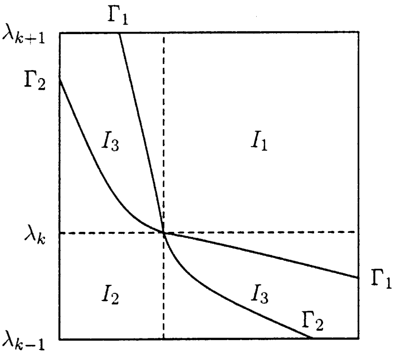

Fig. 1. Fučík spectrum in Q_k.

Then the set $\Gamma = \{(a,b) \in \mathbf{R}^2$: (1.6) has a nontrivial solution$\}$ is called *Fučík spectrum* on Ω. When $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$ are all simple eigenvalues, the structure of the Γ in the square $Q_k = (\lambda_{k-1}, \lambda_{k+1}) \times (\lambda_{k-1}, \lambda_{k+1})$ is known: (see [21,27,29]) there exists two decreasing curves $\Gamma_1 = \gamma_{k,1}(a)$ and $\Gamma_2 = \gamma_{k,2}(a)$ passing through (λ_k, λ_k) such that $Q_k \cap \Gamma = \Gamma_1 \cup \Gamma_2$. Moreover, these two curves divide Q_k into three parts: above the curves (I_1), below the curves (I_2) and between the curves (I_3). (See Fig. 1.)

Our results for jumping nonlinearities are best represented in the form of (1.2) in the following classification.

Theorem 1.1. *Let f satisfy $(f1),(f2)$, and let f be superlinear. Suppose*

$$\lambda_{k-1} < f'(0) < \lambda_k, \quad \sup_{u \in \mathbf{R}} f'(u) < \lambda_{k+1}, \quad \text{and} \quad (f_+, f_-) \in Q_k.$$

Then (1.2)

$$\begin{aligned} &\textit{has no nontrivial solution} && \textit{if } (f_+, f_-) \in I_2 \cup \Gamma_2, \\ &\textit{has exactly one nontrivial solution} && \textit{if } (f_+, f_-) \in I_3 \cup \Gamma_1, \\ \textit{and } &\textit{has exactly two nontrivial solutions} && \textit{if } (f_+, f_-) \in I_1. \end{aligned}$$

Theorem 1.1 can be viewed as an extension and summary of previous results by Ambrosetti and Mancini [2], Castro and Lazer [11] and Các [8]. If $\Omega = (0, \pi)$ and $n = 1$, this result (in fact all the results in this paper) can be extended to the best possible result: a complete classification of solution set for f satisfying (f1), (f2) and f being superlinear or sublinear. (See Theorems 7.1 and 7.7.)

The methods in this paper can also be applied to some other semilinear elliptic equations. Two such examples are

$$\Delta u + \lambda u - f(u) = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where f satisfies (f1), (f2), f is either sublinear or superlinear, and

$$\Delta u + \lambda u - h(x)|u|^{p-1}u = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $p > 1$ and $h(x)$ is a nonnegative smooth function with $m(\{x \in \Omega\colon h(x) = 0\}) \neq 0$. The bifurcation approach has been applied to these problems in Shi and Wang [31], Castro et al. [12] and Ouyang [24]. (See also the references therein.) Also, in our results, the nonlinearity $f(u)$ can be a more general form $f(x,u)$ which depends on the space variable x, and the Laplacian Δ can be replaced by a general self-adjoint second order elliptic operator. In most of the paper, we only work on superlinear f, but all results can be obtained for sublinear f without any difficulty.

We organize our paper in the following way. In Section 2, we give some preliminaries. In Section 3, we prove a more general version of Parts (B) and (C). In Section 4, we study the solution curves bifurcating from the first two eigenvalues. The case of jumping nonlinearity will be treated in Section 5. In Section 6, we convert our results for (1.1) to the nonparameterized version (1.2). We study the special case of $n = 1$ and $\Omega = (0, \pi)$ in Section 7. In the paper, we denote by $||u||_2$ the L^2 norm for $u \in L^2(\Omega)$, and by $m(\Omega)$ the Lebesgue measure of Ω. Also C stands for a generic positive constant.

2. Preliminaries

Let $W(x) \in L^\infty(\Omega)$. Consider an eigenvalue problem:

$$\Delta\phi + W(x)\phi = -\mu_i(W)\phi \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

It is well-known that, for $i = 1, 2, \ldots,$

$$\mu_i(W) = \mathrm{Min}_i\, \mathrm{Max}_i \frac{\int_\Omega (|\nabla z|^2 - W(x)z^2)\,\mathrm{d}x}{\int_\Omega z^2\,\mathrm{d}x}, \tag{2.2}$$

where Max_i is over all $z(\neq 0) \in T_i$, and Min_i is over all linear subspaces T_i of $H_0^1(\Omega)$ of dimension i. For $W_1, W_2 \in L^\infty(\Omega)$ satisfying $W_2(x) \geqslant W_1(x)$ almost everywhere, $\mu_i(W_2) \leqslant \mu_i(W_1)$. If in addition $m(\{W_2 > W_1\}) > 0$, then $\mu_i(W_2) < \mu_i(W_1)$.

Let f satisfy (f1), (f2) and let f be superlinear. Then for $u \in \mathbf{R} \setminus \{0\}$, we have

$$f'(0) < \frac{f(u)}{u} < f'(u) \leqslant \sup_{u\in\mathbf{R}} f'(u),$$
$$\frac{f(u)}{u} < f_+ \quad \text{if } u > 0, \quad \text{and} \quad \frac{f(u)}{u} < f_- \quad \text{if } u < 0.$$

Consequently, if $u(\cdot)$ is a non-trivial solution of (1.1), then

$$\mu_j(\lambda f'(0)) > \mu_j(\lambda f(u)/u) > \mu_j(\lambda f'(u)) \geqslant \mu_j\left(\lambda \sup_{u\in\mathbf{R}} f'(u)\right),$$
$$\mu_j(\lambda f(u)/u) > \min\{\mu_j(\lambda f_+), \mu_j(\lambda f_-)\}. \tag{2.3}$$

As in the introduction, we assume that $f_+ \geqslant f_-$, and we define $I_k = (\lambda_{k,+}^{\infty}, \lambda_k^0)$, and $\widetilde{I_k} = (\lambda_k^M, \lambda_k^0)$. Obviously $I_k \subset \widetilde{I_k}$. The following lemma provides the basic spectral information of a nontrivial solution.

Lemma 2.1. *Suppose that f satisfies $(f1), (f2)$ and f is superlinear.*
(1) *If $\lambda \notin \bigcup_{j\in \mathbf{N}} I_j$, then* (1.1) *has no non-trivial solution.*
(2) *If $\lambda \in I_j \setminus \bigcup_{k\neq j} \widetilde{I_k}$, and u is a non-trivial solution to* (1.1), *then $M(u) = j$ and u is nondegenerate.*
(3) *If $\lambda \in (\bigcap_{k\leqslant i\leqslant j} I_i) \setminus (\bigcup_{i<k, i>j} \widetilde{I_i})$, and u is a nontrivial solution to* (1.1), *then $k \leqslant M(u) \leqslant j$.*

Proof. (1) If u is a nontrivial solution of (1.1), then $0 = \mu_j(\lambda f(u)/u)$ for some $j \geqslant 1$. By (2.3), $\mu_j(\lambda f'(0)) > \mu_j(\lambda f(u)/u) = 0 > \mu_j(\lambda f_+)$, thus $\lambda \in I_j$.

(2) If $\lambda \in I_j$, and $\lambda \notin \widetilde{I_k}$ for any other $k \neq j$, then for any $k < j$, $\mu_k(\lambda f'(0))$ and $\mu_k(\lambda \sup_{u\in\mathbf{R}} f'(u))$ are both nonpositive, and by (2.3), $\mu_k(\lambda f'(u)) < 0$ and $\mu_k(\lambda f(u)/u) < 0$. Similarly, for $k > j$, $\mu_k(\lambda f'(u)) > 0$ and $\mu_k(\lambda f(u)/u) > 0$. On the other hand, $0 = \mu_l(\lambda f(u)/u)$ for some $l \geqslant 1$. Hence $l = j$, and by (2.3) $\mu_j(\lambda f'(u)) < 0$. Therefore $M(u) = j$ and $0 \neq \mu_k(\lambda f'(u))$ for any $k \in \mathbf{N}$. The proof of (3) is similar to that of (2). □

The bifurcation from infinity plays an important role in our bifurcation analysis. We say that λ_* is a point where *a bifurcation from infinity* occurs for (1.1) if there exists a sequence $\lambda^k \to \lambda_*$ as $k \to \infty$ such that u_k is a solution of (1.1) with $\lambda = \lambda^k$ and $||u_k||_2 \to \infty$. Therefore, all the solutions are *a priori* bounded in $L^2(\Omega)$ norm if λ is away from the points where bifurcation from infinity occur. (See [30] for more on other results on bifurcation from infinity.)

Lemma 2.2. *Suppose that f satisfies $(f1)$ and $(f2)$, and λ_* is a point where a bifurcation from infinity occurs. Then the pair $(a,b) = (\lambda_* f_+, \lambda_* f_-) \in \Gamma$.*

Proof. We define $\phi_k(x) = ||u_k||_2^{-1} u_k(x)$, then ϕ_k satisfies

$$\Delta\phi_k + \lambda^k \frac{f(u_k)}{u_k}\phi_k = 0. \tag{2.4}$$

We multiply (2.4) by ϕ_k and integrate over Ω, then we obtain

$$\int_\Omega |\nabla\phi_k|^2 \,\mathrm{d}x - \lambda^k \int_\Omega \frac{f(u_k)}{u_k}\phi_k^2 \,\mathrm{d}x = 0. \tag{2.5}$$

Since $f(u)/u$ is bounded by (f2), then $||\phi_k||_{H_0^1(\Omega)}$ is uniformly bounded. Thus there exists $\phi \in H_0^1(\Omega)$ such that $\{\phi_k\}$ has a subsequence (which we still denote by $\{\phi_k\}$) converging to ϕ strongly in $L^2(\Omega)$, and weakly in $H_0^1(\Omega)$. Let $\Omega^+ = \{x \in \Omega\colon \phi(x) > 0\}$ and $\Omega^- = \{x \in \Omega\colon \phi(x) < 0\}$. Then $u_k(x) = ||u_k||_2\phi_k(x) \to \pm\infty$ as $k \to \infty$ for

$x \in \Omega^+ \cup \Omega^-$, thus

$$\frac{f(u_k(x))}{u_k(x)} \to f_+, x \in \Omega^+, \quad \text{and} \quad \frac{f(u_k(x))}{u_k(x)} \to f_-, x \in \Omega^-, \tag{2.6}$$

by Lebesgue Control Convergence Theorem.

Let $\psi \in C_0^1(\Omega)$. We multiply (2.4) by ψ and integrate over Ω, then we obtain (here $\Omega^0 = \Omega \setminus (\Omega^+ \cup \Omega^-)$)

$$\int_\Omega \nabla\phi_k \cdot \nabla\psi \, dx - \lambda^k \int_{\Omega^+} \frac{f(u_k)}{u_k} \phi_k \psi \, dx$$
$$-\lambda^k \int_{\Omega^-} \frac{f(u_k)}{u_k} \phi_k \psi \, dx - \lambda^k \int_{\Omega^0} \frac{f(u_k)}{u_k} \phi_k \psi \, dx = 0. \tag{2.7}$$

By the weak convergence of ϕ_k and (2.6), we obtain

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, dx - \int_\Omega (\lambda_* f_+ \phi^+ - \lambda_* f_- \phi^-)\psi \, dx = 0, \tag{2.8}$$

and we conclude that ϕ is a weak solution of (1.6) with $(a,b) = (\lambda_* f_+, \lambda_* f_-)$. □

In Section 5, we will use a result by Ruf [28] to show that if $|f_+ - f_-|$ is small enough and all eigenvalues λ_ks are simple, then there exists two $\lambda \in [\lambda_k/f_+, \lambda_k/f_-]$ such that $(\lambda f_+, \lambda f_-) \in \Gamma$, and these two points are both points where a bifurcation from infinity occurs. This is a generalization of bifurcation from a simple eigenvalue and from infinity when $f_+ = f_-$ by Crandall and Rabinowitz [14,26]. In general, it is not known for $(\lambda f_+, \lambda f_-) \in \Gamma$, whether λ is a point where a bifurcation from infinity occurs if it is not a "simple Fučík eigenvalue".

We close this section by a result about the turning directions of the bifurcation curves:

Lemma 2.3. *Suppose that f satisfies $(f1), (f2), f_+ = f_-$ and f is superlinear.*

(1) *If $\lambda_* = \lambda_j^0$ is a point where a bifurcation from the trivial solutions occurs for* (1.1) (*see Theorem* 1.7 *in* [14]), *and* $(\lambda(s), u(s))$, $|s| < \delta$, *is the solution curve of* (1.1) *bifurcating from* $(\lambda, u) = (\lambda_j^0, 0)$, *then* $\lambda(s) < \lambda_*$ *for* $0 < |s| < \delta$.

(2) *If $\lambda_* = \lambda_j^\infty$ is a point where a bifurcation from infinity occurs for* (1.1) (*see Theorem* 1.6 *and Corollary* 1.8 *in* [26]), *and* $(\lambda(s), u(s))$, $|s| > \delta$, *is the solution curve of* (1.1) *bifurcating from* $(\lambda, u) = (\lambda_j^\infty, \infty)$, *then* $\lambda(s) > \lambda*$ *for* $|s| > \delta$.

Proof. We first prove (1). Let $\mu_k(s) = \mu_k(\lambda(s) f'(u(s)))$ and let $\overline{\mu_k}(s) = \mu_k(\lambda(s) f(u(s))/u(s))$ for $k \in \mathbf{N}$. Then by Lemma 2.1, $\mu_k(s) < \overline{\mu_k}(s)$ for $|s|$ sufficiently small. On the other hand, $0 = \overline{\mu_k}(s)$ for some k. Since $\overline{\mu_j}(0) = 0$ and $\overline{\mu_k}(s)$ is continuous with respect to s, then $\overline{\mu_j}(s) = 0$ for $< |s| < \delta$. Thus $\mu_j(s) > 0$ and $M(u(s)) = j - 1$. Now we can apply Theorem 1.16 in [15], and obtain $sign(\lambda'(s)) = sign(s)$ for $0 < |s| < \delta$. In particular, $\lambda(s) > \lambda*$ for $0 < |s| < \delta$. The proof of (2) is similar, but we have to use a version of Theorem 1.16 in [15] for the bifurcation from infinity, which is not found in well-known references. For completeness, we provide such a result in Appendix A. □

3. Exact multiplicity when eigenvalues are separated

Our first main result is

Theorem 3.1. *Suppose that f satisfies* $(f1),(f2), f_+ = f_-$, *and f is superlinear. Then* (1.1) *has only the trivial solution if* $\lambda \notin \bigcup_{k\geqslant 1} I_k$, *and has exactly two nontrivial solutions* $u_k^{\pm}(\lambda,\cdot)$ *if* $\lambda \in I_k \setminus \bigcup_{j\neq k} \tilde{I}_j$. *Moreover*, $u_k^{\pm}(\lambda,\cdot)$ *are nondegenerate and* $M(u_k^{\pm}(\lambda,\cdot)) = k$.

Proof. By Lemma 2.1, (1.1) has only the trivial solution if $\lambda \notin \bigcup_{k\geqslant 1} I_k$. From Lemma 2.1, if u is a nontrivial solution of (1.1) for $\lambda \in I_k \setminus \bigcup_{j\neq k} \tilde{I}_j$, then u is nondegenerate and its Morse index $M(u) = k$. On the other hand, for such λ, the Morse index of the trivial solution is $M(0) = k-1$.

To prove that there are exactly two nontrivial solutions, we use a Leray–Schauder degree argument. Let $X = L^2(\Omega)$. It is well-known that (1.1) is equivalent to $\Psi(u) = u - \lambda LF(u) = 0$, $u \in X$, where $L = (-\Delta)^{-1} : X \to H^2(\Omega) \cap H_0^1(\Omega)$ is a compact operator and F is the Nemiskii operator associated with f. For any nondegenerate nontrivial solution u of (1.1) with $M(u) = k$, there exists a neighborhood N_u of u in X such that $deg(\Psi, N_u, 0) = (-1)^k$. Similarly, $deg(\Psi, N_0, 0) = (-1)^{k-1}$, where N_0 is a small neighborhood of 0 in X.

Next we show that $deg(\Psi, B(R,0), 0) = (-1)^k$, where $B(R,0) = \{x \in X: ||x||_2 \leqslant R\}$, $R > 0$ is a constant depending only on λ and $\lambda \in (\lambda_k^\infty, \lambda_{k+1}^\infty)$. Define a homotopy mapping: $H(t,u) = (1-t)\Psi u + t[u - \lambda f_+ Lu]$. We notice that for $\lambda \in (\lambda_k^\infty, \lambda_{k+1}^\infty)$, $u - \lambda f_+ Lu = 0$ holds only when $u = 0$ and $deg(I - \lambda f_+ L, B(R,0), 0) = (-1)^k$ for any $R > 0$. So it remains to prove $H(t,u) \neq 0$ where $0 \leqslant t \leqslant 1$ and $u \in \partial B(R,0)$ for some $R > 0$. Suppose this is not true, then by taking a subsequence, we obtain a sequence (t_k, u_k), such that $t_k \to t_*, ||u_k||_2 \to \infty$ and $u_k/||u_k||_2 \to u_*$ (weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$) as $k \to \infty$. Then similar to the proof of Lemma 2.2, (t_*, u_*) satisfies $\Delta u_* + (1-t_*)\lambda f_+ u_* + t_*\lambda f_+ u_* = 0$ and $u_* = 0$ on $\partial\Omega$, which is a contradiction to the choice of λ. Therefore, $deg(\Psi, B(R,0), 0) = deg(u - \lambda f_+ Lu, B(R,0), 0) = (-1)^k$. Then the additivity of Leray–Schauder degree implies (1.1) has at exactly two (nontrivial) solutions in $B(R,0) \setminus \overline{N_0}$ for $\lambda \in I_k \setminus \bigcup_{j\neq k} \tilde{I}_j$. □

Now we do a bifurcation analysis to the solutions which we obtain in Theorem 3.1. For the simplicity, we assume that $f_+ = f_- = \sup_{u\in\mathbf{R}} f'(u)$ and $I_k = \tilde{I}_k = (\lambda_k^\infty, \lambda_k^0)$. If there exists $\lambda \in I_k \setminus \bigcup_{j\neq k} I_j$, then $I_{k-1} \cap I_{k+1} = \emptyset$, and there are four possible alignments of I_js $(j = k-1, k, k+1)$:

(1) $I_{k-1} \cap I_k = I_k \cap I_{k+1} = \emptyset$, $\lambda_{k-1}^0 \leqslant \lambda_k^\infty < \lambda_k^0 \leqslant \lambda_{k+1}^\infty$;

(2) $I_{k-1} \cap I_k \neq \emptyset$, $I_k \cap I_{k+1} = \emptyset$, $\lambda_k^\infty < \lambda_{k-1}^0 < \lambda_k^0 \leqslant \lambda_{k+1}^\infty$;

(3) $I_{k-1} \cap I_k = \emptyset$, $I_k \cap I_{k+1} \neq \emptyset$, $\lambda_{k-1}^0 \leqslant \lambda_k^\infty < \lambda_{k+1}^\infty < \lambda_k^0$;

(4) $I_{k-1} \cap I_k \neq \emptyset$, $I_k \cap I_{k+1} \neq \emptyset$, $\lambda_k^\infty < \lambda_{k-1}^0 < \lambda_{k+1}^\infty < \lambda_k^0$.

If $\lambda \in (a,b)$, where $a = \max\{\lambda_{k-1}^0, \lambda_k^\infty\}$ and $b = \min\{\lambda_k^0, \lambda_{k+1}^\infty\}$, then the hypotheses in Theorem 3.1 are satisfied and (1.1) has exactly two nontrivial solutions. In Case (1), $(a,b) = I_k$, so I_k is separated from other I_js, and we obtain a “whole” branch of solu-

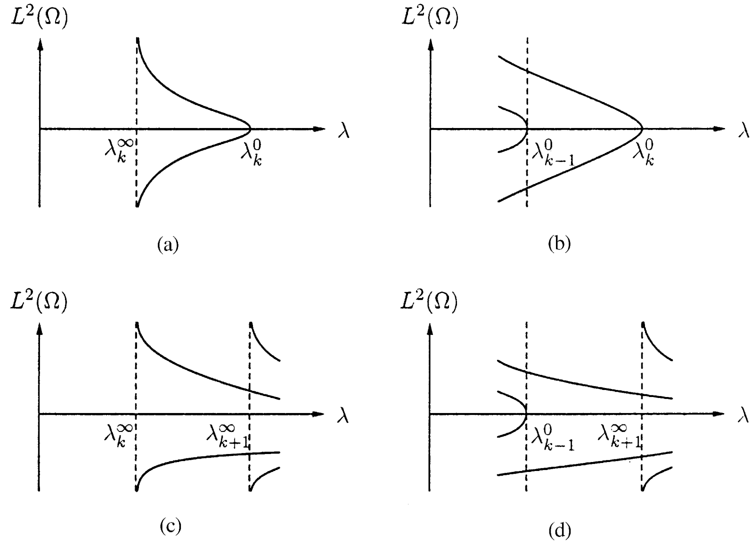

Fig. 2. (A) $(a,b)=(\lambda_k^\infty,\lambda_k^0)$; (B) $(a,b)=(\lambda_{k-1}^0,\lambda_k^0)$; (C) $(a,b)=(\lambda_k^\infty,\lambda_{k+1}^\infty)$; (D) $(a,b)=(\lambda_{k-1}^0,\lambda_{k+1}^\infty)$.

tions, which is the case discussed in [31] (see Fig. 2A); in Case (2), $(a,b)=(\lambda_{k-1}^0,\lambda_k^0)$, there is a bifurcation from the trivial solutions at $\lambda=\lambda_k^0$, and as $\lambda\to(\lambda_{k-1}^0)^+$, the solutions are bounded (see Fig. 2B); in Case (3), $(a,b)=(\lambda_k^\infty,\lambda_{k+1}^\infty)$, there is a bifurcation from infinity occurring at $\lambda=\lambda_k^\infty$, and as $\lambda\to(\lambda_{k+1}^\infty)^-$, the norm of the solutions is bounded from below (see Fig. 2C); and in Case 4, $(a,b)=(\lambda_{k-1}^0,\lambda_{k+1}^\infty)$, there is no bifurcation in either end of (a,b) and all solutions satisfy $m\leqslant\|u\|\leqslant M$ for some $m,M>0$ (see Fig. 2D).

Next we show that in Cases (2), (3) and (4), the loose ends of the bifurcation curves can be extended a little bit, and we obtain exactly four solutions in the extended parts:

Theorem 3.2. *Suppose that the conditions in Theorem* 3.1 *are satisfied. In addition we assume* $f_+=f_-=\sup_{u\in\mathbf{R}}f'(u)$, $\tilde I_{k-1}\cap\tilde I_{k+1}=\emptyset$, *and the bounded interval* (a,b) *is defined as in above. Then there exists* $\varepsilon>0$ *such that*
(1) *if* $a=\lambda_{k-1}^0$, *then for* $\lambda\in(\lambda_{k-1}^0-\varepsilon,\lambda_{k-1}^0)$, (1.1) *has exactly four nontrivial solutions*;
(2) *if* $b=\lambda_{k+1}^\infty$, *then for* $\lambda\in(\lambda_{k+1}^\infty,\lambda_{k+1}^\infty+\varepsilon)$, (1.1) *has exactly four nontrivial solutions.*

Proof. Suppose that $a=\lambda_{k-1}^0$. For $\lambda\in[a,a+\varepsilon_1),\lambda\in I_k\setminus\bigcup_{j\neq k}I_j$. Thus (1.1) has exactly two nontrivial solutions $u_1(\lambda),u_2(\lambda)$ for $\lambda\in[a,a+\varepsilon_1)$ by Theorem 3.1. Since $u_i(a)$ $(i=1,2)$ is nondegenerate, then by the implicit function theorem, there exists $\varepsilon_2>0$ such that (1.1) has exactly one solution $u_i(\lambda)$ near $u_i(a)$ for $\lambda\in(a-\varepsilon_2,a)$. On the other hand, $(a,0)$ is a point where a bifurcation from the trivial solutions occurs, so there exists $\varepsilon_3>0$, such that for $\lambda\in(a-\varepsilon_3,a)$, (1.1) has another two solutions $u_3(\lambda)$ and $u_4(\lambda)$ near $(a,0)$ for $\lambda\in(a-\varepsilon_3,a)$. Note that by Lemma 2.3, the bifurcation at

$(a,0)$ is subcritical. Then for $\varepsilon_4 = \min(\varepsilon_2, \varepsilon_3)$, (1.1) has at least four nontrivial solutions $u_i(\lambda)$ for $\lambda \in (a - \varepsilon_4, a)$.

We claim that there exists $\varepsilon_5 \in (0, \varepsilon_4)$ such that for $\lambda \in (a - \varepsilon_5, a)$, (1.1) has exactly these four nontrivial solutions. Suppose it is not true, then there exist a sequence $\lambda^k \to a^-$, such that (1.1) has another solution, say $u_5(\lambda^k)$. If $\overline{\lim}_{\lambda^k \to a^-} ||u_5(\lambda)||_2 = \infty$, then $\lambda = a$ is a bifurcation point where a bifurcation from infinity occurs. By Lemma 2.2, $a = \lambda_j^\infty$, which is impossible by our assumptions. If $\overline{\lim}_{\lambda \to a^-} ||u_5(\lambda^k)||_2 < \infty$, by taking a subsequence, we can assume that $\overline{\lim}_{\lambda \to a^-} ||u_5(\lambda^k)||_2 = K \geqslant 0$. Since $f(u)/u$ is bounded for all $u \in \mathbf{R}$, then by (2.5), $||u_5(\lambda^k)||_{H_0^1(\Omega)}$ is uniformly bounded for all λ^ks. Therefore, there exists $v \in H_0^1(\Omega)$ such that $u_5(\lambda^k)$ has a subsequence converging to v strongly in $L^2(\Omega)$, and weakly in $H_0^1(\Omega)$. Moreover, $||v||_2 = K$ and v is a weak solution to (1.1), with $\lambda = a$. However, for $\lambda = a$, (1.1) has exactly three solutions $u_1(a)$, $u_2(a)$ and 0. So v is the same as one of these three solutions, and $u_5(\lambda)$ must be identical to one of $u_i(\lambda)$ $(i = 1, 2, 3, 4)$, that reaches a contradiction. The proof of (2) is similar. □

4. On the solutions bifurcating from first two eigenvalues

For superlinear or sublinear f, we have some better results on the first two solution curves bifurcated from λ_1^0 and λ_2^0. Here for simplicity, we assume $f_+ = f_-$. The following result is essentially known, and we use our bifurcation approach to give a simple proof for the superlinear part. The result for superlinear f was proved by Castro et al. [10], and the result for sublinear f is well-known, see for example [1].

Theorem 4.1. *Let f satisfy $(f1)$, $(f2)$, $f_+ = f_-$.*

(1) *Suppose that f is superlinear. For $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$, (1.1) has exact two solutions $u_1(\lambda)$, $u_2(\lambda)$ which change sign, $M(u_i(\lambda)) = 2$, and $u_i(\lambda)$ changes sign exactly once $(i = 1, 2)$;*

(2) *Suppose that f is sublinear. For $\lambda \in I_1 = (\lambda_1^0, \lambda_1^\infty)$, (1.1) has exact two solutions $u_1(\lambda)$, $u_2(\lambda)$ which are of one sign, one positive, one negative, and $M(u_i(\lambda)) = 0$.*

Proof. (1) First we recall a result by Bahri and Lions [4]: if f is superlinear, u is a nontrivial solution of (1.1), and $N(u)$ is the number of nodal domains of u in Ω (a nodal domain is a connected component of $\{x \in \Omega: u(x) \neq 0\}$), then $N(u) \leqslant M(u)$. If $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$, then by Lemma 2.1, $M(u) = 1$ or 2 for any nontrivial solution u. If u is a sign-changing solution, then $N(u) \geqslant 2$, thus $M(u) = N(u) = 2$ for any sign-changing solution u whenever $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$.

By [26], $\lambda = \lambda_2^\infty$ is a point where a bifurcation from infinity occurs, and there are two solution curves $u_1(\lambda)$, $u_2(\lambda)$, for $\lambda \in (\lambda_2^\infty, \lambda_2^\infty + \varepsilon)$ which bifurcate to the right of λ_2^∞. Since $u_i(\lambda)/||u_i(\lambda)||_2 \to \phi_2$, the eigenfunction corresponding to λ_2, as $\lambda \to (\lambda_2^\infty)^-$, then $u_i(\lambda)$ $(i = 1, 2)$ are both sign changing solutions, thus $M(u_i(\lambda)) = N(u_i(\lambda)) = 2$. In particular, $u_i(\lambda)$ are nondegenerate as long as $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$, so the solution curves $\{u_i(\lambda): \lambda \in (\lambda_2^\infty, \lambda_2^\infty + \varepsilon)\}$ can be extended to at least $\lambda = \lambda_3^M$ without any turning point. On the other hand, by a similar argument as in the proof of Theorem 3.2, there is no any other sign-changing solution for $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$. The proof for (2) is well-known, see [1]. □

For superlinear f, and $\lambda \in I_1$, it is well known that (1.1) has one positive solution and one negative solution by Mountain Pass Lemma. So for $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$, there are at least four nontrivial solutions, and by Theorem 3.2, four is the exact number for $\lambda \in (\lambda_2^\infty, \lambda_2^\infty + \varepsilon)$ (and $\lambda \in (\lambda_1^0 - \varepsilon, \lambda_2^0)$ if $\tilde{I}_3$ is disjoint from $I_1 \cap I_2$). It is tempting to think whether four will be the exact number for all $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$. However, we construct an example to show this is not true in general.

Proposition 4.2. *There is $\varepsilon > 0$, a bounded domain Ω and a smooth superlinear function f which satisfies $(f1)$, $(f2)$ such that $(I_1 \cup I_2) \cap (\bigcup_{j>2} \tilde{I}_j) = \emptyset$, and for $\lambda \in (\lambda_2^\infty + \varepsilon, \lambda_1^0 - \varepsilon)$, (1.1) has exactly eight nontrivial solutions. In these eight solutions, three are positive, three are negative, the other two changes sign exactly once, four of them have Morse index $M(u) = 1$ and the other four have Morse index $M(u) = 2$.*

Proof. Our construction follows Dancer [16]. Let $B = B_1 \cup B_2$, where B_1 and B_2 are two disjoint open balls in $\mathbf{R}^n$ with radius R_1 and R_2 ($R_1 > R_2$). Let E be a compact set in $\mathbf{R}^n$ with measure zero such that $\bar{B} \cup E$ is connected. We assume that $\{\Omega_m\}$ ($m \geqslant 1$) is a sequence of connected bounded domains such that: (i) given any compact subset K of B, $\Omega_n \supset K$ for large n and (ii) given any open set O containing $\bar{B} \cup E$, $\Omega_m \subset O$ for large m. Thus Ω_m is a domain close to B with a shape of unequal-sized dumb-bell.

We denote the eigenvalues of $-\Delta$ on $H_0^1(B_1)$ and $H_0^1(B_2)$ by $\lambda_k(B_1)$ and $\lambda_k(B_2)$. The function f is chosen such that (f1), (f2) are satisfied, f is superlinear, $f_+ = f_-$ and

$$\lambda_1^\infty(B_1) < \lambda_1^\infty(B_2) < \lambda_1^0(B_1) < \lambda_1^0(B_2) < \lambda_2^\infty(B_1),$$

where $\lambda_k^\infty(B_i)$, $\lambda_k^0(B_i)$ are defined as in (1.4) for $k > 0$ and $i = 1, 2$. The inequality can be satisfied if we follow such a procedure: first fix R_1, then choose f such that $\lambda_1^0(B_1) < \lambda_2^\infty(B_1)$, and finally choose R_2 such that R_2 is slightly smaller than R_1.

Since $\lambda_1^0(B_1) < \lambda_2^\infty(B_1)$ and $\lambda_1^0(B_2) < \lambda_2^\infty(B_2)$, then by Theorem 3.1 (or Theorem 1.3 in [31]), there is a solution curve Σ_i (with two branches) bifurcating from $(\lambda_1^0(B_i), 0)$, continuing to the left, and eventually blowing up at $(\lambda_1^\infty(B_i), \infty)$, for (1.1) with $\Omega = B_i$, $i = 1, 2$. All other solutions of (1.1) for $\Omega = B_i$, $i = 1, 2$ are separated from these two curves by the strip $\{(\lambda, u)\colon \lambda_1^0(B_2) < \lambda < \lambda_2^\infty(B_1), u \in L^2(\Omega)\}$. Moreover, each Σ_i has two branches $\Sigma_i^\pm$, the solutions on Σ_i^+ are positive and the solutions on Σ_i^- are negative. Therefore, for $\lambda \in (\lambda_1^\infty(B_2), \lambda_1^0(B_1))$, (1.1) with $\Omega = B_1$ has exactly two nontrivial solutions: $u_1^+(\lambda)$ and $u_1^-(\lambda)$; (1.1) with $\Omega = B_2$ has exactly two nontrivial solutions: $u_2^+(\lambda)$ and $u_2^-(\lambda)$. Since $u = 0$ is also a solution to both equations, then (1.1) with $\Omega = B$ has exactly eight solutions which are not entirely vanishing in B.

By Theorem 1 in [16], for m sufficiently large, for any nondegenerate solution u_0 of (1.1) with $\Omega = B$, (1.1) with $\Omega = \Omega_m$ has a solution u_m close to u_0. Moreover, if the solutions of (1.1) have an a priori bound, then the number of the solutions of (1.1) with $\Omega = \Omega_n$ is the same as that of $\Omega = B$ for large m. So if we choose

$\lambda \in (\lambda_1^{\infty}(B_2) + \varepsilon, \lambda_1^0(B_1) - \varepsilon)$, then a uniform a priori bound can be found since we exclúse the bifurcation points. Therefore, for m large enough, with $\Omega = \Omega_m$, (1.1) has exactly eight nontrivial solutions. By Theorem 4.1 (1), only two of them are sign-changing, which can only be the solutions which are close to $u_1^+(\lambda) + u_2^-(\lambda)$ and $u_1^-(\lambda) + u_2^+(\lambda)$. The three solutions which has a positive component have to be positive, and the three which has a negative component have to be negative (this is true here by Theorem 4.1, but a more general proof can be found in Theorem 2 in [16]). The Morse indices of these solutions can also be easily computed since the spectral properties for $\Omega = \Omega_m$ are inherited from that of $\Omega = B$ by Theorem 1 in [16]. □

We conjecture that if λ_1 and λ_2 are both simple eigenvalues, then the number of the nontrivial solutions of (1.1) is *always* between 4 and 8 for $\lambda \in (I_1 \cap I_2) \setminus (\bigcup_{i>2} \tilde{I}_i)$ if f satisfies (f1), (f2) and f is superlinear.

For sublinear f, we can construct a similar example. Here, for $\lambda \in (I_1 \cap I_2) \setminus \bigcup_{i>2} \tilde{I}_i$, we have one positive solution, one negative solution, both stable ($M(u) = 0$); two stable sign-changing solutions ($M(u) = 0$) and four unstable sign-changing solutions ($M(u) = 1$). It is interesting that, in this case, (1.1) possesses some sign-changing solutions which are stable, and they are the local minimizers of the energy functional. It was conjectured by Ni [23] that a stable solution of (1.1) with any f must be of one sign if Ω is convex, and here we provide an example for the necessity of the convexity since Ω_m is dumbbell-shaped.

5. Bifurcation from a simple split eigenvalue

In this section, we assume $f_+ > f_-$. Recall that $I_k = (\lambda_{k,+}^{\infty}, \lambda_k^0)$, and $\tilde{I}_k = (\lambda_k^M, \lambda_k^0)$. Our main result is

Theorem 5.1. *Suppose that f satisfies $(f1)$, $(f2)$, f is superlinear, and $I_k \cap \left(\bigcup_{j \neq k} \tilde{I}_j\right) = \emptyset$. Then there exists $\varepsilon > 0$ such that for $f_+/f_- \in (1, 1+\varepsilon)$, there exist $\lambda_{k,+}, \lambda_{k,-} \in [\lambda_{k,+}^{\infty}, \lambda_{k,-}^{\infty}]$ such that* (1.1) *has only the trivial solution if $\lambda \in (\lambda_k^M, \lambda_{k,+}]$, has exactly one nontrivial solution if $\lambda \in (\lambda_{k,+}, \lambda_{k,-}]$ and has exactly two nontrivial solutions if $\lambda \in (\lambda_{k,-}, \lambda_k^0)$, all these solutions have Morse index $M(u) = k$. Moreover, all nontrivial solutions of* (1.1) *with $\lambda \in I_k$ lie on two smooth curves $\Sigma_k^+ = \{(\lambda, u_i^+(\lambda, \cdot)) \colon \lambda \in (\lambda_{k,+}, \lambda_k^0)\}$ and $\Sigma_k^- = \{(\lambda, u_i^-(\lambda, \cdot)) \colon \lambda \in (\lambda_{k,-}, \lambda_k^0)\}$, which join at $(\lambda_k^0, 0)$, and* (Fig. 3)

$$\lim_{\lambda \to (\lambda_{k,\pm})^+} ||u_k^{\pm}(\lambda, \cdot)||_{L^2(\Omega)} = \infty. \tag{5.1}$$

The most parts of the proof are the same as that of Theorem 1.1 in [31]. We briefly sketch the proof here: λ_k^0 is a point where a bifurcation from the trivial solutions occurs, and the bifurcation is subcritical. So $\Sigma_k^{\pm}$ continues to the left. However, since I_k is separated from other $\tilde{I}_j$'s, $\Sigma_k^{\pm}$ cannot go beyond $\lambda = \lambda_{k,+}^{\infty}$. So both Σ_k^+ and Σ_k^- blow up before $\lambda = \lambda_{k,+}^{\infty}$. By the same arguments as in the proof of Theorem 3.2, we can show there are no solutions other than the ones on $\Sigma_k^{\pm}$. The blow-up points must be

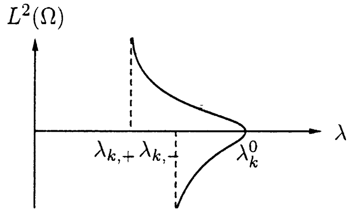

Fig. 3. Bifurcation curves split at ∞.

points where bifurcation from infinity occur, so by Lemma 2.2, if λ_* is such a point, then $(\lambda_* f_\pm, \lambda_* f_-) \in \Gamma$, the Fučík spectrum.

The proof can be completed as long as the following proposition is true.

Proposition 5.2. *Suppose the assumptions in Theorem* 5.1 *hold and* $k \geqslant 2$, *then there exists* $\varepsilon > 0$ *and for each* $f_+/f_- \in (1, 1+\varepsilon)$, *there exist exactly two numbers* $\lambda = \lambda_{k,\pm}$ *in* I_k *such that* $(\lambda f_+, \lambda f_-) \in \Gamma$. *Moreover*, $\lambda_{k,\pm}$ *are both the points where a bifurcation from infinity occur.*

Proof. Let $p = f_+/f_-$. From the properties of Fučík Spectrum, near $(a, b) = (\lambda_k, \lambda_k)$, there exist two decreasing curves $b = \gamma_{k,1}(a)$ and $b = \gamma_{k,2}(a)$ and a neighborhood N of (λ_k, λ_k) such that $\Gamma \cap N = \{(a, \gamma_{k,i}(a))\colon a \in (\lambda_k - \delta, \lambda_k + \delta), i = 1, 2\}$. Thus there exists $\varepsilon > 0$ such that for $p \in (1-\varepsilon, 1+\varepsilon)$, $b = pa$ and $b = \gamma_{k,i}(a)\,(i = 1, 2)$ has exactly one intersection point in N. Moreover, ε, δ and N can be chosen in the way that $(\{(a, pa)\colon a > 0, p \in (1-\varepsilon, 1+\varepsilon)\} \cap \{(a, b)\colon a > \lambda_k, b < \lambda_k\}) \subset N$. It is known [29] that for $P_k = \{(a, b)\colon \lambda_{k-1} < a, b < \lambda_k\}$, $\Gamma \cap (P_{k-1} \cup P_k) = \emptyset$. So inside Q_k, there are exactly two points on $b = pa$ which belong to Γ, say (a_1, pa_1) and (a_2, pa_2). Then $\lambda = \lambda_{k,\pm}$ are defined as $\lambda_{k,+} = a_1/f_+$ and $\lambda_{k,-} = a_2/f_+$. (We can re-index them to make $\lambda_{k,+} > \lambda_{k,-}$.)

Next we prove that the bifurcation from infinity occur at $\lambda_{k,\pm}$. We rewrite (1.1) as

$$\Delta u + \lambda f_+ u^+ + \lambda f_- u^- + N(\lambda, u) = 0,$$

where

$$N(\lambda, u) = \lambda \left[\frac{f(u^+)}{u^+} - f_+ \right] u^+ - \lambda \left[\frac{f(u^-)}{u^-} - f_- \right] u^-.$$

Let $w = u/||u||_2^2$, then the equation of w is

$$\Delta w + \lambda f_+ w^+ + \lambda f_- w^- + N_1(\lambda, w) = 0,$$

where $N_1(\lambda, w) = ||w||_2^2 N(\lambda, ||w||_2^{-2} w)$. It is standard to prove $||N_1(\lambda, w)||_2 = o(||w||_2)$ as $||w||_2 \to 0$ (which we omit here). Then from Theorem 1 in [28], $\lambda = \lambda_{k,\pm}$ are points where a bifurcation from the trivial solutions occurs, which is an extension of the bifurcation from simple eigenvalue by Crandall and Rabinowitz [14]. In [28], it is showed that λ is a bifurcation point for $\Delta u + \lambda u + \gamma u^- + N(\lambda, u) = 0$ in Ω and $u = 0$ on $\partial\Omega$ if $(\lambda, \lambda + \gamma) \in \Gamma$. The result in [28] cannot be directly applied here since the equation is in a slightly different form, but the proof still works without essential change. So we would just refer the proof to [28]. By the inverse transformation $u = w/||w||_2^2$, we conclude that $\lambda_{k,\pm}$ are points where a bifurcation from infinity occurs. □

Now we complete the proof of Theorem 5.1.

Completion of Proof of Theorem 5.1. The solution curves $\Sigma_k^{\pm}$ which bifurcate from $(\lambda_k^0, 0)$ must blow up to ∞ at $\lambda = \lambda_{k,\pm}$. Suppose that the two branches both blow up at the $\lambda = \lambda_{k,+} (< \lambda_{k,-})$. $\lambda_{k,-}$ is still a point where a bifurcation from infinity occurs, and by Lemma 2.3, the bifurcation is subcritical. So there exists a solution curve Σ_* (it is a curve since all solutions are nondegenerate), which continues to the right of $\lambda_{k,-}$. By our assumptions, Σ_* cannot go beyond $\lambda = \lambda_k^0$, and by the same proof as in that of Theorem 3.2, Σ_* cannot meet the line of trivial solutions. So the only possibility is that Σ_* blows up at $\lambda = \lambda_{k,+}$. However, the bifurcation from infinity at $\lambda = \lambda_{k,+}$ has to be subcritical by Lemma 2.3, so that is a contradiction. Therefore, Σ_k^+ and Σ_k^- have to blow up at two different points $\lambda_{k,+}$ and $\lambda_{k,-}$ respectively. There are no other solutions by the previous arguments in the proof of Theorem 3.2. □

6. Equation without the parameter λ

In this section, we restate our main results in the context of (1.2), the nonparameterized version. First, in the context of (1.2), our main results can be summarized as follows (we assume that f satisfies (f1), (f2) and f is superlinear):

(A) If $\lambda_k \leqslant f'(0) < f_\pm \leqslant \lambda_{k+1}$, then (1.2) has only the trivial solution $u = 0$;

(B) If $\lambda_{k-1} < f'(0) < \lambda_k < f_\pm \leqslant \sup_{u\in\mathbf{R}} f'(u) < \lambda_{k+1}$, then (1.2) has exactly two nontrivial solutions which are nondegenerate and with Morse index $M(u) = k$;

(C1) There exists $\varepsilon > 0$, such that if $\lambda_{k-1} - \varepsilon < f'(0) < \lambda_{k-1} < \lambda_k < f_\pm \leqslant \sup_{u\in\mathbf{R}} f'(u) < \lambda_{k+1}$ and $\lambda_k/\lambda_{k-1} < f_\pm/f'(0)$, then (1.2) has exactly four nontrivial solutions;

(C2) There exists $\varepsilon > 0$, such that if $\lambda_{k-1} < f'(0) < \lambda_k < \lambda_{k+1} < f_\pm < \lambda_{k+1} + \varepsilon$ and $\lambda_{k+1}/\lambda_k < f_\pm/f'(0)$, then (1.2) has exactly four nontrivial solutions;

(D) If $f'(0) < \lambda_1 < \lambda_2 < f_\pm \leqslant \sup_{u\in\mathbf{R}} f'(u) < \lambda_3$, then (1.2) has exactly two nontrivial sign-changing solutions which are nondegenerate and with Morse index $M(u) = 2$;

Result (B) here is essentially the same as the results of Castro and Lazer [11] and Ambrosetti and Mancini [2]. The conversion of the results for (1.1) to (1.2) is obvious. Let $g(\lambda, u) = \lambda f(u)$, then $\lambda_k/f_+ < \lambda < \lambda_k/f'(0)$ is equivalent to $g_u(\lambda, 0) < \lambda_k < g_u(\lambda, +\infty)$. So the conditions on the above results can all be converted from the conditions on equation (1.1). Notice that in above $f_+ \neq f_-$ is allowed, which is equivalent $\lambda \in (\max\{\lambda_k/f_+, \lambda_k/f_-\}, \lambda_k^0)$, not in the gap of two Fučík spectrum points. So the conclusions above can be easily drawn from the results in previous section.

Theorem 5.1 can also be restated for (1.2). In fact, we can combine Theorem 5.1 and a continuation argument to prove the following result which completely classifies the solution set when $f'(0) \in (\lambda_{k-1}, \lambda_k)$ and $(f_+, f_-) \in Q_k$:

Theorem 6.1. *Let f satisfy* ($f1$), ($f2$), *and let f be superlinear. Suppose $\lambda_{k-1} < f'(0) < \lambda_k$, $\sup_{u\in\mathbf{R}} f'(u) < \lambda_{k+1}$, and $(f_+, f_-) \in Q_k$. Then* (1.2)

has no nontrivial solution if $(f_+, f_-) \in \overline{I_2}$,
has exactly one nontrivial solution if $(f_+, f_-) \in \overline{I_3} \setminus \overline{I_2}$,
and has exactly two nontrivial solutions if $(f_+, f_-) \in I_1$.

Proof. From (A), (1.2) has no nontrivial solution when $(f_+, f_-) \in P_{k-1} = [\lambda_{k-1}, \lambda_k] \times [\lambda_{k-1}, \lambda_k]$, the lower-left square block of Q_k; and from (B), (1.2) has exactly two nontrivial solutions when $(f_+, f_-) \in P_k = [\lambda_k, \lambda_{k+1}] \times [\lambda_k, \lambda_{k+1}]$, the upper-right square block of Q_k. So we shall only consider $(f_+, f_-) \in Q_k \setminus (P_{k-1} \cup P_k)$. Because of the symmetry, we can further assume that $f_+ > f_-$ so (f_+, f_-) is in the lower-right block $P_{k,k-1} = [\lambda_k, \lambda_{k+1}] \times [\lambda_{k-1}, \lambda_k]$.

Let N be the neighborhood defined in Proposition 5.2. We first assume that $(f_+, f_-) \in P_{k,k-1} \cap N$. We construct a function $g : \mathbf{R} \to \mathbf{R}$ such that g satisfies (f1), (f2) and g is superlinear. Moreover $g_\pm = f_\pm$, $\lambda_k > g'(0) > f'(0)$, $ug''(u) \geqslant 0$ (so $\sup_{u\in\mathbf{R}} g'(u) = f_+$), and

$$\frac{\sup_{u\in\mathbf{R}} g'(u)}{g'(0)} = \frac{g_+}{g'(0)} \leqslant \min\left(\frac{\lambda_{k+1}}{\lambda_k}, \frac{\lambda_k}{\lambda_{k-1}}\right). \tag{6.1}$$

We first consider equation $\Delta u + g(u) = 0$ in Ω, and $u = 0$ on $\partial\Omega$. We embed $\Delta u + g(u) = 0$ into a family of equations $\Delta u + \lambda g(u) = 0$, then Theorem 5.1 can be applied: $I_k \cap (\bigcup_{j\neq k} \tilde{I}_j) = \emptyset$ by (6.1). Thus for $\lambda \in [\lambda_k/f_+, \lambda_k/f_-]$, there exists $\lambda_{k,\pm}$ such that $\Delta u + \lambda g(u) = 0$ has no nontrivial solution when $\lambda \in [\lambda_k/f_+, \lambda_{k,+}]$, has exactly one nontrivial solution when $\lambda \in (\lambda_{k,+}, \lambda_{k,-}]$, and has exactly two nontrivial solutions when $\lambda \in (\lambda_{k,-}, \lambda_k/f_-]$. Notice that when λ increases from λ_k/f_+ to λ_k/f_-, the pair $(\lambda f_+, \lambda f_-)$ slides along the ray $b = pa$ ($p = f_+/f_-$) crossing the Fučík spectrum curves twice. So the result in Theorem 1.1 is true for $g(u)$ and $(f_+, f_-) \in P_{k,k-1} \cap N$.

Next, we prove it is true for $f(u)$ and $(f_+, f_-) \in P_{k,k-1} \cap N$. Define a homotopy: $H_1(t,u) = t\Psi(u) + (1-t)\Phi(u)$,where $t \in [0,1]$, $\Psi(u) = u - LF(u)$, $\Phi(u) = u - LG(u)$, $L = (-\Delta)^{-1}$ as in the proof of Theorem 3.2, F and G are the Nemiskii operator associated with f and g, respectively. For any nontrivial solution u of $\Psi(u) = 0$, $deg(\Psi, N_u, 0) = (-1)^k$ for a small neighborhood N_u of u since u is nondegenerate and has Morse index k. And the same is true for Φ. Also, we have $deg(\Psi, N_0, 0) = deg(\Phi, N_0, 0) = (-1)^{k-1}$. For $R > 0$ large enough, $H(t,u) \neq 0$ for $t \in [0,1]$ and $u \in \partial B(R,0)$, since for $h(t,u) = tf(u) + (1-t)g(u)$, $h'(t,\pm\infty) = f_\pm$, $(h'(t,+\infty), h'(t,-\infty)) \notin \Gamma$, so all solutions are a priori bounded by the same argument in the proof of Theorem 3.2. Therefore the number of the solutions for (1.2) is the same as that of $\Delta u + g(u) = 0$.

Finally we prove it is true for $f(u)$ and $(f_+, f_-) \in P_{k,k-1} \setminus N$. First, we assume that $(f_+, f_-) \in I_3$. Let (a_1, b_1) and (a_2, b_2) be two points in $I_3 \cap P_{k,k-1}$ such that $t(a_1, b_1) + (1-t)(a_2, b_2) \subset I_3 \cap P_{k,k-1}$. Let g_i ($i = 1,2$) be two functions satisfying (f1), (f2) and g_i be superlinear. Moreover $g_{i+} = a_i$, $g_{i-} = b_i$, $\lambda_k > g_i'(0) > \lambda_{k-1}$, $i = 1,2$. Define a homotopy: $H_1(t,u) = t\Psi_1(u) + (1-t)\Psi_2(u)$, where $t \in [0,1]$, $\Psi_i(u) = u - LG_i(u)$, G_i is the Nemiskii operator associated with g_i for $i = 1,2$. Then similar to above, $deg(\Psi_i, N_u, 0) = (-1)^k$ for any small neighborhood of the nontrivial solution

u and $deg(\Psi_i, N_0, 0) = (-1)^{k-1}$ for any small neighborhood of the trivial solution 0. And for $R > 0$ large enough, $H_1(t,u) \neq 0$ for $t \in [0,1]$ and $u \in \partial B(R,0)$ since the path avoids the Fučík spectrum curves, so all solutions are a priori bounded by Lemma 2.2. In particular, the number of the solutions for $\Delta u + g_1(u) = 0$ is the same as that of $\Delta u + g_2(u) = 0$. Since $I_3 \cap P_{k,k-1}$ is path-connected, then for any $(a,b) \in I_3 \cap P_{k,k-1}$, there exists $(a_1,b_1) \in I_3 \cap N$ and a piecewise linear path which is entirely inside $I_3 \cap P_{k,k-1}$ connecting (a,b) and (a_1,b_1). Then for any $(a,b) \in I_3 \cap P_{k,k-1}$, if $(f_+, f_-) = (a,b)$, then (1.2) has exactly one nontrivial solution. The proof for $(f_+, f_-) \in I_1$ or I_2 is similar. □

7. The special case of $n = 1$

In this section, we consider (1.1) for $n = 1$ and $\Omega = (0, \pi)$:

$$u'' + \lambda f(u) = 0, \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0. \tag{7.1}$$

We would completely classify all the solutions of (7.1) for f satisfying (f1), (f2) and f being superlinear or sublinear. Our approach here is similar to [31] Section 4. Our main result is

Theorem 7.1. *Suppose that f satisfies ($f1$), ($f2$), and f is superlinear. For $k \in \mathbf{N}$, there exists $\lambda_{k,+}, \lambda_{k,-} \in [\lambda_{k,+}^{\infty}, \lambda_{k,-}^{\infty}]$ such that the set of nontrivial solutions Σ of* (1.1) *satisfies*

$$\Sigma = \bigcup_{k \in \mathbf{N}} \Sigma_k^{\pm}, \quad \Sigma_k^{\pm} = \{(\lambda, u_k^{\pm}(\lambda, \cdot)) \colon \lambda \in (\lambda_{k,\pm}, \lambda_k^0)\}. \tag{7.2}$$

Moreover, $\partial_x u_k^+(\lambda, 0) > 0$ (resp. $\partial_x u_k^-(\lambda, 0) < 0$), and $u_k^+(\lambda, \cdot)$ (resp $u_k^-(\lambda, \cdot)$) has exactly $k-1$ zeros in $(0,\pi)$; all solutions of (1.1) *which have exactly $k-1$ zeros lie on the curve $\Sigma_k^{\pm}$, $M(u_k^{\pm}) = k$, and there are no turning points on $\Sigma_k^{\pm}$. Furthermore, $\lambda_{k,+} = \lambda_{k,-}$ if k is even, and $\lambda_{k,+} \neq \lambda_{k,-}$ if k is odd.*

The theorem gives the exact count of the nontrivial solutions of (1.1) for any $\lambda > 0$. In fact if $f_+ = f_-$, then (1.1) has exactly $2p$ nontrivial solutions if λ belongs to exactly p intervals I_j's. The number of solutions can be an odd number for some λ if $f_+ \neq f_-$. In fact, in the proof we would show that if $f'(0)$, f_+ and f_- are given, then $\lambda_{k,+}$ and $\lambda_{k,-}$ can be explicitly calculated, so the exact number of solutions for any $\lambda > 0$ can always be determined.

To prove the theorem, we need the following preliminaries. First,

$$\phi'' + \lambda \phi = 0, \quad x \in (0,\pi), \quad \phi(0) = \phi(\pi) = 0 \tag{7.3}$$

possesses a sequence of eigenvalues $\{\lambda_j = j^2\}$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty$ as $j \to \infty$, λ_j is a simple eigenvalue, any eigenfunction ϕ_j corresponding to λ_j has exactly $j-1$ zeros in $(0,\pi)$ and all zeros of ϕ_j in $[0,\pi]$ are simple. (A simple zero of ϕ_j is a point $\in[0,\pi]$ such that $\phi_j(x)=0$ and $\phi_j'(x)\neq 0$). We define

$$S_j^+=\{v\colon v(0)=v(\pi)=0,\ v_x(0)>0,\ v \text{ has exactly } j-1 \text{ zeros in } (0,\pi), \text{ and all zeros of } v \text{ in } [0,\pi] \text{ are simple}\},$$

$$S_j^- = -S_j^+, \text{ and } S_j=S_j^+\cup S_j^-.$$

Then Theorem 2.3 of [25] can be applied to (7.1), and we have the following lemma.

Lemma 7.2. *For any $k\in\mathbf{N}$, (7.1) possesses a continuum of solutions Σ_k in $\mathbf{R}\times E$ with $\Sigma_k\subset(\mathbf{R}\times S_k)\cup\{(\lambda_k^0,0)\}$ and Σ_k is unbounded, where $E=C^2[0,\pi]$.*

Let $\Sigma_k^+=\Sigma_k\cap S_k^+$ and $\Sigma_k^-=\Sigma_k\cap S_k^-$. To prove Theorem 7.1, we need to find a global parameter for each component $\Sigma_k^\pm$. We show we can use the $\max u(\lambda,\cdot)$ to parameterize Σ_k^+, and $\min u(\lambda,\cdot)$ to parameterize Σ_k^-.

If $u(\lambda,\cdot)\in S_k$ is a nontrivial solution of (7.1) with $k>2$, then $u(\lambda,\cdot)$ is a rescaling and periodic extension of a positive solution and a negative solution. In fact, (u,u_x) is a solution of a first order system:

$$u'=v,\quad v'=-\lambda f(u),\quad u(0)=0,\quad v(0)\neq 0. \tag{7.4}$$

For the function f which we consider here, each solution orbit of (7.4) in (u,v) plane is a periodic orbit centered at origin from the phase portrait analysis of (7.4).

Lemma 7.3. *Suppose that f satisfies the conditions in Theorem 7.1. Given $k\in\mathbf{N}$, for any $d>0$ there exists exactly one $\lambda>0$ such that (1.1) has a solution $u(\lambda,\cdot)\in S_k^+$ and $\max_{x\in[0,\pi]}u(\lambda,x)=d$. Similar result hold for $d<0$ and S_k^-.*

Proof. We prove the lemma for $k=2m+1$ where $m\in\mathbf{N}$. The case when k is an even number is similar. Consider the initial value problem

$$u'=v,\quad v'=-f(u),\quad u(0)=d,\quad v(0)=0. \tag{7.5}$$

Let $(u(x),v(x))$ be the unique solution of (7.5). For any $d>0$, there exists a unique $T_1=T_1(d)>0$ such that $u(T_1)=0$, $u(x)>0$ and $v(x)<0$ for $x\in[0,T_1)$, and there exists a unique $T_2=T_2(d)$ such that $v(T_1+T_2)=0$, $u(x)<0$ and $v(x)<0$ for $x\in(T_1,T_1+T_2)$. Let $v(T_1)=-v_0<0$ for some $v_0>0$. Then

$$u'=v,\quad v'=-f(u),\quad u(0)=0,\quad v(0)=v_0 \tag{7.6}$$

has a unique solution $(u_1(x),v_1(x))$ for $x>0$. In particular, $(u_1(x),v_1(x))=(u(x+T_1),v(x+T_1))$ from the property of (7.6). Therefore, $u_1(x)$ with $x\in[0,2(m+1)T_1+2mT_2]$ is a solution of $u''+f(u)=0$, $u(0)=0$, $u(2(m+1)T_1+2mT_2)=0$, and u_1 has exactly $2m+1$ zeros in $(0,2(m+1)T_1+2mT_2)$. Let $T=2(m+1)T_1+2mT_2$, $u_2(x)=u_1(Tx)$, then u_2 is a solution of (7.1) with $\lambda=T^2\pi^{-2}$. Since T_1 and T_2 are uniquely determined by $d>0$, thus T and λ are also uniquely determined by d. $\square$

Lemma 7.3 implies that Σ_k^+ can be written as a graph $\{(\lambda(d),d)\colon d>0\}$ in $\mathbf{R}^+\times\mathbf{R}^+$. Another reason that the solutions of (7.1) can be classified is that the Fučík spectrum in the case of ordinary differential equation is completely known. In fact, Fučík (see [20, Lemma 42.2, p. 323]) proved

Lemma 7.4. *Consider*

$$\phi'' + a\phi^+ - b\phi^- = 0, \ x \in (0,\pi), \ \phi(0) = \phi(\pi) = 0. \tag{7.7}$$

Let $\Gamma = \{(a,b) \in \mathbf{R}^2$: (7.7) *has a nontrivial solution*$\}$. *Then* $\Gamma = \bigcup_{i=1}^{5} \Gamma_i$, *where* $\Gamma_1 = \{(1,b)\colon b \in \mathbf{R}\}$, $\Gamma_2 = \{(a,1)\colon a \in \mathbf{R}\}$,

$$\Gamma_3 = \bigcup_{k\in\mathbf{N}} \Gamma_{3,k}, \ \Gamma_{3,k} = \left\{(a,b)\colon \frac{a^{1/2}b^{1/2}}{a^{1/2}+b^{1/2}} = k, \ a,b > 1\right\},$$

$$\Gamma_4 = \bigcup_{k\in\mathbf{N}} \Gamma_{4,k}, \ \Gamma_{4,k} = \left\{(a,b)\colon \frac{a^{1/2}(b^{1/2}-1)}{a^{1/2}+b^{1/2}} = k, \ a,b > 1\right\},$$

and

$$\Gamma_5 = \bigcup_{k\in\mathbf{N}} \Gamma_{5,k}, \ \Gamma_{5,k} = \left\{(a,b)\colon \frac{(a^{1/2}-1)b^{1/2}}{a^{1/2}+b^{1/2}} = k, \ a,b > 1\right\}.$$

Relating Lemma 7.4 to Fig. 1, we find that for each $k \in \mathbf{N}$, $\Gamma_{3,k}$ is a decreasing curve emerging from $(a,b) = (2k,2k)$, $\Gamma_{4,k}$ and $\Gamma_{5,k}$ are two decreasing curves emerging from $(a,b) = (2k+1, 2k+1)$. In particular, it implies, for an even number $2k$, the two curves of Fučík spectrum are coincident, and for an odd number $2k+1 \geqslant 3$, the two curves are distinct.

The last tool we need is a comparison argument based on Sturm comparison lemma. Following [31], we consider the solution $w(u,\cdot)$ of

$$w'' + \lambda f'(u(\lambda,x))w = 0, \quad x \in (0,\pi), \quad w(0) = 0, \quad w'(0) = 1, \tag{7.8}$$

where $u = u(\lambda,\cdot)$ is a solution of (7.1). The following lemma is similar to Lemma 4.3 in [31], so we omit the proof.

Lemma 7.5. *Suppose that* $u(\lambda,\cdot)$ *is a solution of* (7.1), *and* $w(u,\cdot)$ *is the solution of* (7.8), *then* $M(u(\lambda,\cdot)) = k$ *if and only if* $w(\lambda,\cdot)$ *has exactly* k *zeros in* $(0,\pi)$.

The key result for proving the nonexistence of turning points is the following lemma.

Lemma 7.6. *Suppose that* f *satisfies the conditions in Theorem* 7.1. *If* $(\lambda, u(\lambda,\cdot)) \in S_k$ *is a solution of* (7.1), *then* $M(u(\lambda,\cdot)) = k$, *and* $w(u,\pi) \neq 0$. *In particular*, $u(\lambda,\cdot)$ *is nondegenerate.*

Proof. Since $u \in S_k$, then all zeros of $u(\lambda,\cdot)$ are simple. By the maximum principle and that $f(u)u > 0$ for $u \neq 0$, $u(\lambda,\cdot)$ has no positive local minimum and negative local maximum. Thus $u(x) = u(\lambda,x)$ has $k-1$ zeros in $(0,\pi)$ and $u_x(x) = u_x(\lambda,x)$ has k zeros in $(0,\pi)$. The functions $u(x)$, $w(x) = w(\lambda,x)$ and $u_x(x)$ satisfy the following equations, respectively

$$u'' + \lambda \frac{f(u)}{u} u = 0,\ w'' + \lambda f'(u) w = 0,\ u_x'' + \lambda f'(u) u_x = 0.$$

Since f is superlinear, then $f'(u) > f(u)/u$ for $u \neq 0$. By the Sturm comparison lemma, between any two consecutive zeros of u, there exists at least one zero of w. Since $u(0) = u(\pi) = 0$, then u has $k+1$ zeros in $[0, \pi]$. Thus w has at least k zeros in $(0, \pi)$. On the other hand, between any two consecutive zeros of w, there exists at least one zero of u_x. So if w has at least $k+1$ zeros in $(0, \pi)$, plus $w(0) = 0$, then u_x has at least $k+1$ zeros, which contradicts with u_x having only k zeros. Thus w has at most k zeros in $(0, \pi)$. Therefore, w has exactly k zeros in $(0, \pi)$. $w(\pi) = 0$ will lead to a similar contradiction. So $w(\pi) \neq 0$ and by Lemma 7.5, $M(u(\lambda, \cdot)) = k$. □

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. By Lemma 7.2, for $k \in \mathbf{N}$, there exists a solution curve Σ_k bifurcating from $(\lambda_k^0, 0)$, and the solution $u(\lambda, \cdot)$ on Σ_k has exactly $k-1$ zeros in $(0, \pi)$. By Lemma 7.6, $M(u(\lambda, \cdot)) = k$, then the k-th eigenvalue $\mu_k(u(\lambda, \cdot)) > 0$. Therefore, by Lemma 2.3, the solution curve Σ_k bends to the left of λ_k^0 since f is superlinear. By Lemma 7.3, $\Sigma_k^+ = \{(\lambda_k(d), d)\colon d > 0\}$. By Lemma 7.6, (7.1) has no nontrivial degenerate solution, thus $\lambda_k'(d) \neq 0$ for any $d > 0$ and in fact $\lambda_k'(d) < 0$ since it is true for $d > 0$ small. So Σ_k^+ can also be parameterized by λ. Let $\lambda_* = \inf\{\lambda\colon (\lambda, d) \in \Sigma_k^+\}$. $\lambda_* > 0$ since f satisfies (f2), then (7.1) has only trivial solution if $\lambda > 0$ is small. Therefore λ_* must a point where Σ_k^+ blows up. From Lemma 2.2, $(\lambda_* f_+, \lambda_* f_-) \in \Gamma$, and since $\Sigma_k^+ \subset S_k^+$, then the "eigenfunction" ϕ corresponding to $(\lambda_* f_+, \lambda_* f_-)$ also belongs to S_k^+. From Lemma 7.4, there is exactly one $\lambda = \lambda_{k,+} \in (0, \lambda_k^0)$ such that $(\lambda_{k,+} f_+, \lambda_{k,+} f_-) \in \Gamma$ and the corresponding "eigenfunction" $\phi \in S_k^+$. Hence, $\lambda_* = \lambda_{k,+}$. The proof for Σ_k^- is similar. By Lemma 7.3, there are no other solutions of (7.1) which have $k-1$ zeros in $(0, \pi)$ except the ones on Σ_k. The last statement follows directly from Lemma 7.4. □

A complete classification of solutions for (1.2) can also be obtained when $n = 1$, which is a generalization of Theorem 6.1 for this special case. We consider

$$u'' + f(u) = 0, \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0. \tag{7.9}$$

From Lemma 7.4, we know that for $k \geqslant 2$, there exists two decreasing curves $b = \gamma_{k,1}(a)$ and $b = \gamma_{k,2}(a)$ emerging from (λ_k, λ_k), and we assume that $\gamma_{k,1}(a) \geqslant \gamma_{k,2}(a)$. When k is even, $\gamma_{k,1}(a) = \gamma_{k,2}(a)$, and when k is odd, $\gamma_{k,1}(a) > \gamma_{k,2}(a)$. For $k \geqslant 2$, we define

$$J_k = \{(a, b)\colon \gamma_{k,1}(a) > b > \gamma_{k,2}(a)\},\ I_k = \{(a, b)\colon \gamma_{k+1,2}(a) > b > \gamma_{k,1}(a)\},$$

$$\gamma_{k,1} = \{(a, b)\colon b = \gamma_{k,1}(a)\},\ \gamma_{k,2} = \{(a, b)\colon b = \gamma_{k,2}(a)\}.$$

And

$$J_1 = \{(a, b)\colon a > \lambda_1, 0 < b < \lambda_1\} \cup \{(a, b)\colon b > \lambda_1, 0 < a < \lambda_1\},$$

$$I_1 = \{(a, b)\colon a > \lambda_1, b > \lambda_1, b < \gamma_{2,2}(a)\},$$

$$\gamma_{1,1} = \{(a,b)\colon a \geqslant \lambda_1, b = \lambda_1\} \cup \{(a,b)\colon b \geqslant \lambda_1, a = \lambda_1\},$$

$$\gamma_{1,2} = \{(a,b)\colon \lambda_1 \geqslant a > 0, b = \lambda_1\} \cup \{(a,b)\colon \lambda_1 \geqslant b > 0, a = \lambda_1\}.$$

We also assume that $\lambda_0 = 0$ and $I_0 = \{(a,b)\colon 0 < a < \lambda_1, 0 < b < \lambda_1\}$. Note that $J_k = \emptyset$ if k is even. Then we have the following exact multiplicity result.

Theorem 7.7. *Let f satisfy $(f1)$, $(f2)$, and let f be superlinear. Suppose that $\lambda_{k-1} \leqslant f'(0) < \lambda_k$ for $k \geqslant 2$, and $\lambda_{k-1} < f'(0) < \lambda_k$ for $k = 1$.*
(1) *If $(f_+, f_-) \in J_m \cup (\gamma_{m,1} \setminus \{(\lambda_m, \lambda_m)\})$, then (7.9) has exactly $2(m-k)+1$ nontrivial solutions for $m \geqslant k$.*
(2) *If $(f_+, f_-) \in I_m \cup \gamma_{m+1,2}$, then (7.9) has exactly $2(m-k)+2$ nontrivial solutions for $m \geqslant k-1$.*

In Theorem 7.7, $f'(0)$ can be any positive number, and since f is superlinear, then $f_+ > f'(0)$, $f_- > f'(0)$, our result completely classifies the solution set for these f_+, f_-.

Proof. Similar to the proof in Section 6, we embed equation (7.9) into a one-parameter family of equations (7.1). Then (7.9) is equivalent to (7.1) with $\lambda = 1$. For $\lambda = 1$, $\lambda_{k-1} \leqslant f'(0) < \lambda_k$ implies $\lambda_{k-1}/f'(0) \leqslant 1 < \lambda_k/f'(0)$.

Let $(f_+, f_-) \in J_m \cup (\gamma_{1,m} \setminus \{(\lambda_m, \lambda_m)\})$ for some $m \geqslant k$, and $p = f_+/f_- > 0$. In this case $p \neq 1$. We consider $\lambda \in \mathbf{R}^+$, then $(\lambda f_+, \lambda f_-)$ moves along the ray $b = pa$, $(a > 0)$. The ray $b = pa$ intersect each $\gamma_{j,i}$ $(i = 1, 2, j \geqslant 1)$ exactly once. We denote the intersection points of $b = pa$ and $\gamma_{m,i}$, $(i = 1, 2)$ by $\lambda_{m,+}, \lambda_{m,-}$. Then

$$\lambda_{1,-} < \lambda_{1,+} < \lambda_{2,-} = \lambda_{2,+} < \cdots < \lambda_{2k-1,-} < \lambda_{2k-1,+} < \lambda_{2k,-} = \lambda_{2k,+} < \cdots.$$

Since $(f_+, f_-) \in J_m \cup \gamma_{m,1}$, then $\lambda_{m,+} \geqslant 1 > \lambda_{m,-}$. By Theorem 7.1, $\Sigma_j^{\pm}$ exists for $\lambda \in I_j^{\pm} \equiv (\lambda_{j,\pm}, \lambda_j^0)$. Thus, $1 \in I_{m,-}$ but $1 \notin I_{m,+}$. On the other hand, for any j such that $k \leqslant j < m$, $1 \in I_{j,\pm}$ since $1 < \lambda_k^0 \leqslant \lambda_j^0$ and $1 > \lambda_{m,-} > \lambda_{j,\pm}$. Hence for $\lambda = 1$, (7.3) has exactly $2(m-k)+1$ solutions, which are in $\Sigma_j^{\pm}$, $k \leqslant j < m$ and Σ_m^-. The proof for $(f_+, f_-) \in I_m \cup \gamma_{m+1,2}$ is similar. □

We remark that Theorem 7.1 is also true if (f2) is replaced by $f(u)/u \to \infty$ as $u \to \pm\infty$. (So f is no longer asymptotic linear but asymptotic superlinear.) In this case, all Σ_ks continue left to $\lambda = 0^+$ and there is no turning points on any of Σ_k except $(\lambda, u) = (\lambda_k^0, 0)$. So for any $\lambda > 0$, (7.1) has infinite many solutions. In [30], we consider the case only one of the limit (f_+ and f_-) is ∞ and the other is finite. There is a new type of bifurcation from infinity in that case.

Appendix A. Stability of the solutions bifurcating from infinity

The bifurcation from infinity for an asymptotic linear operator is well-known. The following result is due to Rabinowitz [26] and Dancer [17].

Theorem 8.1. *Let* $\lambda_0(\neq 0)\in\mathbf{R}$ *and let* $F:\mathbf{R}\times X\to X$ *be a continuous mapping such that* $F(\lambda,u)=-u+\lambda Bu+H(\lambda,u)$, *where* B *is a continuous linear operator on* X, λ_0^{-1} *is an isolated point of the spectrum of* B *such that* λ_0^{-1} *has algebraic multiplicity* 1, w_0 *is the eigenvector corresponding to* λ_0^{-1}, *and* H *satisfies* $||H(\lambda,u)||/||u||\to 0$ *as* $||u||\to\infty$, *uniformly for* λ *near* λ_0, *For any* $\varepsilon>0$, *there exist* M *and* $\gamma>0$ *such that, if* $|\lambda_1-\lambda_0|\leqslant\gamma$ *and* $|\lambda_2-\lambda_0|\leqslant\gamma$, $u=\alpha w_0+w, v=\alpha w_0+z$ *where* $|\alpha|\geqslant M, ||w||\leqslant\gamma|\alpha|$, *and* $||z||\leqslant\gamma|\alpha|$, *then*

$$||H(\lambda_1,u)-H(\lambda_2,v)||\leqslant\varepsilon[||u-v||+(||u||+||v||)|\lambda_1-\lambda_2|].$$

If Z *is a complement of* $\mathrm{span}\{w_0\}$ *in* X, *then there exists* $N>0$ *and continuous mappings* $\lambda:\{s\colon |s|\geqslant N\}\to\mathbf{R}$ *and* $\psi:\{s\colon |s|\geqslant N\}\to Z$ *such that* $\lambda(s)\to\lambda_0$ *and* $||\psi(s)||\to 0$ *as* $|s|\to\infty$ *and* $F(\lambda(s),sw_0+s\psi(s))=0$. *Moreover, there exist* C *and* $\rho>0$ *such that each solution* (λ,u) *of* $F(\lambda,u)=0$ *with* $|\lambda-\lambda_0|\leqslant\rho$ *and* $||u||\geqslant C$ *has the above form.*

In the context of Theorem 8.1, we have the following result concerning the stability of the solutions on the bifurcation curve which we obtain in Theorem 8.1. (Recall that μ is a K-simple eigenvalue of a linear operator L if there exists w such that $Lw=\mu Kw$, where K is also a linear operator.)

Theorem 8.2. *Let* F, B, H, Z, λ_0 *and* w_0 *be as in Theorem* 8.1, *and let* $(\lambda(s),u(s))$, $|s|\geqslant\delta$, *be the solution curve in Theorem* 8.1. *In addition, we assume that* $H(\lambda,u)$ *is twice continuously differentiable, for* $\lambda\in\mathbf{R}$, $||H_u(\lambda,u)||=o(||u||)$ *as* $||u||\to\infty$; K *is a bounded linear operator from* X *to* X, *and* 0 *is a* K*-simple eigenvalue of* $F_u(\lambda_0,\infty)=-I+\lambda_0 B$. *Then there exist* $\varepsilon>0$, C^1 *functions* $\gamma:(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\to\mathbf{R}$, $\mu:\{s\colon |s|\geqslant\delta\}\to\mathbf{R}$, $v:(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\to X$, $w:\{s\colon |s|\geqslant\delta\}\to X$ *such that*

$$F_u(\lambda,\infty)v(\lambda)=\gamma(\lambda)Kv(\lambda)\quad \mathit{for}\ \lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon), \tag{8.1}$$

$$F_u(\lambda(s),u(s))w(s)=\mu(s)Kw(s)\quad \mathit{for}\ s\in\{s\colon |s|\geqslant\delta\}, \tag{8.2}$$

$\gamma(\lambda_0)=\lim_{|s|\to\infty}\mu(s)=0$, $v(\lambda_0)=\lim_{|s|\to\infty}w(s)=w_0$, *and* $v(\lambda)-w_0\in Z$, $w(s)-w_0\in Z$. *Moreover*, $\gamma'(\lambda_0)\neq 0$, *and near* $s=\infty$ *the functions* $\mu(s)$ *and* $-s\lambda'(s)\gamma'(\lambda_0)$ *have the same zeros, and, whenever* $\mu(s)\neq 0$, *the same sign. More precisely*,

$$\lim_{|s|\to\infty,\mu(s)\neq 0}\frac{-s\lambda'(s)\gamma'(\lambda_0)}{\mu(s)}=1. \tag{8.3}$$

Proof. Since

$$\begin{aligned}&||F_u(\lambda(s),u(s))-F_u(\lambda_0,\infty)||\\&\quad\leqslant|\lambda(s)-\lambda_0|\cdot||F_u(\lambda_0,\infty)||+|\lambda_0|\cdot||H_u(\lambda(s),u(s))||\to 0\end{aligned}$$

as $|s|\to\infty$, then by Lemma 1.3 of [15], (8.1) and (8.2) hold, and $\gamma(\lambda_0)=\lim_{|s|\to\infty}\mu(s)=0$, $v(\lambda_0)=\lim_{|s|\to\infty}w(s)=w_0$, and $v(\lambda)-w_0\in Z$, $w(s)-w_0\in Z$. The other parts of the proof are similar to the proof of Theorem 1.16 in [15]. □

Theorems 8.1 and 8.2 can be directly applied to (1.1) for $f_+=f_-$. It is possible to generalize them to the case of $f_+\neq f_-$ if some appropriate cones in X are defined.

We do not attempt to do such a generalization here. But we point out that Theorem 8.2 holds as long as the solutions near (λ_0,∞) form a curve, the same proof can be carried over for $Bu=au^+ - bu^-$. In this paper, when $f_+\neq f_-$, we only show that λ_* is a bifurcation point if $(\lambda_* f_+,\lambda_* f_-)\in\Gamma$. With a little more careful analysis, we can show the solutions actually form a curve. But we do not need to do that, since in all the situations, the solutions are nondegenerate, so they form a curve automatically. Thus Theorem 8.2 can be applied even to the case of $f_+\neq f_-$.

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