DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **7**, Number **3**, July **2001**

Website: http://math.smsu.edu/journal

pp. **559–571**

# EXACT MULTIPLICITY OF SOLUTIONS FOR CLASSES OF SEMIPOSITONE PROBLEMS WITH CONCAVE-CONVEX NONLINEARITY

Junping Shi[1]

Department of Mathematics, Tulane University
New Orleans, LA 70118, USA.

Ratnasingham Shivaji[2]

Department of Mathematics, Georgia Southern University
Statesboro, GA 30460, USA.

1. **Introduction.** Consider the boundary value problem:

$$u'' + \lambda f(u) = 0, \;\; x \in (-1,1), \;\; u(-1) = u(1) = 0, \tag{1.1}$$

where $\lambda$ is a positive parameter. The nonlinearity $f(u)$ is called semipositone if $f(0) < 0$. In this paper we will only consider the positive solutions of (1.1). Semipositone problems were introduced by Castro and Shivaji in [CS1], and they arise from various disciplines, like astrophysics and population dynamics. (see [CMS] for more details.)

It is possible that (1.1) has non-negative solutions with interior zeros (see [CS1]). This is not the case when $f(0) \geq 0$, where any non-negative solution of (1.1) is strictly positive in $(0,1)$. Note that any solution of (1.1) is symmetric with respect to any point $x_0 \in (-1,1)$ such that $u'(x_0) = 0$, so any positive solution of (1.1) is a reflection extension of a monotone decreasing solution of

$$\begin{cases} u'' + \lambda f(u) = 0, \;\; x \in (0,1), \\ u'(0) = u(1) = 0, \\ u'(x) < 0, \;\; x \in (0,1). \end{cases} \tag{1.2}$$

So the study of all positive solutions is reduced to the study of (1.2). On the other hand, all solutions of (1.2) can be parameterized by their initial values $u(0) = \rho$. In fact, by integrating the equation, we obtain

$$u'(x) = -\sqrt{2\lambda[F(\rho) - F(u(x))]}, \;\; x \in (0,1), \tag{1.3}$$

where $\rho = u(0)$, and

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\rho \frac{du}{\sqrt{F(\rho) - F(u)}} := G(\rho). \tag{1.4}$$

1991 *Mathematics Subject Classification.* 34B18, 34C23.

*Key words and phrases.* Exact Multiplicity, Bifurcation, Semilinear Elliptic Equation, Semipositone Problems.

[1]Current address: Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA.

[2]Current address: Department of Mathematics, Mississippi State University, Mississippi State, MS 39762, USA.

So for each $\rho > 0$, there is at most one (if the integral in (1.4) is well defined and convergent) $\lambda$ such that (1.2) has a solution. Thus the solution set of (1.2) can be represented by $\lambda = \lambda(\rho) = [G(\rho)]^2$, which we call *bifurcation diagram*.

In this paper, we study the precise behavior of the bifurcation diagram of (1.2) for a class of semipositone nonlinearities. One way of studying such problems is the quadrature method which is based on the estimate of $G(\rho)$ defined in (1.4). (See for examples, [SW], [Sc].) Our approach uses a bifurcation result of Crandall and Rabinowitz [CR] and some integral comparison arguments first found in Korman, Li and Ouyang [KLO]. More integral comparison type results were found in [OS1], [OS2], [KL] and [KS]. The proofs of our main results will use some delicate integral comparison inequalities in [KL] and [KS]. The motivation of such an attempt is that the quadrature methods based on (1.4) is hard to be generalized to higher dimensional problem similar to (1.2), which is the equation:

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \;\; u'(0) = u(1) = 0. \tag{1.5}$$

The solutions to equation (1.5) are the radially symmetric solutions of a semilinear partial differential equation of elliptic type:

$$\Delta u + \lambda f(u) = 0, \;\; x \in \Omega, \;\; u = 0, \;\; x \in \partial\Omega, \tag{1.6}$$

where $\Omega = B^n$, the unit ball in $\mathbf{R}^n$. Though we only consider the one dimensional problem as (1.2) in this paper, we believe that our method has great potential to be generalized to higher dimensional equation (1.5).

We shall assume that $f(\cdot)$ satisfies

**(f1)** $f \in C^2[0,\infty)$, $f(0) < 0$, $f(u) < 0$ for $u \in (0,b)$ for some $b > 0$, $f(b) = 0$ and $f'(b) > 0$;

**(f2)** There exists $\theta > b$ such that

$$f(\theta) > 0, \; F(u) < 0 \text{ for } u \in (0,\theta), \text{ and } F(\theta) = 0, \tag{1.7}$$

where $F(u) = \int_0^u f(t)dt$;

**(f3)** There exists $\alpha > \theta$ such that

$$f''(u) \le 0, \quad \text{for } u \in [0,\alpha], \quad \text{and } f''(u) \ge 0, \quad \text{for } u \in [\alpha,\infty). \tag{1.8}$$

The nonlinearity $f$ satisfying (f3) is referred as a concave-convex function. Moreover, we assume that $f$ also satisfies one of following two conditions:

**(f4a)** $f(u) > 0$ for all $u > b$, or

**(f4b)** There exists $d > c > \theta$ such that $f(c) = f(d) = 0$, $f(u) > 0$ for $u \in (b,c) \bigcup (d,\infty)$, $f(u) < 0$ for $u \in (c,d)$, $f'(c) < 0$ and $f'(d) > 0$.

The graphs of $f$ satisfying (f1)-(f4) are illustrated in Figures 1 and 2.

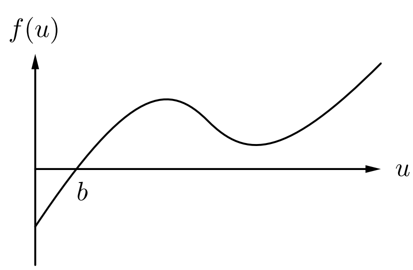


Fig. 1: $f$ satisfying (f4a)

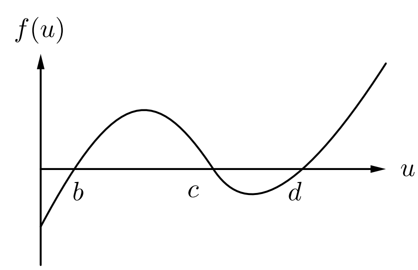


Fig. 2: $f$ satisfying (f4b)

Under some other appropriate conditions, we prove that the bifurcation diagram of (1.2) looks exactly like one of the following two graphs: (respectively for $f$ satisfying (f4a) or (f4b))

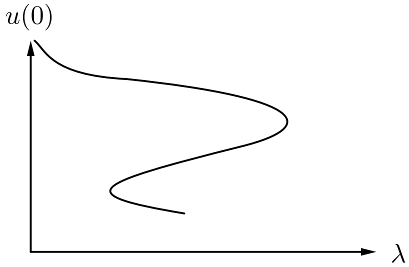


Fig. 3: Reversed $S$-shaped curve

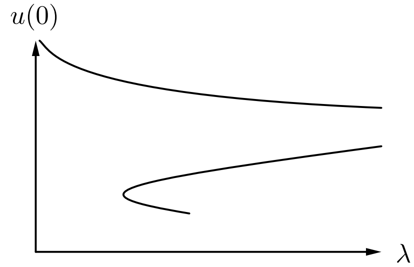


Fig. 4: Broken reversed $S$-shaped curve

Castro and Shivaji [CS1] first used quadrature methods to analyze the bifurcation diagrams of (1.2) with $f$ satisfying (f1-3) and (f4a). In particular, they showed that for $\lambda$ in certain range, (1.2) has at least three solutions. Khamayseh and Shivaji [KhS] studied the bifurcation diagrams for $f$ satisfying (f1), (f2), and $f$ evolving from (f4a) type to (f4b) type. Our results here improve these earlier results since we prove the exact shape of the bifurcation diagrams. And the evolution of the bifurcation diagrams are made more precise since we obtain the exact number of turning points. More (exact) multiplicity results on semipositone problems for balls or intervals can be found in [ACS], [CG], [CGS], [CS2], [OS2], [CMS]. Shi [S1] proved the exactness of a reversed $S$-shaped curve for a class of semipositone problem. One of his examples is $f(u) = (u+\varepsilon)^3 - b(u+\varepsilon)^2 + c(u+\varepsilon)$, where $b, c > 0$, $3c > 4b^2$, and $\varepsilon \in (-\varepsilon_0, 0)$ for some $\varepsilon_0 > 0$. The result in [S1] does not require the condition (3.2), but it needs $f$ being a form $f(u) = g(u+\varepsilon)$, and $g(\cdot)$ is a positone function. And the method in [S1] is based on a perturbation and continuation argument, while the method in this paper is more direct.

A related problem is the exactness of $S$-shaped curve for a class of positone problem, especially for the Perturbed Gelfand equation which arises from combustion theory

$$\Delta u + \lambda \exp[-1/(u+\varepsilon)] = 0,\ x \in B^n,\ u(x) = 0,\ x \in \partial B^n, \tag{1.9}$$

where $B^n$ is the unit ball in $n$-dimensional space. The study of the $S$-shaped curves can be found in [BIS], [HM], [W], [KL], [DL], [S1] and [KS]. Our techniques in this paper is similar to those in [KL] and [KS].

In Section 2, we set up a framework of the bifurcation approach to (1.2). Then in Section 3, we study the bifurcation diagrams of (1.2) for (f4a) type nonlinearities, and in Section 4, we consider the corresponding problem for (f4b) type nonlinearities. We will discuss an example in Section 5. In the paper, we shall use $u_x$ or $u'$ to denote the derivative with respect to the spatial variable $x$.

**Acknowledgement**: We would like to thank referee for the careful reading of the manuscript and some helpful suggestions.

2. **Bifurcation approach.** In this section, we briefly describe a bifurcation approach to (1.2) which was developed in recent years. The details and the omitted

proofs of all quoted facts can be found in [KLO], [OS1], [OS2]. Similar bifurcation analysis are also found in [ACS], [CG] and [CS2].

From the uniqueness of ordinary differential equation (or from (1.4)), for any $\rho > 0$, there is at most one $\lambda(\rho) > 0$ such that (1.2) has a positive solution $u(\cdot, \rho)$ with $\lambda = \lambda(\rho)$ and $u(0) = \rho$. So the solution set of (1.2) can be globally parameterized by $\rho = u(0)$, and it is a curve of the form $\{(\lambda(\rho), \rho)\}$, where $\rho > 0$ belongs to a certain admissible set. In fact, we have the following description of this admissible set:

**Lemma 2.1.** *Suppose that $f$ satisfies (f1) and (f2). And we also assume that $\theta$ defined in (f2) is unique for all $u > 0$.*

1. (1.2) *has a solution satisfying $u_x(1, \rho) = 0$ if and only if $\rho = \theta$ and*
$$\lambda(\theta) = \frac{1}{2}\left(\int_0^\theta \frac{du}{\sqrt{-F(u)}}\right)^2. \tag{2.1}$$
2. (1.2) *has a solution $u(\cdot, \rho)$ with $u(0, \rho) = \rho$ ($\rho \neq \theta$) if and only if $f(\rho) > 0$ and $F(\rho) > F(u)$ for $u \in [0, \rho)$.*

*Proof.* These facts can be observed from the phase portrait of the first order system
$$u' = v, \quad v' = -\lambda f(u). \tag{2.2}$$
Here we give a proof using shooting arguments. We consider an initial value problem:
$$u'' + f(u) = 0, \quad x > 0, \quad u'(0) = 0, \quad u(0) = \rho > 0. \tag{2.3}$$
We denote the solution of (2.3) by $U(x, \rho)$. Note that if there is a $x(\rho) > 0$ such that $U(x(\rho), \rho) = 0$ and $U(x, \rho) > 0$ for $0 \leq x < \rho$, then $u(x, \rho) = U(x(\rho)x, \rho)$ is a solution of (1.2). We define
$$D = \{\rho > 0 : \text{ (1.2) has a solution } u(\cdot, \rho) \text{ such that } u(0, \rho) = \rho\}.$$
If $f(\rho) \leq 0$, we claim $U(x, \rho) \geq \rho > 0$ for all $x > 0$. If $f(\rho) = 0$, then $U(x, \rho) \equiv \rho$. So we only consider the case of $f(\rho) < 0$. At $x = 0$, $U_x(0, \rho) = 0$ and $U_{xx}(0, \rho) = -f(U(0, \rho)) > 0$, thus $U(x, \rho) > \rho$ for small $x > 0$. Suppose that there is $x_0 > 0$ such that $U(x_0, \rho) < \rho$, then there must be a $x_1 \in (0, x_0)$ such that $U(x_1, \rho) = \rho$ and $(x - x_1)[U(x, \rho) - \rho] < 0$ for $x$ near $x_1$. But from the equation, we obtain
$$\frac{1}{2}[U_x(x_1, \rho)]^2 + \int_0^{x_1} f(U(x, \rho))U_x(x, \rho)dx = \frac{1}{2}[U_x(x_1, \rho)]^2 = 0.$$
So $U_x(x_1, \rho) = 0$, and again by the equation, $U_{xx}(x_1, \rho) = -f(U(x_1, \rho)) > 0$, so $U(x, \rho) > \rho$ for $x$ near $x_1$. That is a contradiction. So if $\rho \in D$, then $f(\rho) > 0$. On the other hand, by (1.3), $F(\rho) \geq F(U(x, \rho))$ for $x > 0$. If $F(U(x_0, \rho)) = F(\rho)$ for some $x_0 > 0$, and $U(x_0, \rho) > 0$, then $U_x(x_0, \rho) = 0$ by a modified (1.3), and by the same argument as above, the solution has to "turn back" at $x = x_0$ and $U(x, \rho) > 0$ for all $x > 0$. Thus for any $\rho \in D$, either $F(\rho) > F(u)$ for all $u \in [0, \rho)$, or $F(\rho) > F(u)$ for $u \in (0, \rho)$ and $F(\rho) = F(0)$.

Now we prove that if $f(\rho) > 0$, $F(\rho) > F(u)$ for all $u \in [0, \rho)$, then $\rho \in D$. Since $f(\rho) > 0$, $U_{xx}(x, \rho) < 0$ for $x \in (0, \varepsilon)$ with some small $\varepsilon > 0$, thus $U(x, \rho) > 0$ and $U_x(x, \rho) < 0$ for $x \in (0, \varepsilon)$. Then we define (following [OS2])
$$R(\rho) = \sup\{x > 0 : U(s, \rho) > 0 \text{ and } U_x(s, \rho) < 0, \quad s \in (0, x)\}. \tag{2.4}$$

There are four possibilities at $x = R(\rho)$ if $f(\rho) > 0$, and we define

$$\begin{aligned} N &= \{\rho > 0 : R(\rho) < \infty, U(R(\rho),\rho) = 0, U_x(R(\rho),\rho) < 0\}, \\ P &= \{\rho > 0 : R(\rho) < \infty, U(R(\rho),\rho) > 0, U_x(R(\rho),\rho) = 0\}, \\ G &= \{\rho > 0 : R(\rho) = \infty\}, \\ B &= \{\rho > 0 : R(\rho) < \infty, U(R(\rho),\rho) = 0, U_x(R(\rho),\rho) = 0\}. \end{aligned} \tag{2.5}$$

Clearly $D = N \bigcup B$. If $\rho \in P$, then $F(\rho) = F(U(R(\rho),\rho))$ and vice versa. So we only need to prove that $\rho \notin G$. Suppose that $\rho \in G$. Then by integrating the equation, we obtain

$$\frac{1}{2}[U_x(x,\rho)]^2 = F(\rho) - F(U(x,\rho)). \tag{2.6}$$

Since $R(\rho) = \infty$, $U(x,\rho) > 0$ and $U_x(x,\rho) < 0$ for all $x > 0$, then $\lim_{x\to\infty} U(x,\rho) = \rho_0 \geq 0$ exists and $\lim_{x\to\infty} U_x(x,\rho) = 0$, and that leads to a contradiction in (2.6) if $F(\rho) > F(\rho_0)$. So $\rho \in N$ if and only if $f(\rho) > 0$, $F(\rho) > F(u)$ for all $u \in [0,\rho)$.

Finally $B = \{\theta\}$ follows from (1.3). And (1.4) indicates the expression in (2.1), which completes the proof of Lemma 2.1. □

**Remark.** From the discussion in the proof of Lemma 2.1, $G = \emptyset$ if $f$ satisfies (f4a). If $f$ satisfies (f4b), and there exists $\theta_1 > d$ such that $F(\theta_1) = F(c)$, then $G = \{\theta_1\}$ since we can show that $\theta_1$ does not belong to $P \bigcup N \bigcup B$ as in the proof of Lemma 2.1. In fact, $\lim_{x\to\infty} U(x,\theta_1) = c$. We will need this fact in Section 4.

Recall that $D$ is the admissible set for $u(0)$ defined in the proof of Lemma 2.1. Then the bifurcation diagram can be written as

$$\Sigma = \{(\lambda(\rho),\rho) : \rho \in D\}. \tag{2.7}$$

If $\lambda'(\rho) \neq 0$, then the corresponding solution $u(\cdot,\rho)$ is nondegenerate, while if $\lambda'(\rho) = 0$, then the solution is degenerate. At a nondegenerate solution, we can continue the bifurcation diagram by the implicit function theorem. At a degenerate solution, we can show that ([CR], [S1])

$$\lambda''(\rho) = \frac{-\lambda(\rho)\int_0^1 f''(u(x,\rho))w^3(x)dr}{\int_0^1 f(u(\rho,x))w(x)dx}, \tag{2.8}$$

where $w$ is a nontrivial solution of the linearized equation

$$w'' + \lambda f'(u)w = 0, \quad x \in (0,1), \quad w'(0) = w(1) = 0. \tag{2.9}$$

Thus if $\lambda''(\rho) \neq 0$, then the bifurcation diagram is parabola-like near the degenerate solution, and that is also the reason we call the degenerate solution a *turning point* sometimes. An important property of a degenerate solution is that $w$ is of one sign.

**Lemma 2.2.** *Suppose that* $(\lambda(\rho), u(\cdot,\rho))$ *is a degenerate solution of* (1.2), *and* $w$ *is the corresponding solution of linearized equation* (2.9). *Then* $w(x) \neq 0$ *for* $x \in [0,1)$, *so we can choose* $w$ *as positive in* $[0,1)$.

*Proof.* The function $u_x(x,\rho)$ satisfies

$$v'' + \lambda f'(u)v = 0, \quad x \in (0,1), \quad v(0) = 0, \quad v'(x) < 0, \quad x \in (0,1). \tag{2.10}$$

Suppose that $w$ has a zero $x_0 \in (0,1)$. Since $w$ and $u_x$ satisfy the same differential equation (not the same boundary conditions), then by the Sturm comparison lemma, there is a zero of $u_x$ in the interval $(x_0, 1)$, that is a contradiction. So $w$ is of one sign in $[0, 1)$. ☐

For the denominator in (2.8), we can show that (see [KLO], [OS1])

$$\int_0^1 f(u(x,\rho))w(x)dx = \frac{u'(1)w'(1)}{2\lambda(\rho)}. \tag{2.11}$$

Although it is possible that $u'(1) = 0$ for a solution $u(\cdot)$ of (1.2) (in fact, $u_x(1,\rho) = 0$ if and only if $\rho = \theta$ by Lemma 2.1), we can show that $u'(1) < 0$ if $u(\cdot)$ is a degenerate solution.

**Lemma 2.3.** *Suppose that* $(\lambda(\rho), u(\cdot,\rho))$ *is a degenerate solution of* (1.2)*, and* $w$ *is the corresponding solution of linearized equation* (2.9)*. Then* $u_x(1,\rho) < 0$*. In particular,*

$$\int_0^1 f(u(x,\rho))w(x)dx > 0. \tag{2.12}$$

*Proof.* Suppose that $u_x(1,\rho) = 0$. We extend $w$ evenly to $[-1, 1]$ and extend $u_x$ oddly to $[-1, 1]$. Then $w$ and $u_x$ are both the solutions of a linear boundary value problem:

$$v'' + \lambda f'(u)v = 0, \ \ x \in (-1, 1), \ \ v(-1) = v(1) = 0. \tag{2.13}$$

However (2.13) has at most one solution (up to a scalar multiplier) from the uniqueness of solution, so we have a contradiction. Therefore we must have $u_x(1,\rho) < 0$ at a degenerate solution. From Lemma 2.2, we can assume that $w(x) > 0$ for $x \in [0, 1)$ and in that case, $w'(1) < 0$ since if $w'(1) = 0$ then $w \equiv 0$. Thus we get (2.12) by using (2.11). ☐

**Remark.** Lemma 2.3 tells us that $u(\cdot,\theta)$ can not be a degenerate solution of (1.2). However $u(\cdot,\theta)$ is a degenerate solution of (1.1) since $u_x(\cdot,\theta)$ solves the linearized equation

$$w'' + \lambda f'(u)w = 0, \ \ x \in (-1, 1), \ \ \ w(-1) = w(1) = 0, \tag{2.14}$$

where $u$ is the solution of (1.1) by evenly extending $u(\cdot,\theta)$. Thus a symmetry-breaking pitchfork bifurcation will occur at $(\lambda(\theta), u(\cdot,\theta))$. (See [K]).

So, the direction of the turn of the bifurcation diagram is now determined by the integral $\int_0^1 f''(u(x))w^3(x)dx$. Here we recall the following results from [OS2] Theorems 3.12 and 3.13:

**Lemma 2.4.** *Suppose that* $(\lambda(\rho), u(\rho))$ *is a degenerate solution of* (1.2)*, and* $w$ *is the corresponding solution of linearized equation* (2.9)*.*

1. *If* $f''(u) > 0$ *for* $u > 0$*, then* $\lambda''(\rho) < 0$*;*
2. *If* $f''(u) < 0$ *for* $u > 0$*, then* $\lambda''(\rho) > 0$*;*
3. *If* $f(0) \geq 0$*, and there exists* $\alpha > 0$ *such that* $f''(u) < 0$ *for* $u$ *in* $(0, \alpha)$*, and* $f''(u) > 0$ *for* $u$ *in* $(\alpha, \infty)$*, then* $\lambda''(\rho) > 0$*;*
4. *If* $f(0) \leq 0$*, there exists* $\alpha > 0$ *such that* $f''(u) > 0$ *for* $u$ *in* $(0, \alpha)$*, and* $f''(u) < 0$ *for* $u$ *in* $(\alpha, \infty)$*, then* $\lambda''(\rho) < 0$*.*

We also recall that a solution $(\lambda(\rho), u(\cdot,\rho))$ of (1.2) is stable if the principal eigenvalue $\mu_1$ of

$$\phi'' + \lambda(\rho)f'(u(\cdot,\rho))\phi = -\mu_1\phi, \ \ x \in (0, 1), \ \ \phi'(0) = \phi(1) = 0, \tag{2.15}$$

is non-negative, otherwise it is unstable. From Corollary 5.6 in [OS2], we have

**Lemma 2.5.** *Suppose that* $\{(\lambda(\rho), u(\cdot, \rho)) : \rho \in D\}$*, is the solution curve of* (1.2)*. Then* $\lambda'(\rho) > 0$ *if and only if* $\mu_1 > 0$*, and* $\lambda'(\rho) \le 0$ *if and only if* $\mu_1 \le 0$*.*

Finally we prove that

**Lemma 2.6.** *Suppose that* $f$ *satisfies (f1) and* ($f2$)*. Then* $u(\cdot, \theta)$ *is a unstable solution.*

*Proof.* Let $\phi$ be the eigenfunction corresponding to $\mu_1$, the principal eigenvalue for $u = u(\cdot, \theta)$. From the equation of $u_x$ and (2.15), we obtain

$$[\phi' u_x - u_x' \phi]|_0^1 + \mu_1 \int_0^1 \phi u_x dx = 0. \tag{2.16}$$

Using the boundary conditions and $u_x(1, \theta) = 0$, we have

$$u_{xx}(0, \theta)\phi(0) + \mu_1 \int_0^1 \phi u_x dx = 0. \tag{2.17}$$

We can assume that $\phi(x) > 0$ for $x \in [0, 1)$, and we also have $u_{xx}(0, \theta) = -\lambda f(\theta) < 0$ and $u_x \le 0$, thus $\mu_1 < 0$. □

3. **Reversed $S$-shaped solution curve.** In this section, we consider the bifurcation diagram of (1.2) for $f$ satisfying (f1)-(f3) and (f4a). An associated function of $f(\cdot)$ is

$$H(u) = 2F(u) - uf(u), \tag{3.1}$$

which plays an important role in our classification of bifurcation diagrams. Recall that $\alpha$ is the point where $f''(u)$ changes sign (see (f3)). Our result is

**Theorem 3.1.** *Suppose that* $f$ *satisfies (f1)-(f3) and (f4a). In addition, we assume that*

$$H(\alpha) \ge 0, \tag{3.2}$$

*and* $f$ *is asymptotic superlinear, i.e.*

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty. \tag{3.3}$$

*Then the admissible set* $D = [\theta, \infty)$*, and the bifurcation diagram* $\lambda(\rho)$ *is exactly reversed* $S$*-shaped (see Figure 3). More precisely,* (1.2) *has no solution with* $0 < \rho < \theta$*, the solution curve starts from* $\rho = \theta$*, continues to the left, turns back at* $\rho = \rho_1$*, continues to the right to the second degenerate solution at* $\rho = \rho_2$*, then continues to the left without any more turns, and*

$$\lim_{\rho \to \infty} \lambda(\rho) = 0.$$

*In particular,* (1.2) *has exactly two degenerate solutions* $(\lambda(\rho_1), u(\cdot, \rho_1))$ *and* $(\lambda(\rho_2), u(\cdot, \rho_2))$*, and* (1.2) *has exactly three solutions when* $\lambda \in (\lambda(\rho_1), \min(\lambda(\rho_2), \lambda(\theta)))$*.*

To prove the theorem, we first prove two key lemmas.

**Lemma 3.2.** *Suppose that* $f$ *satisfies (f1)-(f2). In addition, we assume that there exists* $\alpha > \theta$ *such that* $f(u) > 0$ *for* $u \in [\theta, \alpha]$*,* $f''(u) \le 0$ *for* $u \in (0, \alpha)$*, and* (3.2) *is satisfied. Then* $u(\cdot, \alpha)$ *is a stable solution of* (1.2) *with* $\mu_1 > 0$*, the portion of the solution curve* $(\lambda(\rho), \rho)$*,* $\rho \in [\theta, \alpha]$*, makes exactly one turn at some* $\rho_1 \in (\theta, \alpha)$*, and* $(\lambda(\rho), \rho)$ *is exactly* $\subset$*-shaped.*

*Proof.* Let $u(x) = u(x, \alpha)$. We have $H(0) = 0$, $H'(u) = f(u) - uf'(u)$, $H''(u) = -uf''(u) \geq 0$, $H'(0) = f(0) < 0$. It follows that $H(u)$ is convex on $[0, \alpha]$, and it takes its negative minimum at some $u = \beta$. Define $x_0 \in (0,1)$ by $u(x_0) = \beta$. We then conclude

$$\begin{aligned} &f(u(x)) - u(x)f'(u(x)) \geq 0 \ \text{ on } (0, x_0), \\ &f(u(x)) - u(x)f'(u(x)) \leq 0 \ \text{ on } (x_0, 1). \end{aligned} \tag{3.4}$$

We also notice that by the condition (3.2)

$$\int_0^1 [f(u) - uf'(u)]\, u'(x)dx = \int_0^1 \frac{d}{dx} H(u(x))dx = -H(\alpha) \leq 0. \tag{3.5}$$

Assume that $\mu_1 \leq 0$ in (2.15). Without loss of generality, we assume that the principal eigenfunction $\phi > 0$ in $[0, 1)$. Since $\alpha > \theta$, then $u'(1) < 0$ and near $x = 1$ we have $-u'(x) > \phi(x)$. Since $-u'(0) = 0$, while $\phi(0) > 0$, the functions $\phi(x)$ and $-u'(x)$ change their order at least once on $(0, 1)$. We claim that the functions $\phi(x)$ and $-u'(x)$ change their order exactly once on $(0, 1)$. Observe that $-u'(x)$ satisfies

$$(-u')'' + \lambda f'(u)(-u') = 0, \ \ x \in (0, 1). \tag{3.6}$$

If $\mu_1 = 0$, then $\phi$ and $-u'$ satisfy the same differential equation, and $\phi > 0$, hence $\phi$ and $-u'$ can not change sign more than once. next we assume $\mu_1 < 0$. Suppose $\phi$ and $-u'$ change their order more than once on $(0, 1)$. Let $x_3 \in (0, 1)$ be the largest point where $\phi(x)$ and $-u'(x)$ change the order. Assuming the claim to be false, let $x_2$, with $0 < x_2 < x_3$, be the next point where the order changes. We have $\phi > -u'$ on $(x_2, x_3)$, and the opposite inequality to the left of $x_2$. Since $\phi(0) > -u'(0)$, there is another point $x_1 < x_2$, where the order is changed. We multiply (2.15) by $-u'$, multiply (3.6) by $\phi$, subtract and integrate from $x_1$ to $x_2$, then we obtain

$$\begin{aligned} &\phi(x_2)[\phi'(x_2) + u''(x_2)] - \phi(x_1)[\phi'(x_1) + u''(x_1)] \\ &+ \mu_1 \int_{x_1}^{x_2} (-u'(x))\phi(x)dx = 0, \end{aligned} \tag{3.7}$$

since $\phi(x_i) = -u'(x_i)$ for $i = 1, 2$. Let $t(x) = \phi(x) - (-u'(x))$. Then $t(x) \leq 0$ for $x \in (x_1, x_2)$ and $t(x) \geq 0$ for $x \in (x_2, x_3)$. Thus $t'(x_1) = \phi'(x_1) + u''(x_1) \leq 0$ and $t'(x_2) = \phi'(x_2) + u''(x_2) \geq 0$. Because $\phi(x) > 0$ and $-u'(x) > 0$ on $(0, 1)$, we get a contradiction in (3.7) with $\mu_1 < 0$.

Since the point of changing of order is unique, by scaling $\phi(x)$ we can achieve

$$\begin{aligned} &-u'(x) \leq \phi(x) \quad \text{on } (0, x_0), \\ &-u'(x) \geq \phi(x) \quad \text{on } (x_0, 1). \end{aligned} \tag{3.8}$$

Using (3.4), (3.8), and also (3.5), we have

$$\int_0^1 [f(u) - uf'(u)]\, \phi(x)\, dx > \int_0^1 [f(u) - uf'(u)]\, (-u'(x))\, dx \geq 0. \tag{3.9}$$

On the other hand, multiplying the equation (2.15) by $u$, the equation (1.2) by $\phi$, subtracting and integrating over $(0, 1)$, we have

$$\int_0^1 [f(u) - uf'(u)]\, \phi dx = \frac{\mu_1}{\lambda} \int_0^1 u\phi dx \leq 0,$$

which contradicts (3.9). So $\mu_1 > 0$.

From Lemma 2.1, $[\theta, \alpha] \subset D$, $u(\cdot, \alpha)$ is a stable solution, and from Lemma 2.6, $u(\cdot, \theta)$ is unstable. Thus by Lemma 2.5 and intermediate value theorem, there is a $\rho_1 \in (\theta, \alpha)$ such that $\lambda'(\rho_1) = 0$. Since $f''(u) \le 0$ for $u \in (0, \alpha)$, then at a degenerate solution $(\lambda(\rho), u(\cdot, \rho))$, we must have $\int_0^1 f''(u(x,\rho))w^3(x)dx < 0$, where $w$ is a solution of (2.9) and $w > 0$. Thus $\lambda''(\rho) > 0$ for any degenerate solution by (2.8) and Lemma 2.3, which implies there is a unique $\rho_1 \in (\theta, \alpha)$ such that $\lambda'(\rho_1) = 0$ and $\lambda''(\rho_1) > 0$, and the portion of bifurcation diagram between $\rho = \theta$ and $\rho = \alpha$ is exactly $\subset$-shaped. □

Lemma 3.2 is similar to Lemma 1 in [KS] where a positone problem was studied. The next lemma is essentially from Lemma 2.1 of [KL], so we refer the proof to [KL].

**Lemma 3.3.** *Suppose that $f \in C^2[0,\infty)$, $(\lambda(\rho), u(\cdot,\rho))$ is a degenerate solution of* (1.2)*, and $w$ is the corresponding solution of linearized equation* (2.9)*. Then*

$$\int_0^1 f''(u(x,\rho))u_x(x,\rho)w^2(x)dx = \frac{w^2(0)}{2F(\rho)}I(\rho), \tag{3.10}$$

*where $\rho = u(0,\rho)$, and*

$$I(\rho) = f^2(\rho) - 2F(\rho)f'(\rho). \tag{3.11}$$

Finally we have

**Lemma 3.4.** *Suppose $f \in C^1[0,\infty)$. If there exists $\rho \in (0,\infty)$ such that for $u \in [0,\rho)$, $H(u) >\ (<)\ H(\rho)$, and $(\lambda(\rho), u(\cdot,\rho))$ is a solution of* (1.2)*, then $\lambda'(\rho) <\ (>)\ 0$.*

*Proof.* Differentiating (1.4), we obtain

$$\frac{1}{2\sqrt{\lambda(\rho)}}\lambda'(\rho) = \frac{1}{2\sqrt{2}}\int_0^\rho \frac{[H(\rho) - H(u)]du}{\rho[F(\rho) - F(u)]^{3/2}}, \tag{3.12}$$

then the result follows from the given condition. □

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* From Lemma 2.1, there is no solution of (1.2) with $u(0) < \theta$. From Lemma 3.2 and Lemma 2.5, we know that the bifurcation diagram starts from $\rho = \theta$, $\lambda'(\theta) < 0$ and the portion of $\rho \in [\theta, \alpha]$ is exactly $\subset$-shaped with one turning point at $\rho_1 \in (\theta, \alpha)$ and $\lambda''(\rho_1) > 0$.

Consider the function $H(u) = 2F(u) - uf(u)$. We have $H''(u) \ge 0$ for $u \in [0, \alpha]$, $H''(u) \le 0$ for $u \in [\alpha, \infty)$ and $H(\alpha) > 0$. Since $f$ is asymptotic superlinear, and

$$\left[\frac{f(u)}{u}\right]' = -\frac{H'(u)}{u^2}, \tag{3.13}$$

then there exists $\gamma > \alpha$ such that $H'(\gamma) < 0$ and $H'(u) < H'(\gamma)$ for all $u > \gamma$. Thus $H(u) = H(\gamma) + \int_\gamma^u H'(t)dt < H(\gamma) + (u - \gamma)H'(\gamma) \to -\infty$ as $u \to \infty$.

From the properties of $H(u)$, there exists a unique $\eta_1 \in (\alpha, \gamma)$ such that $H'(\eta_1) = 0$ and $H(\eta_1) > 0$. For any $\rho \in [\alpha, \eta_1]$, the proof of Lemma 3.2 can be applied to prove that $u(\cdot, \rho)$ is a stable solution, and thus $\lambda'(\rho) > 0$ by Lemma 2.5. On the other hand, there exists $\eta_2 > \eta_1$, such that for any $u > \eta_2$, $H(u) < H(v)$ for any $v \in [0, u)$ since $\lim_{u\to\infty} H(u) = -\infty$, so $\lambda'(\rho) < 0$ for $\rho > \eta_2$ by Lemma 3.4. Obviously there is at least another turning point $\rho = \rho_2$ between $\rho = \eta_1$ and $\rho = \eta_2$. It remains to show that there is exactly one turning point. At $\rho = \eta_1$, $H'(\eta_1) = f(\eta_1) - \eta_1 f'(\eta_1) = 0$. Then $I(\eta_1) = \eta_1 f'(\eta_1)f(\eta_1) - 2F(\eta_1)f'(\eta_1) =$

$-f'(\eta_1)H(\eta_1) < 0$. Moreover $I'(u) = -2F(u)f''(u) < 0$ for $u > \eta_1$, hence $I(u) < 0$ for all $u \geq \eta_1$.

Suppose that $(\lambda(\rho), u(\cdot,\rho))$ is a degenerate solution of (1.2) such that $\rho > \eta_1$, and $w$ is the corresponding solution of linearized equation (2.9). Then $w > 0$ is the principal eigenfunction for (2.15). From the proof of Lemma 3.2, we know that $w$ and $-u_x$ $(-u_x(x,\rho))$ changes order exactly once, and we can assume that (3.8) holds with $\phi = w$ and $x_0$ satisfying $u(x_0,\rho) = \alpha$. Therefore

$$\begin{aligned} 0 < -\frac{w^2(0)}{2F(\rho)} I(\rho) &= \int_0^1 f''(u(x))(-u_x(x))w^2(x)dx \\ &< \int_0^1 f''(u(x))w^3(x)dx. \end{aligned} \tag{3.14}$$

And by (2.8) and Lemma 2.3, we have $\lambda''(\rho) < 0$. Hence there is exactly one turning point above the level $\rho = \eta_1$ on the bifurcation diagram $(\lambda(\rho), \rho)$.

Finally $\lim_{\rho\to\infty} \lambda(\rho) = 0$ since $f$ is asymptotic superlinear, and by estimating $G(\rho)$ in (1.4), we can show that. (See [CS1], [S2] for details.) □

Next we discuss the behavior of bifurcation diagrams if $f$ satisfies all conditions in Theorem 3.1 except (3.3). Instead, we assume that

$$\lim_{u\to\infty} \frac{f(u)}{u} = k \geq 0. \tag{3.15}$$

Note that $\lim_{u\to\infty} f(u)/u$ always exists if (f3) is satisfied.

**Case 1:** $H'(u) \geq 0$ for all $u > \alpha$. Then $H(u) > 0$ for all $u > \alpha$, and Lemma 3.2 can be applied to all $\rho > \alpha$. So $\lambda'(\rho) > 0$ for all $\rho > \alpha$. Notice that $f(u)/u$ is decreasing for $u > \alpha$, so $k$ is either 0 or a positive number. If $k = 0$, then $f$ is asymptotic sublinear, and $\lim_{\rho\to\infty} \lambda(\rho) = \infty$. (See Figure 5.) If $k > 0$, then $f$ is asymptotic linear, and $\lim_{\rho\to\infty} \lambda(\rho) = \pi/k$, the point where a bifurcation from infinity occurs. (See Figure 6.) The proofs of these statements can be found in [CGS] and [S2], and we omit them here.

**Case 2:** There exists $\gamma > \alpha$ such that $H'(\gamma) < 0$, then $H'(u) < H'(\gamma)$ for all $u > \gamma$ since $f''(u) \geq 0$ for $u \geq \alpha$. In particular $f(u)/u$ is increasing for $u > \alpha$, so $k$ is either $\infty$ (which we have considered in Theorem 3.1) or a positive number. So we assume $k > 0$. Then again, we know that $\lim_{\rho\to\infty} \lambda(\rho) = \pi/k$. Also $\lim_{u\to\infty} H(u) = -\infty$, then $\lambda'(\rho) < 0$ for large $\rho > 0$. So there is another turning point on the bifurcation diagram, and the bifurcation diagram is as Figure 7.

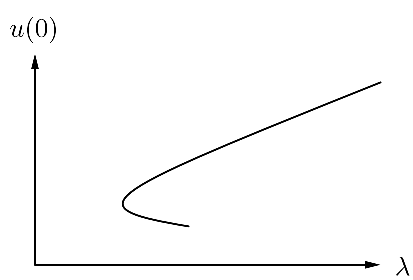

Fig. 5: Case 1a

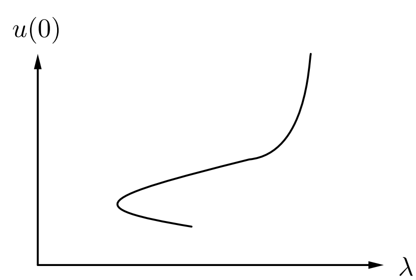

Fig. 6: Case 1b

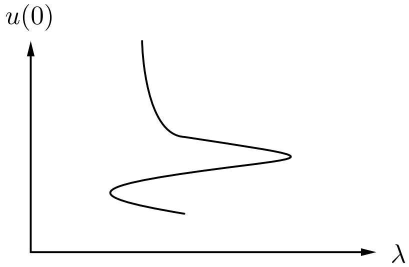


Fig. 7: Case 2

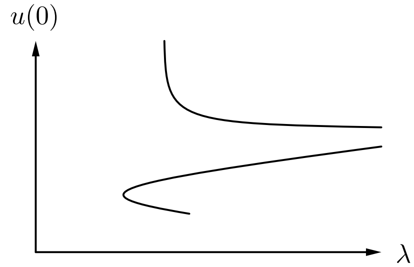


Fig. 8: Variant of Theorem 4.1

4. **Broken reversed $S$-shaped solution curve.** In this section we consider the bifurcation diagrams of (1.2) with $f$ satisfying (f1)-(f3) and (f4b). From Lemma 2.1, we know that if $F(u) < F(c)$ for all $u > d$, then (1.2) has no solution with $u(0) > c$. So we assume that $f$ also satisfies

(f5) There exists $\theta_1 > d$ such that $F(u) > F(c)$ for $u > \theta_1$ and $F(\theta_1) = F(c)$.

Our main result is

**Theorem 4.1.** *Suppose that $f$ satisfies (f1)-(f3), (f4b) and (f5). In addition, we assume that* (3.2) *and* (3.3) *hold, and*

$$I(\theta_1) \leq 0, \tag{4.1}$$

*where $\theta_1$ is defined in (f5) and $I(\rho)$ is defined in* (3.11) *Then the admissible set $D = [\theta, c) \bigcup (\theta_1, \infty)$, and the bifurcation diagram $\lambda(\rho)$ is exactly like Figure 4. More precisely, the bifurcation diagram has two connected components, the lower branch is exactly $\subset$-shaped with one turning point, and the upper branch is a monotone decreasing curve satisfying*

$$\lim_{\rho \to \theta_1^+} \lambda(\rho) = \infty, \quad \lim_{\rho \to \infty} \lambda(\rho) = 0. \tag{4.2}$$

*Proof.* From Lemma 2.1, we have $D = [\theta, c) \bigcup (\theta_1, \infty)$. So we call the connected component of solution curve with $\rho \in [\theta, c)$ the lower branch, and the one with $\rho \in (\theta_1, \infty)$ the upper branch.

If $\alpha > c$, then $f$ is concave in $[0, c]$, and the lower branch is therefore $\subset$-shaped from the result of [OS2]. If $\alpha \leq c$, for the portion $\rho \in [\theta, \alpha]$ on the lower branch, we can still use the proof of Theorem 3.1 to show that it is $\subset$-shaped. Furthermore, $H(u) > 0$ for any $u \in (\alpha, c)$, so by Lemma 3.2, all solutions with $u(0) \in (\alpha, c)$ are stable. Thus $\lambda'(\rho) > 0$ for $\rho \in (\alpha, c)$. So the lower branch is also $\subset$-shaped in this case.

For the upper branch, similar to the proof of Theorem 3.1, we have $\lambda'(\rho) < 0$ for $\rho > 0$ large enough, and $\lim_{\rho \to \theta_1} \lambda(\rho) = \infty$ because of the continuity of $R(\rho)$ defined in (2.4). By (4.1) and the proof of Theorem 3.1, $\lambda''(\rho) < 0$ at any degenerate solution if $\rho > \theta_1$, which implies there is no any degenerate solution with $u(0) > \theta_1$. The other parts of proof is same as that of Theorem 3.1. □

**Remark.**

1. If $f$ satisfies all conditions in Theorem 4.1 except (3.3), but it satisfies (3.15), then the bifurcation diagram will look like Figure 8.
2. The condition (4.1) holds if the following condition is true:

$$F(d) \geq 0. \tag{4.3}$$

In fact, if (4.3) is true, then $I(d) \le 0$ and $I'(u) \le 0$ for $u > d$, so $I(\theta_1) \le 0$. We conjecture that (4.1) is not necessary for the results in Theorem 4.1.

5. **An example.** We consider an example from [CS1]:

$$f(u) = u^3 - Au^2 + Bu - C, \tag{5.1}$$

where $A, B, C > 0$. We can easily verify that (f1) and (f3) are satisfied for this $f(u)$. (f3) is satisfied if $F(\alpha) = F(A/3) > 0$, that is equivalent to

$$B > \frac{A^3 + 36C}{6A}. \tag{5.2}$$

Since

$$H(u) = 2F(u) - uf(u) = -\frac{u^4}{2} + \frac{Au^3}{3} - Cu, \tag{5.3}$$

then (3.2) is satisfied if

$$A^3 \ge 54C. \tag{5.4}$$

For this $f(u)$, either (f4a) or (f4b) is satisfied. If

$$A^2 \le 3B, \tag{5.5}$$

then $f'(u) \ge 0$ for all $u \ge 0$, and (f4a) is satisfied. If

$$A^2 > 3B, \tag{5.6}$$

then $f'(u) = 0$ at

$$u = u_\pm = \frac{A \pm \sqrt{A^2 - 3B}}{3}. \tag{5.7}$$

$f(u_-) > 0$ is always true since $f(u_-) > f(\alpha) > 0$. And $f(u_+) > 0$ if

$$f(u_+) = \frac{2}{9}(3B^2 - A)u_+ + \frac{AB}{9} - C > 0, \tag{5.8}$$

that is equivalent to

$$K(A,B,C) = (9AB - 2A^2 - 27C)^2 - 4(A^2 - 3B)^3 > 0. \tag{5.9}$$

Therefore Theorem 3.1 holds if $f$ satisfies (5.2), (5.4), either (5.5), or (5.6) and (5.9).

If (5.6) holds, $f(u_+) < 0$, that is

$$K(A,B,C) = (9AB - 2A^2 - 27C)^2 - 4(A^2 - 3B)^3 < 0, \tag{5.10}$$

and (4.1), then Theorem 4.1 holds. (4.1) is harder to verify, but we point out that when $K(A,B,C) \in (-\varepsilon, 0)$ for some $\varepsilon > 0$, (4.3) (thus (4.1)) is satisfied.

## REFERENCES


[ACS] Ali, Ismael; Castro, Alfonso; Shivaji, R., Uniqueness and stability of nonnegative solutions for semipositone problems in a ball. *Proc. Amer. Math. Soc.* **117** (1993), no. 3, 775–782.

[BIS] Brown, K. J.; Ibrahim, M. M. A.; Shivaji, R., $S$-shaped bifurcation curves. *Nonlinear Anal.* **5** (1981), no. 5, 475–486.

[CG] Castro, Alfonso; Sudhasree, Gadam, Uniqueness of stable and unstable positive solutions for semipositone problems. *Nonlinear Anal.* **22** (1994), no. 4, 425–429.

[CGS] Castro, Alfonso; Gadam, Sudhasree; Shivaji, R., Branches of radial solutions for semipositone problems. *J. Differential Equations* **120** (1995), no. 1, 30–45.

[CS1] Castro, Alfonso; Shivaji, R., Nonnegative solutions for a class of nonpositone problems. *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), no. 3-4, 291–302.

[CS2] Castro, Alfonso; Shivaji, R., Positive solutions for a concave semipositone Dirichlet problem. *Nonlinear Anal.* **31** (1998), no. 1-2, 91–98.

[CMS] Castro, Alfonso; Maya, C.; Shivaji, R., Nonlinear eigenvalue problems with semipositone structure. To appear in *Elec. Jour. Diff. Equa.*, (2000).

[CR] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.* **52** (1973), 161–180.

[DL] Du, Yihong; Lou, Yuan, Proof of a conjecture for the perturbed Gelfand equation from combustion theory. (preprint), (1997).

[HM] Hastings, S. P.; McLeod, J. B., The number of solutions to an equation from catalysis. *Proc. Roy. Soc. Edinburgh Sect. A* **101** (1985), no. 1-2, 15–30.

[KhS] Khamayseh, A. K.; Shivaji, R., Evolution of bifurcation curves for semipositone problems when nonlinearities develop multiple zeroes. *Appl. Math. Comput.* **52** (1992), no. 2-3, 173–188.

[K] Korman, Philip, Curves of sign-changing solutions for semilinear equations, (submitted).

[KL] Korman, Philip; Li, Yi, On the exactness of an S-shaped bifurcation curve, *Proc. Amer. Math. Soc.* **127(4)**, 1011-1020, (1999).

[KLO] Korman, Philip; Li, Yi; Ouyang, Tiancheng, Exact multiplicity results for boundary-value problems with nonlinearities generalising cubic, *Proc. Royal Soc. Edinburgh, Ser. A.* **126A**, 599-616, (1996).

[KS] Korman, Philip; Shi, Junping, Instability and exact multiplicity of solutions of semilinear equations. To appear in *Elec. Jour. Diff. Equa.*, (2000).

[OS1] Ouyang Tiancheng; Shi, Junping, Exact multiplicity of positive solutions for a class of semilinear problem. *Jour. Diff. Equa.* **146**, 121-156, (1998).

[OS2] Ouyang Tiancheng; Shi, Junping, Exact multiplicity of positive solutions for a class of semilinear problems: II, *Jour. Diff. Equa.* **158**, 94-151, (1999).

[Sc] Schaaf, R., Global solution branches of two-point boundary value problems. *Lecture Notes in Mathematics*, **1458**. Springer-Verlag, Berlin, (1990).

[S1] Shi, Junping, Persistence and bifurcation of degenerate solutions, *Jour. Func. Anal.* **169**, 494-531, (1999).

[S2] Shi, Junping, Blow-up Points of Solution Curves for a Semilinear Problem. To appear in *Topo. Meth. of Nonlinear Anal.*, (2000).

[SW] Smoller, J.; Wasserman, A., Global bifurcation of steady-state solutions. *J. Differential Equations* **39** (1981), no. 2, 269–290.

[W] Wang, Shin Hwa, On $S$-shaped bifurcation curves. *Nonlinear Anal.* **22** (1994), no. 12, 1475–1485.


Received July 2000; revised October 2000.

*E-mail address*: shij@math.wm.edu
*E-mail address*: shivaji@ra.msstate.edu