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# Instability and exact multiplicity of solutions of semilinear equations *

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*Dedicated to Alan Lazer on his 60th birthday*

**Abstract**

For a class of two-point boundary-value problems we use bifurcation theory to show that a solution is unstable under a simple, geometric in nature, assumption on the non-linear term. As an application we obtain some new exact multiplicity results.

## 1 Introduction

It is well known that for convex $f(u)$, with $f(0) > 0$, the set of all positive solutions for the boundary-value problem (depending on a parameter $\lambda > 0$)

$$u'' + \lambda f(u) = 0 \quad \text{for } a < x < b, \quad u(a) = u(b) = 0 \tag{1}$$

is relatively simple. Indeed, if we use $(\lambda, u(0))$ to depict the solutions, then the set of all positive solutions consists of one curve, which admits at most one turn, see T. Laetsch [6]. Under what condition does the solution curve turn? T. Laetsch [6] showed that this is the case if $f(u)/u$ is eventually strictly increasing. Another well-known condition is $f(u) \geq c_1 u^p + c_2$ for all $u > 0$, with constants $c_1, c_2 > 0$, $p > 1$, see e.g. [4]. Both conditions restrict behavior of $f(u)$ for large $u > 0$.

A condition not restricting behavior at infinity came up in the work on the $S$-shaped curves, see the references in [2]. Let $F(u) = \int_0^u f(t)\,dt$, $h(u) = 2F(u) - uf(u)$. The condition in question is that $h(\alpha) < 0$ for some $\alpha > 0$. One shows that the solution with maximal value equal to $\alpha$ is unstable, from which one concludes that a turn must occur, since the solutions with small maximal value are stable. Previous proofs of this (in e.g. [2]) involved phase-plane analysis. In the present work we present a bifurcation theory proof of

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instability. One advantage of bifurcation theory is its flexibility. In Section 4 we extend our results to a class of non-autonomous problems, where phase-plane analysis is clearly not applicable. Inequality $h(\alpha) < 0$ implies that the area under the graph of $f(u)$ is smaller than that of the triangle with vertices $(0,0)$, $(0,u)$ and $(u, f(u))$. It means that $f(u)$ is "sufficiently convex" on $(0,\alpha)$. As an application we obtain some new exact multiplicity results for both autonomous and non-autonomous problems.

## 2 Instability and multiplicity

Without loss of generality we may work on the interval $(-1,1)$, and consider positive solutions of the boundary-value problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{2}$$

For any solution $u(x)$ of (2) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0. \tag{3}$$

Recall that solution $u(x)$ of (2) is called unstable if $\mu < 0$, otherwise it is stable. Recall also that a solution of (2) is called degenerate (or singular) if for $\mu = 0$ there is a non-trivial solution of (3). It is easy to see that for a positive degenerate solution any solution $w$ of (3) is of one sign, i.e. $\mu = 0$ can only be the principal eigenvalue. In fact, if $u$ is a positive degenerate solution, then $u$ is an even function, $u' < 0$ in $(0,1)$ and $u'$ satisfies $(u')'' + \lambda f'(u)u' = 0$. Then by Sturm comparison lemma, $w$ must be of one sign. It follows that unstable solutions are non-degenerate.

Let $F(u) = \int_0^u f(t)\,dt$, $h(u) = 2F(u) - uf(u)$. Our main instability result is

**Theorem 1** *Assume that $f \in C^1[0,\infty)$, $f(0) > 0$, and for some $\alpha > \beta > 0$ we have:*

$$h'(u) \geq 0 \quad \text{for } 0 < u < \beta, \quad h'(u) \leq 0 \quad \text{for } \beta < u < \alpha, \tag{4}$$

$$h(\alpha) \leq 0\,. \tag{5}$$

*Then the solution of (2) with $u(0) = \alpha$ is unstable if it exists.*

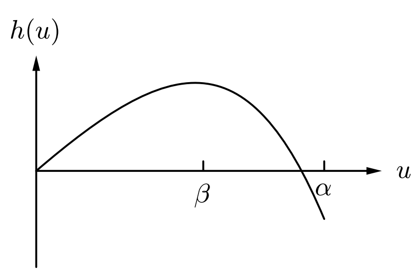


Figure 1: The graph of $h(u)$

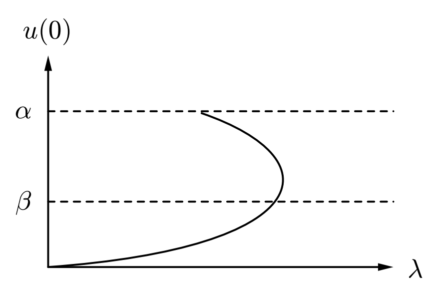


Figure 2: Bifurcation Diagram

**Proof:** We have $h(0) = 0$, $h'(u) = f(u) - uf'(u)$, $h'(0) = f(0) > 0$. It follows that $h(u)$ has a unique critical point (a local maximum) on $[0, \alpha]$, and it takes its positive maximum at $u = \beta$. (See Figure 1.) Define $x_0 \in (0, 1)$ by $u(x_0) = \beta$. We then conclude

$$\begin{aligned} f(u(x)) - u(x)f'(u(x)) &\le 0 \quad \text{on } (0, x_0), \\ f(u(x)) - u(x)f'(u(x)) &\ge 0 \quad \text{on } (x_0, 1). \end{aligned} \tag{6}$$

We also remark that by the condition (5),

$$\int_0^1 [f(u) - uf'(u)]\, u'(x)\, dx = \int_0^1 \frac{d}{dx} h(u(x))\, dx = -h(\alpha) \ge 0. \tag{7}$$

Assume now that $u(x)$ is stable, i.e. $\mu \ge 0$ in (3). Without loss of generality, we assume that $w > 0$ in $(-1, 1)$. By the maximum principle, $u'(1) < 0$, so near $x = 1$ we have $-u'(x) > w(x)$. Since $-u'(0) = 0$, while $w(0) > 0$, the functions $w(x)$ and $-u'(x)$ change their order at least once on $(0, 1)$. We claim that the functions $w(x)$ and $-u'(x)$ change their order exactly once on $(0, 1)$. Observe that $-u'(x)$ satisfies

$$(-u')'' + \lambda f'(u)(-u') = 0 \quad \text{on } (0, 1), \tag{8}$$

Let $x_3 \in (0, 1)$ be the largest point where $w(x)$ and $-u'(x)$ change the order. Assuming the claim to be false, let $x_2$, with $0 < x_2 < x_3$, be the next point where the order changes. We have $w > -u'$ on $(x_2, x_3)$, and the opposite inequality to the left of $x_2$. Since $w(0) > -u'(0)$, there is another point $x_1 < x_2$, where the order is changed. We multiply (3) by $-u'$, multiply (8) by $w$, subtract and integrate from $x_1$ to $x_2$, then we obtain

$$\begin{aligned} w(x_2)[w'(x_2) + u''(x_2)] - w(x_1)[w'(x_1) + u''(x_1)] & \\ + \mu \int_{x_1}^{x_2} (-u'(x)) w(x) dx &= 0\,, \end{aligned} \tag{9}$$

since $w(x_i) = -u'(x_i)$ for $i = 1, 2$. Let $t(x) = w(x) - (-u'(x))$. Then $t(x) \le 0$ for $x \in (x_1, x_2)$ and $t(x) \ge 0$ for $x \in (x_2, x_3)$. Thus $t(x_1) = w'(x_1) + u''(x_1) \le 0$ and $t(x_2) = w'(x_2) + u''(x_2) \ge 0$. Because $w(x) > 0$ and $-u'(x) > 0$ on $(0, 1)$, we get a contradiction in (9).

Since the point of changing of order is unique, by scaling $w(x)$ we can achieve

$$\begin{aligned} -u'(x) &\le w(x) \quad \text{on } (0, x_0), \\ -u'(x) &\ge w(x) \quad \text{on } (x_0, 1). \end{aligned} \tag{10}$$

Using (6), (10), and also (7), we have

$$\int_0^1 [f(u) - uf'(u)]\, w(x)\, dx < \int_0^1 [f(u) - uf'(u)]\, (-u'(x))\, dx \le 0. \tag{11}$$

On the other hand, multiplying the equation (3) by $u$, the equation (2) by $w$, subtracting and integrating over $(0,1)$, we have

$$\int_0^1 [f(u) - uf'(u)]\, w(x)\, dx = \frac{\mu}{\lambda} \int_0^1 uw\, dx \geq 0,$$

which contradicts (11). So $\mu < 0$. ♢

**Remarks.**

1. Theorem 1 is stated in a way that we assume the existence of a solution with $u(0) = \alpha$. In fact, if $f(u) > 0$ for all $u \in [0, \alpha]$, then for any $d \in (0, \alpha]$, there exists a unique $\lambda(d)$ such that (2) has a positive solution with $\lambda = \lambda(d)$ and $u(0) = d$, see e.g. [4]. (See Figure 2.)

2. It is easy to see that the condition (4) holds if

$$f''(u) > 0 \quad \text{for } 0 < u < \alpha, \tag{12}$$

   and (5) is also satisfied. So Theorem 1 is true if we replace (4) by (12).

3. It is well-known that if for some $\beta > 0$, $f(u) > 0$ and $h'(u) \geq 0$ for $0 \leq u \leq \beta$, then the solutions of (2) with $u(0) = d$ and $0 < d \leq \beta$ are all stable. (See for example [8] Theorem 6.2.) Thus Theorem 1 implies that if $f$ is convex and positive, and satisfies (5), then the unique degenerate solution $u$ satisfies $\beta < u(0) < \alpha$. (See Figure 2.)

Our first application of Theorem 1 is to fully exclude the phase plane arguments from the proof of exact $S$-shapedness in P. Korman and Y. Li [2]. The main result of that paper showed that solution curve is exactly S-shaped, provided that the conditions of the Theorem 1 and (12) hold, and in addition $f''(u) < 0$ for $u > \alpha$, and $\lim_{u \to \infty} f(u)/u = 0$. The proof used a bifurcation theoretic approach, except at one point, when the phase plane argument was used to show that when $u(0) = \alpha$ the solution curve travels to the left, i.e. $\lambda$ is decreasing when $u(0)$ is increasing (see formula (2.13) in [2]). Theorem 1 above provides an alternative proof of this fact. Indeed, the solution curve starts at $\lambda = 0, u = 0$, which is a stable solution (the principal eigenvalue of the corresponding linearized problem $= \pi^2/4$). As we increase $\lambda$, the solutions on the curve continue to be stable until a degenerate solution is reached. Since $f(u)$ is convex for $u < \alpha$, a standard bifurcation analysis shows that a turn to the left occurs at a degenerate solution, see e.g. [3]. Hence the solution curve admits at most one turn for $u(0) < \alpha$, and since by Theorem 1 the solution at $u(0) = \alpha$ is unstable, this turn has already occurred, and so the solution curve travels to the left. The rest of the proof follows [2]. (See Figure 3.)

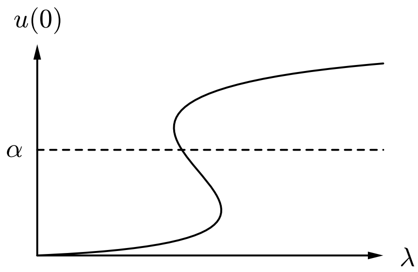


Figure 3: $S$-shaped curve

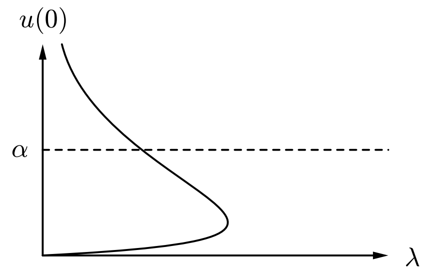


Figure 4: $\supset$-shaped curve

We derive next several new exact multiplicity results, where the nonlinear term $f(u)$ does not have to be convex.

**Theorem 2** *Assume $f \in C^1[0,\infty)$, $f(u) > 0$ for all $u > 0$, and assume that for some $\alpha > 0$ we have (5), (12), and*

$$h'(u) = f(u) - uf'(u) < 0 \quad \textit{for all } u > \alpha, \tag{13}$$

*Then there exist two constants $0 \le \bar\lambda < \lambda_0$ so that the problem (2) has no solution for $\lambda > \lambda_0$, exactly two solutions for $\bar\lambda < \lambda < \lambda_0$, and in case $\bar\lambda > 0$ it has exactly one solution for $0 < \lambda < \bar\lambda$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 4.) If we moreover assume that*

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty \tag{14}$$

*then $\bar\lambda = 0$.*

**Proof:** By the implicit function theorem there is a curve of positive solutions of (2), starting at $\lambda = 0$, $u = 0$. This curve continues for increasing $\lambda$, with $u(0)$ increasing, see e.g. [3]. From the Theorem 1 and the above remarks on [2], we know that by the time the solution curve reaches $u(0) = \alpha$ it has made exactly one turn, and it travels to the left. If we can show that any solution with $u(0) > \alpha$ is also unstable (and in particular non-degenerate), the proof will follow, since the solution curve always travels to the left.

To show that any solution with $u(0) > \alpha$ is unstable, we notice that from (13), $h'(u) \le 0$ for $u > \alpha$, then $h(u) \le h(\alpha) < 0$ for all $u > \alpha$. Thus Theorem 1 can be applied to any $u > \alpha$ as well. Therefore we conclude that for all $u(0) > \alpha$ the solution $u(x)$ is unstable, and the solution curve always moves to the left. Finally, it is well known that the condition (14) prevents the solution curve from going to infinity at a positive $\bar\lambda$, see e.g. [4]. $\diamondsuit$

**Remark.** The result in Theorem 3 is well-known under the conditions $f(u) > 0$ and $f''(u) > 0$ for all $u > 0$. (See [6].) Here we only assume (13) for $u > \alpha$, so $f''$ can change sign for $u > \alpha$. We conjecture that this result is true without any convexity condition but just assuming $f(u) > 0$ for all $u > 0$, $h(\alpha) < 0$ and

$$h'(u) \ge 0 \quad \text{for } 0 < u < \beta, \quad h'(u) \le 0 \quad \text{for } \beta < u < \infty. \tag{15}$$

In our proof the convexity of $f(u)$ for $u$ in $(0,\alpha)$ is still needed to conclude that there is only one turning point in the portion $u(0) \in (0,\alpha)$ of the solution curve.

In the following result we allow $f(u)$ to be concave for small $u$.

**Theorem 3** *Assume $f \in C^2[0,\infty)$, $f(u) > 0$ for all $u > 0$, and assume that for some $\alpha > \beta > 0$ we have (5), (13) and*

$$f''(u) < 0 \quad \text{for } 0 < u < \beta, \quad f''(u) > 0 \quad \text{for } \beta < u < \alpha, \tag{16}$$

*(If $\alpha = \infty$, the condition (13) can be omitted.) Then there exist two constants $0 \le \bar{\lambda} < \lambda_0$, so that the problem (2) has no solution for $\lambda > \lambda_0$, exactly two solutions for $\bar{\lambda} < \lambda < \lambda_0$, and in case $\bar{\lambda} > 0$ it has exactly one solution for $0 < \lambda < \bar{\lambda}$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 4.) If we moreover assume that (14) holds then $\bar{\lambda} = 0$.*

**Proof:** We follow the same scheme as in the Theorem 2. Two things need to be checked: that when $u(0) < \alpha$ a turn to the left occurs on any degenerate solution, while for $u(0) \ge \alpha$ any solution of (2) is unstable.

To see that only a turn to the left can occur when $u(0) < \alpha$, we follow the approach in P. Korman, Y. Li and T. Ouyang [3] (see also T. Ouyang and J. Shi [8]). Assume $(\lambda, u(x))$ is a degenerate solution of (2), i.e. the problem (3) has a non-trivial solution $w(x)$ at $\mu = 0$. Differentiating the equation (2), we have

$$u_x^{''} + \lambda f'(u) u_x = 0. \tag{17}$$

Similarly, differentiating the linearized equation for (2)

$$w_x^{''} + \lambda f'(u) w_x + \lambda f''(u) u_x w = 0. \tag{18}$$

Multiply the equation (18) by $u_x$, the equation (17) by $w_x$, integrate over $(0,1)$ and subtract. After expressing from the corresponding equations $w''(1) = -\lambda f'(u(1))w(1) = 0$, and $u''(1) = -\lambda f(u(1)) = -\lambda f(0)$, we obtain

$$\int_0^1 f''(u) u_x^2 w \, dx = -f(0) w'(1) > 0. \tag{19}$$

Define $x_0 \in (0,1)$ by $u(x_0) = \beta$. Arguing as in [3] (see also the Theorem 1 of the present work) by scaling $w(x)$ we can achieve the inequalities (10) above. Using our condition (16), and the inequality (19), we have

$$\int_0^1 f''(u) w^3 \, dx > \int_0^1 f''(u) u_x^2 w \, dx > 0. \tag{20}$$

This implies that only turns to the left are possible, see [3] for more details.

We claim next for that $u(0) \ge \alpha$ any solution of (2) is unstable. We apply the Theorem 1 to any solution $u$ with $u(0) \ge \alpha$. Since $h(0) = 0$, $h'(0) = f(0) > 0$, and $h''(u) = -uf''(u) > 0$ for $u \in (0,\beta)$, it follows that the function $h(u)$ is positive and increasing on $(0,\beta)$. Since $h(u)$ is concave for $u > \beta$, and eventually $h(u)$ is non-positive (at $u = \alpha$), it follows that $h(u)$ must have a unique point of maximum at some $u_1 > \beta$, and then decrease for $u_1 < u < \alpha$. For $u \geq \alpha$, $h'(u) \leq 0$ and $h(u) \leq h(\alpha) < 0$. (See Figure 5.) We conclude that the inequalities (4) hold even if we replace $\alpha$ in (4) by any $u > \alpha$, and so the Theorem 1 can be applied in case $u(0) > \alpha$, to prove the instability of the solution. ♢

**Example** Consider the problem

$$u'' + \lambda(u^3 - au^2 + bu + c) = 0 \quad \text{for } -1 < x < 1,$$
$$u(-1) = u(1) = 0,$$

with positive constants $a$, $b$ and $c$. Here concavity changes at $\beta = a/3$, while $h(u) = -(1/2)u^4 + (a/3)u^3 + cu$ becomes negative at some $\alpha > \beta$. So we only need to assume that $b > a^2/4$ (to assure that $u^3 - au^2 + bu > 0$ (so $f(u) > 0$) for all $u > 0$) for the Theorem 3 to apply (with $\bar{\lambda} = 0$). A similar result was previously obtained by S.-H. Wang and D.-M. Long [9], who required that $b > 49a^2/160$, and $c$ small enough. We remark though that the nice result of [9] has some advantages over our Theorem 3. Indeed, the result of [9] does not involve the second derivative assumptions (they assume basically that the function $h(u)$ has properties similar to ours, and some technical conditions), and they can allow some $f(u)$ that change concavity more than once.

A special case of the Theorem 3 is when $f''(u) > 0$ for $u > \alpha$, then (13) is satisfied in that case. For that nonlinearity and the higher dimensional analog of (2), T. Ouyang and J. Shi proved the results in the Theorem 3. (See Theorem 6.21 in [8].) Here we do not need to assume the convexity of $f$ for $u > \alpha$. On the other hand, $f(0) > 0$ can be replaced by $f(0) = 0$ and $f'(0) \geq 0$. (See details in [8].)

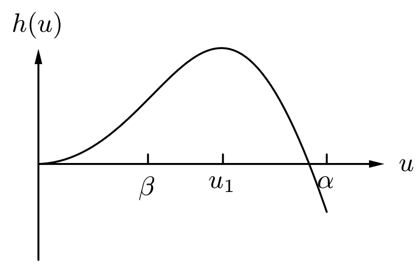


Figure 5: The graph of $h(u)$

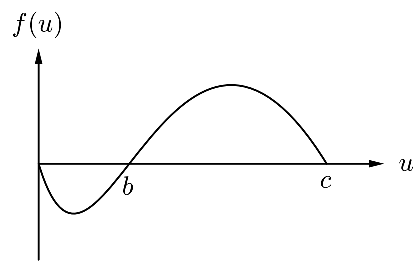


Figure 6: The graph of $f(u)$

## 3 A dual version of instability result

The instability result in Theorem 1 has a dual counterpart in the following theorem:

**Theorem 4** *Assume that $f \in C^1[0,\infty)$ and for some $\alpha > \beta > 0$ we have:*

$$h'(u) \le 0 \quad \text{for } 0 < u < \beta, \quad h'(u) \ge 0 \quad \text{for } \beta < u < \alpha, \tag{21}$$

$$h(\alpha) \ge 0\,. \tag{22}$$

*(see Figure 7 for the graph of $h(u)$.) Then the solution of (2) with $u(0) = \alpha$ is strictly stable if it exists (i.e. we have $\mu > 0$ in (3)).*

The proof of Theorem 4 is exactly same as Theorem 1 except switching all $\le$ and $<$ by $\ge$ and $>$. As an application, we prove a result which generalizes one of the main results of [3].

**Theorem 5** *Assume $f \in C^2[0,\infty)$, $f(0) = 0$, $f(u) < 0$ for $u \in (0,b) \cup (c,\infty)$, and $f(u) > 0$ for $u \in (b,c)$, where $c > b > 0$. Assume that for some $c > \alpha > \beta > b$ we have*

$$f''(u) > 0 \quad \text{for } 0 < u < \beta, \quad f''(u) < 0 \quad \text{for } \beta < u < \alpha, \tag{23}$$

$$h(\alpha) > 0, \tag{24}$$

$$h'(u) = f(u) - uf'(u) > 0 \quad \text{for all } u > \alpha. \tag{25}$$

*(See the graph of $f(u)$ in Figure 6.) Then there exists a constant $\lambda_0 > 0$, so that the problem (2) has no solution for $\lambda < \lambda_0$, exactly two solutions for $\lambda > \lambda_0$, and exactly one solution for $\lambda = \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 8.)*

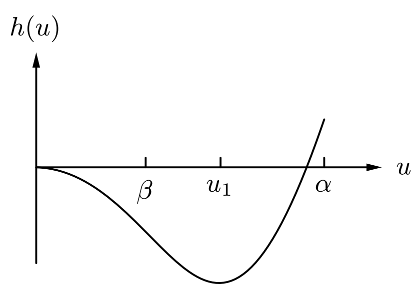


Figure 7: The graph of $h(u)$

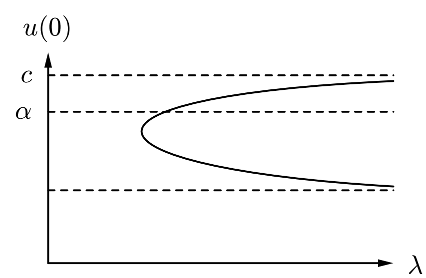


Figure 8: ⊂-shaped curve

**Proof:** Our proof combines the main ingredients of the proof in [3] and that of the Theorem 2. Since $h(0) = 0$, $h'(0) = 0$ and $h''(u) = -uf''(u) < 0$ for $u$ near 0, then $h(u) < 0$ and $h'(u) < 0$ near $u = 0$. Since $h$ is concave for $u \in (0,\beta)$, we have $h'(u) < 0$ for $u \in (0,\beta)$. But $h(\alpha) > 0$, then $h$ must have a local minimum in $(\beta,\alpha)$, which is unique since $h''(u) > 0$ in $(\beta,\alpha)$. Let the unique minimum of $h$ in $(\beta,\alpha)$ be $u_1$. Then $h'(u) > 0$ in $(u_1,\alpha)$, and $h(u) > 0$, $h'(u) > 0$ for $u > \alpha$ by (25). In particular, for any $u \ge \alpha$, we have (21), and so the Theorem 4 applies.

On the other hand, in [3], it is proved that for large $\lambda > 0$, (2) has a positive solution $u(\lambda, \cdot)$ which satisfies $\lim_{\lambda \to \infty} u(\lambda, 0) = c$. Thus there exists a small $\delta > 0$ such that for any $d \in (c-\delta, c)$, (2) has a positive solution $u$ with $u(0) = d$. From Theorem 4, as long as $c - \delta > \alpha$, that solution is strictly stable. So the solutions with $u(0) \in (c-\delta, c)$ are on a smooth curve, which continues to left as $\lambda$ and $u(0)$ decrease. We can continue the curve without any turns to the level $u(0) = \alpha$, since all the solutions with $\alpha \leq u(0) < c$ are strictly stable (hence non-degenerate). For the rest of the proof, we can exactly follow the proof in [3]. $\diamondsuit$

Other applications of Theorem 4 are also possible. For example, Theorems 1.2 and 1.3 in [7] (see also Theorems 6.18 and 6.19 in [8] which allows $f(0) < 0$) can be generalized in a similar way, by replacing $f''(u) < 0$ in $(\alpha, \infty)$ or $(\alpha, c)$ by merely assuming $h'(u) > 0$ in those intervals. We would leave the details to readers.

# 4 A class of symmetric nonlinearities

For the autonomous equation (2) both phase-plane analysis and bifurcation theory apply. If we allow explicit dependence of the nonlinearity on $x$, i.e. consider

$$u'' + \lambda f(x, u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0 \tag{26}$$

then the problem becomes much more complicated. In a series of papers P. Korman and T. Ouyang have considered a class of $f(x, u)$ for which the theory of positive solutions is very similar to that for the autonomous case, see e.g. [4], [5]. Namely, they assumed that $f \in C^2$ satisfies

$$f(-x, u) = f(x, u) \quad \text{for all } -1 < x < 1, \text{ and } u > 0, \tag{27}$$

$$f_x(x, u) < 0 \quad \text{for all } 0 < x < 1, \text{ and } u > 0. \tag{28}$$

Recall that under the above conditions any positive solution of (26) is an even function, with $u'(x) < 0$ for all $x \in (0, 1]$, see [1]. As before the linearized problem

$$w'' + \lambda f_u(x, u) w = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0 \tag{29}$$

will be important for the multiplicity results. In [5] it was shown that under the conditions (27), and (28) any non-trivial solution of (29) is of one sign. We now add another general result for this class of equations.

**Theorem 6** *In addition to (27) and (28) assume that*

$$f(x, u) > 0 \quad \textit{for all } -1 < x < 1, \textit{ and } u > 0. \tag{30}$$

*Then the set of positive solutions of (26) can be parameterized by their maximum values $u(0)$. (I.e. $u(0)$ uniquely determines the pair $(\lambda, u(x))$.)*

**Proof:** Let on the contrary $v(x)$ be another solution of (26) corresponding to some parameter $\mu \geq \lambda$, but $u(0) = v(0)$. The case of $\mu = \lambda$ is not possible in view of uniqueness of initial value problems, so assume that $\mu > \lambda$. Then $v(x)$ is a supersolution of (26), i.e.

$$v'' + \lambda f(x, v) < 0 \quad \text{for } -1 < x < 1, \quad v(-1) = v(1) = 0. \tag{31}$$

Since $v''(0) < u''(0)$, it follows that $v(x) < u(x)$ for $x > 0$ small. Let $0 < \xi \leq 1$ be the first point where the graphs of $u(x)$ and $v(x)$ intersect (i.e. $v(x) < u(x)$ on $(0, \xi)$). We now multiply the equation (26) by $u'$, and integrate over $(0, \xi)$. Denoting by $x_2(u)$ the inverse function of $u(x)$ on $(0, \xi)$, we have

$$\frac{1}{2} u'^2(\xi) + \lambda \int_{u(0)}^{u(\xi)} f(x_2(u), u)\, du = 0. \tag{32}$$

Similarly denoting by $x_1(u)$ the inverse function of $v(x)$ on $(0, \xi)$, we have from (31)

$$\frac{1}{2} v'^2(\xi) + \lambda \int_{u(0)}^{u(\xi)} f(x_1(u), u)\, du > 0. \tag{33}$$

Subtracting (33) from (32), noticing that $x_2(u) > x_1(u)$ for all $u \in (u(\xi), u(0))$, and using the condition (28), we have

$$\frac{1}{2} \left[ u'^2(\xi) - v'^2(\xi) \right] + \lambda \int_{u(\xi)}^{u(0)} \left[ f(x_1(u), u) - f(x_2(u), u) \right] du < 0. \tag{34}$$

Since both terms on the left are positive, we obtain a contradiction. $\diamondsuit$

Next we consider positive solutions of the boundary-value problem

$$u'' + \lambda b(x) f(u) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{35}$$

We assume that $b(x)$ satisfies $b(x) > 0$ for $x \in [0, 1]$, $b(x) = b(-x)$, $b'(x) < 0$ for $x \in (0, 1)$, and $f(u) > 0$, so that this problem belongs to the class discussed above. For any solutions $u(x)$ let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$\begin{gathered} w'' + \lambda b(x) f'(u) w + \mu w = 0 \quad \text{for } -1 < x < 1, \\ w(-1) = w(1) = 0. \end{gathered} \tag{36}$$

**Theorem 7** *Assume $f \in C^2[0, \infty)$, $f(u) > 0$, $f'(u) > 0$ for all $u > 0$, and for some $\alpha > 0$ the conditions (5) and (12) are satisfied. Then the solution of (35) with $u(0) = \alpha$ is unstable if it exists.*

**Proof:** In the proof of Theorem 1, (6) and (7) are still true. Assume now that $u(x)$ is stable, i.e. $\mu \geq 0$ in (36). Then $w(x)$ is a positive solution of the problem

$$w'' + g(x, w) = 0 \quad \text{for } -1 < x < 1, \quad w(-1) = w(1) = 0, \tag{37}$$

with $g(x,w) = \lambda b(x) f'(u(x))w + \mu w$. Since $g(x,w)$ is even in $x$, and

$$g_x = \lambda b'(x) f'(u)w + \lambda b(x) f''(u)u'w < 0 \quad \text{on } (0,1),$$

the theorem of B. Gidas, W.-M. Ni and L. Nirenberg [1] applies to (37). It follows that $w(x)$ is an even function with $w'(x) < 0$ on $(0,1)$. Recall that $w(x)$ is determined up to a constant multiple. Since $w(x)$ is decreasing, while $-u'(x)$ is increasing on $(0,1)$, by scaling $w(x)$ we can achieve (10). Using (6), (10), and also (7), we have (11).

Since $b(x) > 0$, $b'(x) < 0$ in $(0,1)$ using (6) and (11), we have

$$\begin{aligned}
&\int_0^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx \qquad (38)\\
&= \int_0^{x_0} b(x)\,[f(u) - uf'(u)]\,w(x)\,dx + \int_{x_0}^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx \\
&< \int_0^{x_0} b(x_0)\,[f(u) - uf'(u)]\,w(x)\,dx + \int_{x_0}^1 b(x_0)\,[f(u) - uf'(u)]\,w(x)\,dx \\
&= b(x_0)\int_0^1 [f(u) - uf'(u)]\,w(x)\,dx \le 0\,.
\end{aligned}$$

On the other hand, multiplying the equation (36) by $u$, the equation (35) by $w$, subtracting and integrating over $(0,1)$, we have

$$\int_0^1 b(x)\,[f(u) - uf'(u)]\,w(x)\,dx = \frac{\mu}{\lambda}\int_0^1 uw\,dx \ge 0. \qquad (39)$$

We reach a contradiction by combining (38) and (39). ♢

As an application we have the following exact multiplicity result. It extends the corresponding result in [4] by not restricting the behavior of $f(u)$ at infinity. Its proof is similar to that of the Theorem 2. Theorem 6 above allows us to conclude the uniqueness of the solution curve.

**Theorem 8** *Assume $f \in C^2[0,\infty)$, $f(u) > 0$, $f'(u) > 0$ and $f''(u) > 0$ for all $u > 0$, while $h(\alpha) \le 0$ for some $\alpha > 0$. Then there exist two constants $0 \le \bar{\lambda} < \lambda_0$, so that the problem (35) has no solution for $\lambda > \lambda_0$, exactly two solutions for $\bar{\lambda} < \lambda < \lambda_0$, and in case $\bar{\lambda} > 0$ it has exactly one solution for $0 < \lambda < \bar{\lambda}$. Moreover, all solutions lie on a unique smooth solution curve. If we moreover assume that (14) holds then $\bar{\lambda} = 0$.*

**Example.** Theorem 8 applies (with $\bar{\lambda} = 0$) to an example from combustion theory

$$u'' + \lambda b(x)e^u = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0,$$

where $b(x)$ satisfies the above conditions.

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