Asymptotic Spatial Patterns and Entire Solutions of Semilinear Elliptic Equations *†

Junping Shi
Department of Mathematics, College of William and Mary, Williamsburg, VA 23185
and, Department of Mathematics, Harbin Normal University, Harbin, Heilongjiang, P.R.China
Email: shij@math.wm.edu

In this notes we survey some old and new results on the entire solutions of semilinear elliptic equations, *i.e.* the solutions of

$$\Delta u + f(u) = 0, \quad x \in \mathbf{R}^n, \tag{1}$$

where $n \geq 1$, and $f(u)$ is a smooth function. The solutions of (1) are related to the equilibrium solutions of a singularly perturbed reaction-diffusion equation or system, for example,

$$\varepsilon^2 \Delta u_\varepsilon + f(u_\varepsilon) = 0, \quad x \in \Omega, \qquad Bu = 0, \quad x \in \partial\Omega, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, Ω is a smooth bounded domain in $\mathbf{R}^n$, and Bu is an appropriate boundary condition. The connection of (1) and (2) are made by a typical technique called blowup method. Suppose that $\{u_\varepsilon\}$ is a family of solutions of (2). The simplest setup of the blowup method is to choose $P_\varepsilon \in \overline{\Omega}$, and define $v_\varepsilon(y) = u_\varepsilon(\varepsilon y + P_\varepsilon)$, for $y \in \Omega_\varepsilon = \{y : \varepsilon y + P_\varepsilon \in \Omega\}$. Then usually in a proper sense, $v_\varepsilon(y) \to U(y)$, as $\varepsilon \to 0$, where U is an entire solution if P_ε is not too close to $\partial\Omega$. (See for example, Gidas and Spruck [GS2], and Ni and Takagi [NT1, NT2].) Thus the *local* spatial pattern of the equilibrium solution to the reaction-diffusion equation is governed by the spatial pattern of the entire solution. On the other hand, if u_ε is bounded and nonnegative, then it is also natural to require the entire solution to be bounded and nonnegative.

A related question is the connection of the solutions of (2) to the equation on a half-space:

$$\Delta u + f(u) = 0, \quad x \in \mathbf{R}^n_+ = \{x_n > 0\}, \qquad u = 0, \quad x \in \{x_n = 0\}, \tag{3}$$

and the boundary blowup of (2) when $Bu = u$ (Dirichlet boundary) and $P_\varepsilon \in \partial\Omega$ (or near $\partial\Omega$). Under very general conditions, it has been shown that the positive solutions of (3) are functions with form $u = u(x_n)$, thus (3) is reduced to an ODE $u'' + f(u) = 0$. More precisely, the following result has been proved by Dancer [D1] and Berestycki, Caffarelli, and Nirenberg [BCN2]: (see also earlier result by Angenent [A] and the survey paper by Berestycki [B])

Theorem 1. *Suppose that f is Lipschitz countinuous, and u is a bounded positive solution of* (3). *Then*

1. *If $f(0) \geq 0$, then $\dfrac{\partial u}{\partial x_n} > 0$ for $x \in \mathbf{R}^n_+$;*

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2. *If* $f(0) \geq 0$ *and* $n = 2$ *or* 3*, then* $u(x', x_n) = u(x_n)$*, i.e.* u *is a function of* x_n *alone, and* $u'(x_n) > 0$*;*

3. *If* $f(\sup u) \leq 0$*, then* $u(x', x_n) = u(x_n)$*, i.e.* u *is a function of* x_n *alone, and* $u'(x_n) > 0$*.*

From Theorem 1, we can conclude that there are not many spatial patterns for the half-space problem in general. on the other hand, we also notice that the case of $f(0) < 0$ and $n \geq 4$ for (3) is still partially open.

For the whole space problem (1), the earliest result on entire solutions is the classical theorem of Liouville in 1831 (see [GT]):

Theorem 2.

1. *Every solution of* $\Delta u = 0$ *which is bounded from below or above on* $\mathbf{R}^n$ *is a constant.*

2. *Every bounded subharmonic function in* $\mathbf{R}^2$ *is a constant. Thus when* $n = 2$*, if* $f(u) \geq$ *or* $(\leq)$ 0 *for all* u*, then any bounded solution of* (1) *is a constant.*

It is not very hard to show that when $n \geq 3$, there exists non-constant bounded subharmonic function defined on $\mathbf{R}^n$. But results of Liouville-type have been found for $n \geq 3$, though usually with much more advanced methods. An example is the solutions of

$$\Delta u + u^p = 0, \quad x \in \mathbf{R}^n, \quad n \geq 3, \quad p > 0, \quad u > 0. \tag{4}$$

Gidas and Spruck [GS1] prove that when $1 < p < (n+2)/(n-2)$ (it can also be extended to $p \in (0, 1]$), then the solution u of (4) must be $u \equiv 0$; and it has been observed that when $p \geq (n+2)/(n-2)$, (4) does have infinite many radially symmetric positive solutions. Later Chen and Li [CL] proves the same result but with a much shorter proof. Here the Liouville-type theorem is proved by showing the radial symmetry of the solution, which was first used in Gidas, Ni and Nirenberg [GNN], where they also prove a result similar to that of [GS1] but with a decaying condition on u at ∞. More recently, such techniques are also adapted to the case of super critical exponent $p \geq (n+2)/(n-2)$. A summary of the radial symmetry of the solutions of (4) is as follows:

Theorem 3. *Suppose that* u *is a positive solution of* (4)*.*

1. ([GS1, CL]) *If* $0 < p < \dfrac{n+2}{n-2}$*, then* $u \equiv 0$*;*

2. (Caffarelli, Gidas and Spruck [CGS]) *If* $p = \dfrac{n+2}{n-2}$*, then* u *must be radially symmetric;*

3. (Zou [Z1, Z2]) *If* $\dfrac{n+2}{n-2} < p < \dfrac{n+1}{n-3}$ $(< \infty$ *if* $n = 3)$*, and* $|x|^\alpha u(x) \leq C$ *for* $C > 0$*, where* $\alpha = 2/(p-1)$*, then* u *must be radially symmetric;*

4. (Guo [G]) *If* $\dfrac{n+1}{n-3} < p < \dfrac{n}{n-4}$ $(< \infty$ *if* $n = 4)$*, then* u *is radially symmetric if and only if* $\lim\limits_{|x| \to \infty} |x|^\alpha u(x) = \lambda$*, where* $\lambda = [\alpha(n-2-\alpha)]^{1/(p-1)}$*, and* $\lim\limits_{|x| \to \infty} |x|^{1-(\mu+n)/2}(|x|^\alpha u(x) - \lambda) = 0$ *where* $\mu = 2\alpha + 4 - 2n$*;*

5. (Guo [G]) *If* $p \geq \dfrac{n}{n-4}$, *Then u is radially symmetric if and only if* $\lim_{|x|\to\infty} |x|^\alpha u(x) = \lambda$.

Theorem 3 shows that Liouville theorem holds when $n \geq 3$ and the function has a subcritical growth rate, and when the function has a supercritical growth, the radially symmetric patterns emerge. We should also mention that when $n \geq 4$, (4) also has positive non-radial solutions $u(x', x_n) = U(x')$ when $p \geq \dfrac{n+1}{n-3}$, and $U(x')$ is a radially symmetric solution of (4) in $\mathbf{R}^{n-1}$, $x' \in \mathbf{R}^{n-1}$. Such solutions have the spatial patterns in a lower dimensional space. It would be interesting to know if solutions with other spatial pattern exist. It is also curious if Liouville-type theorem holds for other positive $f(u)$ in (4). The only other known result is for logistic type growth function, see Du and Ma [DM], and Dancer and Du [DD].

A lesson we can learn from Liouville-type theorems above is that "normally" we need a sign-changing $f(u)$ to have a non-trivial spatial pattern for (1). It is worthwhile to take a look of the patterns of (1) in $\mathbf{R}^1$. In fact, a bounded solution of $u'' + f(u) = 0$ defined on $\mathbf{R}$ for any C^1 function $f(u)$ must be one of the following:

1. A constant;
2. A periodic function;
3. (Hetroclinic solution) A monotone function $u(x)$ with $\lim_{x\to-\infty} u(x) = m$ and $\lim_{x\to\infty} u(x) = M$;
4. (Homoclinic solution) A symmetric function—without loss of generality, we assume it is an even function, such that $u(x) = u(-x)$, u is monotone on $(0, \infty)$, $\lim_{x\to\pm\infty} u(x) = M$.

While all these patterns still exist on a higher dimensional space as lower dimensional patterns, we would like to know for a sign-changing $f(u)$, what are the other possible spatial patterns in higher dimensions?

Here we have to limit our attention to a more special class of nonlinearity $f(u)$. We assume that f satisfies

(f1) f is smooth, $f(\alpha) = 0$ and $f'(\alpha) > 0$;

(f2) There exists $m < \alpha$ such that $f(m) = 0$, $f'(m) < 0$, and $f(u) < 0$ for $u \in (m, \alpha)$;

and one of the following:

(Balanced) There exists $M > \alpha$ such that $f(M) = 0$, $f'(M) < 0$, and $f(u) > 0$ for $u \in (\alpha, M)$. Moreover $\int_m^M f(u)du = 0$;

(Unbalanced 1) There exists $M > \alpha$ such that $f(M) = 0$, $f'(M) < 0$, and $f(u) > 0$ for $u \in (\alpha, M)$. Moreover $\int_m^M f(u)du > 0$; or

(Unbalanced 2) $f(u) > 0$ for all $u > \alpha$, and $\int_m^M f(u)du > 0$.

An example of balanced nonlinearity is $f(u) = u - u^3$, which is in the Allen-Cahn-Ginzburg-Landau equation; an example of unbalanced type 1 nonlinearity is $f(u) = u(u+1)(a-u)$, for $a > 1$, which can be thought as Allen-Cahn-Ginzburg-Landau with potential function with unequal well depths, or logistic growth with strong Allee effect; and for unbalanced type 2 nonlinearity, a typical example is $f(u) = -u + u^p$, $(n+2)/(n-2) > p > 1$, and $u > 0$, which arises from models of pattern formations in morphogenesis and chemotaxis, and has been studied extensively in recent years (See for example, Ni [N], and the references in [BDS, BS].)

We notice that in the above classification of bounded solutions of $u'' + f(u) = 0$, when f is balanced, there is a hetroclinic solution but no homoclinic solution, and when f is unbalanced, there is a homoclinic solution but no hetroclinic solution. This gives an indication of the difference between the solution sets of (1) for balanced and unbalanced nonlinearities f. In the following we will browse through the gallery of spatial patterns for both nonlinearities.

Unbalanced Case:

For the unbalanced f, the first n-dimensional (cannot be reduced to a lower-dimensional one) pattern is the radially symmetric solution:

Theorem 4. *Suppose that $f(u)$ is unbalanced, and either it is bounded or it is unbounded but subcritical, then* (1) *has a least energy solution $U \in H^2(\mathbf{R}^n) \bigcap C^2(\mathbf{R}^n)$, and U is radially symmetric.*

The earliest existence result seems to be in Strauss [St], and more general existence results are proved in Berestycki and Lions [BL] with a variational approach, and in Berestycki, Lions and Peletier [BLP] with a shooting method approach. The symmetry is prove in [GNN]. When f also satisfies certain convex conditions, the radial solution can be shown to be unique. In particular, when $f(u) = -u + u^p$, $(n+2)/(n-2) > p > 1$, the uniqueness is proved in Kwong [K] and also see a more general result in Kwong and Zhang [KZ]; when $f(u) = -u(u-b)(u-c)$, $0 < 2b < c$, the uniqueness is proved in Ouyang and Shi [OS], and Dancer [D2, D3].

Is the radially symmetric solution the only n-dimensional non-periodic pattern? There is no certain answer yet. However, the indication is yes, from the following result:

Theorem 5. *Suppose that $f(u)$ is unbalanced, and either it is bounded or unbounded but subcritical, and $u \in C^2(\mathbf{R}^n)$ is a solution of* (1)*. Then u must be radially symmetric if one of the following is true:*

1. (Farina [Fa]) *The nodal set $\{u = \alpha\}$ is bounded;*
2. (Dancer [D5]) *The Morse index of u is finite ($n = 2, 3$);*
3. (Shi [S3]) *u is symmetric with respect to each $x_i = 0$, and $\partial_{x_i} u < 0$ in $\{x > 0\}$.*

Hence if there is non-radial solution, then it must have unbounded nodal set and infinite Morse index (for $n = 2$ or 3), and cannot be symmetric. On the other hand, there are many spatial-periodic solutions of (1). In [S1], the author studies a boundary value problem

$$u_{xx} + u_{yy} + \lambda f(u) = 0, \;\; (x, y) \in \Omega, \;\; \frac{\partial u}{\partial n} = 0, \;\; (x, y) \in \partial\Omega, \tag{5}$$

where $\lambda > 0$ and $\Omega = (0, 1) \times (0, b)$, a rectangle. (5) has a one-dimensional solution $v(\lambda, x, y) = u(\lambda, x)$, where u satisfies

$$u'' + \lambda f(u) = 0, \;\; u'(0) = u'(1) = 0, \;\; u'(x) < 0. \tag{6}$$

In fact (5) has a solution curve $\Sigma_1 = \{v(\lambda, x, y) = u(\lambda, x) : \lambda > \lambda_*, u'(x) < 0\}$, for some $\lambda_* > 0$ (see Fig. 1.)

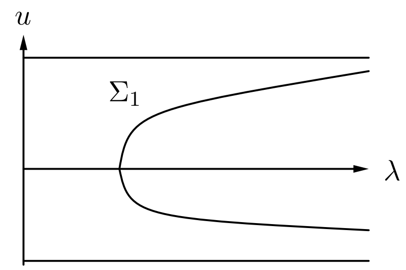

Fig. 1: Branch of 1-d solutions

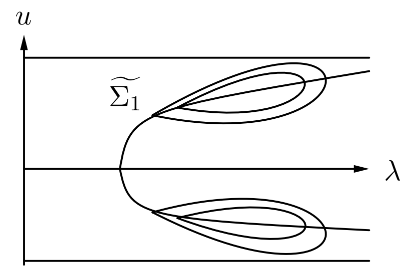

Fig. 2: Mushroom for balanced f

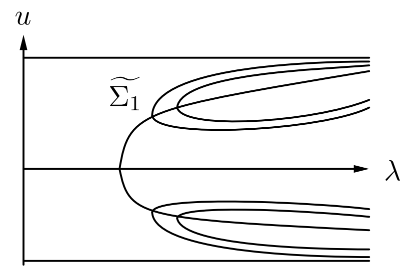

Fig. 3: Tree for unbalanced f

Theorem 6. (Shi [S1]) *Let f be a unbalanced nonlinearity. Then for* (5)*, there exist infinite many bifurcation points $(\Lambda_k, v(\Lambda_k, \cdot)) \in \Sigma_1$ where pitchfork bifurcations occur. All secondary branches are unbounded (see Fig. 3.)*

The solutions of (5) can be re-scaled, reflected and periodic extended to a doubly-periodic solution of (1) with periods $T_x = 2\sqrt{\lambda}$, and $T_y = 2b\sqrt{\lambda}$. Thus the results of Theorem 6 imply that for any $b = T_x/T_y > 0$, when $T_xT_y = \lambda$ is large enough, there is a doubly periodic solution $u(x, y)$ of (1) with (minimal) x-period T_x, and (minimal) y-period T_y; and when $T_xT_y = \lambda$ is small, there is no such solution. Moreover we can show that when T_xT_y is large, u is a solution with infinitely many spikes (which approximates the radially symmetric solution in $\mathbf{R}^2$) evenly distributed on a rectangular checker board, and the nodal lines $\{u = \alpha\}$ are nearly circles. Another type of solutions with periodic structure is found by Dancer [D4]:

Theorem 7. (Dancer [D4]) *Suppose $f(u) = -u + u^p$. Then there exist solutions u of* (1) *such that u is periodic in x_n and decays in x'. More precisely for*

$$\Delta u + \lambda f(u) = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega,$$

where $\Omega = \mathbf{R}^{n-1} \times (0, 1)$, a trivial solution branch is $\Sigma_2 = \{v(\lambda, x', x_n) = u(\lambda^{1/2}x')\}$, where u is the radially symmetric solution on $\mathbf{R}^{n-1}$. There is a bifurcation point $\lambda_{**} > 0$ such that a global branch emerging from Σ_2, on which the solutions depend on variable x_n.

Balanced Case:

The studies of (1) with balanced nonlinearity is strongly guided by the famous De Giorgi's conjecture [DG]:

Let u be a solution of $\Delta u + 2u - 2u^3 = 0, \;\; x \in \mathbf{R}^n$, such that $|u| \le 1, \;\; \dfrac{\partial u}{\partial x_n} > 0, \quad$ for all $x \in \mathbf{R}^n$. Is it true that all level sets $\{u = k\}$ of u are hyperplanes, at least if $n \le 8$?

The conjecture was first proved by Ghoussoub and Gui [GG] for the case of $n = 2$, with a key idea from Berestycki, Caffarelli and Nirenberg [BCN2]. A weak version called Gibbons' conjecture, which assumes that u approaches ± 1 uniformly as $x_n \to \pm\infty$, was proved independently by Barlow, Bass and Gui [BBG], Berestycki, Hamel and Monneau [BHM] and Farina [Fa]. The case of $n = 3$ was proved by Ambrosio and Cabré [AC], and Alberti, Ambrosio and Cabré [AAC]. Very recently, Savin [Sa] proved the original conjecture for $n \le 8$. While most of these results can be generalized from $f(u) = 2u - 2u^3$ to more general bistable nonlinearities, the result of [GG] for $n = 2$ holds for any smooth function f.

There also exist doubly periodic solutions of (1) for balanced f. But compared to Theorem 6, we have

Theorem 8. (Shi [S1]) *Let f be a balanced nonlinearity. Then for* (5), *for any positive integer N, there exists $b_N > 0$ such that for almost all $b > b_N$, there are $2N$ bifurcation points $(\Lambda_k^{\pm}, v(\Lambda_k^{\pm}, \cdot)) \in \Sigma_1$ where pitchfork bifurcations occur (see Fig. 2).*

Similar to Theorem 6, Theorem 8 implies that (1) has a doubly periodic solution with (minimal) x-period $T_x = 2\sqrt{\lambda}$, and (minimal) y-period $T_y = 2b\sqrt{\lambda}$ for any $b = T_y/T_x > k_0$ (or $b^{-1} > k_0$), only when $S_1 > T_xT_y > S_2$. On the other hand, in [S2], the author shows that if $b = T_x/T_y = 1$ (Ω is a square), when $T_x^2 > S_2$, there is a doubly periodic solution $u(x, y)$ with (minimal) x-period T_x, and (minimal) y-period T_x; and when T_x is large, the nodal lines are orthogonal curves which are nearly straight lines. The latter doubly periodic pattern is related to a new non-periodic pattern on $\mathbf{R}^2$: the saddle solution.

Theorem 9. (Shi [S2]) *Suppose that f is a balanced nonlinearity. Then* (1) *has a unique solution $u \in C^2(\mathbf{R}^2)$ satisfying*

$$\begin{aligned}
&u(x,y) = \alpha \quad \text{if } xy = 0,\\
&M > u(x,y) > \alpha \quad \text{if } xy > 0,\\
&\alpha > u(x,y) > m \quad \text{if } xy < 0,\\
&u(y,x) = u(x,y), \;\; u(-y,-x) = u(x,y).
\end{aligned}$$

The saddle solution was first found by Dang, Fife and Peletier [DFP] in 1992 with the extra conditions that f is odd and $f(u)/u$ is decreasing; A paper by Alama, Bronsard and Gui [ABG] studies the vector version of (1) with the condition of f being odd, and their method can also be adapted to proving the existence of saddle solution in the scalar case. The uniqueness of the saddle solution is proved in Berestycki, Caffarelli and Nirenberg [BCN1]. In [Sc], Schatzman proves that the Morse index of the saddle solution is exactly one when $f(u) = 2u - 2u^3$, and the saddle solution is always unstable from the result in [BCN2] and [D5]. It is conjectured that a more degenerate saddle solution u_n exists for any integer n and at least when f is an odd function so that the the nodal set of u_n consists of $\{(x, y) : x + iy \in te^{k\pi i/n}, k = 0, 1, 2, \cdot, 2n - 1\}$. Then the saddle solution in Theorem 9 is u_2, and the monotone one-dimensional solution is u_1. Such solution can be named as a "pizza solution". To close the door of our spatial pattern gallery, we mention a non-existence result in [S2]:

Proposition 10.

1. *For balanced f, there is no radially symmetric solution in $\mathbf{R}^n$ for $n \geq 1$.*
2. *For unbalanced f, there is no saddle solution for $n = 2$.*

In some sense, the radially symmetric solutions and the saddle solution in $\mathbf{R}^2$ are just like the homoclinic and hetroclinic solutions in $\mathbf{R}^1$ for unbalanced and balanced nonlinearities respectively, and they shows the characters of these two types of nonlinearities.

From the above overview of the spatial patterns exhibited by the solutions of (1), we believe that a *classification* of bounded solutions of (1) is possible, at least for the balanced and unbalanced nonlinearities which we defined here. The possible schemes of the classification are according to the number of critical points or the Morse indices of the solutions. The *Morse index* of a bounded solution u of (1) is defined as the dimension of negative space of the functional $E(\phi) = \int_{\mathbf{R}^n}[|\nabla\phi|^2 - f'(u)\phi^2]dx$, where $\phi \in C_0^\infty(\mathbf{R}^n)$, and u is said to be *weakly stable* if $E(\phi) \geq 0$ for all $\phi \in C_0^\infty(\mathbf{R}^n)$. A recent result by Dancer [D5] and the earlier result of [GG] about De Giorgi's conjecture in $\mathbf{R}^2$ can be summarized as the following theorem:

Theorem 11. *Suppose that u is a bounded smooth solution of (1) in $\mathbf{R}^2$. Then the following statements are equivalent:*

1. *u is weakly stable;*
2. *there exists a unit vector $\mathbf{v} \in \mathbf{R}^2$ such that $\nabla u \cdot v > 0$ for any $x \in \mathbf{R}^2$;*
3. *$u(x, y) = w(v_1 x + v_2 y)$, where $\mathbf{v} = (v_1, v_2)$, and w is a bounded solution of $w'' + f(w) = 0$ in $\mathbf{R}$ such that $w'(z) > 0$ for all $z \in \mathbf{R}$.*

Notice that Theorem 11 has no requirements on the nonlinearity f except assuming f is C^1. Dancer [D5] also characterizes the finite Morse index solutions of (1) for the unbalanced f. To conclude our survey, we list the known non-constant spatial patterns in the following table:

Number of critical points	balanced	unbalanced
0	De Giorgi's conjecture 1-d monotone pattern;	generalized De Giorgi's conjecture not exist $n = 2, 3$
1	saddle solution (and pizza solution?)	radially symmetric solution
≥ 2 and $< \infty$	not exist?	not exist?
∞	doubly-periodic	doubly-periodic, periodic-radially symmetric

A similar table can be listed according to the Morse indices, and probably a relation can be established between the Morse indices and the number of critical points.

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References

[ABG] Alama, Stanley; Bronsard, Lia; Gui, Changfeng, Stationary layered solutions in $\mathbf{R}^2$ for an Allen-Cahn system with multiple well potential. *Calc. Var. Partial Differential Equations* **5** (1997), no. 4, 359–390.

[AAC] Alberti, G.; Ambrosio, L.; Cabré, X. On a long standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. *Acta. Appl. Math.* **65** (2001), no. 1-3, 9–33.

[AC] Ambrosio, L.; Cabré, X. Entire solutions of semilinear elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi. *J. Amer. Math. Soc.* **13** (2000), no. 4, 725–739.

[A] Angenent, S. B. Uniqueness of the solution of a semilinear boundary value problem. *Math. Ann.* **272** (1985), no. 1, 129–138.

[BBG] Barlow, Martin T.; Bass, Richard F.; Gui, Changfeng, The Liouville property and a conjecture of De Giorgi. *Comm. Pure Appl. Math.*, **53**, (2000), no. 8, 1007–1038.

[BDS] Bates, Peter W.; Dancer, Norman E.; Shi, Junping, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimensional and instability. *Advances of Differential Equations*, **4**, (1999), no. 1, 1–69.

[BS] Bates, Peter W.; Shi, Junping, Existence and instability of spike layer solutions to singular perturbation problems. *J. Funct. Anal.* **196** (2002), no. 2, 211–264.

[B] Berestycki, H., Qualitative properties of semilinear elliptic equations in unbounded domains. *Progress in elliptic and parabolic partial differential equations* (Capri, 1994), 19–42, *Pitman Res. Notes Math. Ser.*, **350**, Longman, Harlow, 1996.

[BCN1] Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Monotonicity for elliptic equations in unbounded Lipschitz domains. *Comm. Pure Appl. Math.* **50** (1997), no. 11, 1089–1111.

[BCN2] Berestycki, Henri; Caffarelli, Luis; Nirenberg, Louis, Further qualitative properties for elliptic equations in unbounded domains. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25**, (1997), no. 1-2, 69–94 (1998).

[BHM] Berestycki, H.; Hamel, F.; Monneau, R. One-dimensional symmetry of bounded entire solutions of some elliptic equations. *Duke Math. J.* **103** (2000), no. 3, 375–396.

[BL] Berestycki, H.; Lions, P.-L., Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* **82** (1983), no. 4, 313–345.

[BLP] Berestycki, H.; Lions, P.-L.; Peletier, L. A. , An ODE approach to the existence of positive solutions for semilinear problems in $\mathbf{R}^N$. *Indiana Univ. Math. J.* **30** (1981), no. 1, 141–157.

[CGS] Caffarelli, Luis A.; Gidas, Basilis; Spruck, Joel, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42** (1989), no. 3, 271–297.

[CL] Chen, Wen Xiong; Li, Congming, Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63** (1991), no. 3, 615–622.

[D1] Dancer, E. N. Some notes on the method of moving planes. *Bull. Austral. Math. Soc.* **46** (1992), no. 3, 425–434.

[D2] Dancer, E. N. On positive solutions of some singularly perturbed problems where the nonlinearity changes sign. *Topol. Methods Nonlinear Anal.* **5** (1995), no. 1, 141–175.

[D3] Dancer, E. N. A note on asymptotic uniqueness for some nonlinearities which change sign. *Bull. Austral. Math. Soc.* **61** (2000), no. 2, 305–312.

[D4] Dancer, Edward Norman, New solutions of equations on $\mathbf{R}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), no. 3-4, 535–563 (2002).

[D5] Dancer, E. N. Stable and finite Morse index solutions on $\mathbf{R}^n$ or on bounded domains with small diffusion. To appear in *Tran. Amer. Math. Soc.*, (2004).

[DD] Dancer, E. N.; Du, Yihong, Some remarks on Liouville type results for quasilinear elliptic equations. *Proc. Amer. Math. Soc.* **131** (2003), no. 6, 1891–1899.

[DFP] Dang, Ha; Fife, Paul C.; Peletier, L. A., Saddle solutions of the bistable diffusion equation. *Z. Angew. Math. Phys.*, **43**, (1992), no. 6, 984–998.

[DG] De Giorgi, E. Convergence problems for functionals and operators. *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, 131–188, Pitagora, Bologna, 1979.

[DM] Du, Yihong; Ma, Li, Logistic type equations on $\mathbf{R}^N$ by a squeezing method involving boundary blow-up solutions. *J. London Math. Soc. (2)* **64** (2001), no. 1, 107–124.

[Fa] Farina, Alberto, Symmetry for solutions of semilinear elliptic equations in $\mathbf{R}^{\mathbf{N}}$ and related conjectures. *Ricerche Mat.* **48** (1999), suppl., 129–154.

[GG] Ghoussoub, N.; Gui, C., On a conjecture of De Giorgi and some related problems. . *Math. Ann.*, **311** (1998), no. 3, 481–491.

[GNN] Gidas, B.; Ni, Wei Ming; Nirenberg, L., Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. *Mathematical analysis and applications, Part A*, pp. 369–402, *Adv. in Math. Suppl. Stud.*, **7a**, Academic Press, New York-London, 1981.

[GS1] Gidas, B.; Spruck, J. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1981), no. 4, 525–598.

[GS2] Gidas, B.; Spruck, J. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations* **6** (1981), no. 8, 883–901.

[GT] Gilbarg, David; Trudinger, Neil S., Elliptic partial differential equations of second order. Second edition. Springer-Verlag, Berlin-New York, (1983).

[G] Guo, Zongming, On the symmetry of positive solutions of the Lane-Emden equation with supercritical exponent. *Adv. Differential Equations* **7** (2002), no. 6, 641–666.

[K] Kwong, Man Kam, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.* **105** (1989), no. 3, 243–266.

[KZ] Kwong, Man Kam; Zhang, Li Qun, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus. *Differential Integral Equations* **4** (1991), no. 3, 583–599.

[N] Ni, Wei-Ming, Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* **45** (1998), no. 1, 9–18.

[NT1] Ni, Wei-Ming; Takagi, Izumi, On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.*, **44**, (1991), no. 7, 819–851.

[NT2] Ni, Wei-Ming; Takagi, Izumi, Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.*, **70**, (1993), no. 2, 247–281.

[OS] Ouyang Tiancheng; Shi, Junping, Exact multiplicity of positive solutions for a class of semilinear problem. *Jour. Diff. Equa.*, **146**, (1998), no. 1, 121–156.

[Sa] Savin, Ovidiu, preprint, (2003).

[Sc] Schatzman, M. On the stability of the saddle solution of Allen-Cahn's equation. *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), no. 6, 1241–1275.

[S1] Shi, Junping, Global Bifurcation of Semilinear Neumann Boundary Problem. *Tran. Amer. Math. Soc.* **354** (2002), no. 8, 3117–3154.

[S2] Shi, Junping, Saddle solutions of the balanced bistable diffusion equation. *Comm. Pure Appl. Math.* (2002), no. 7, 815–830.

[S3] Shi, Junping, Entire solutions and solutions to singularly perturbed semilinear problems with unbalanced nonlinearities. In preparation.

[St] Strauss, Walter A., Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* **55** (1977), no. 2, 149–162.

[Z1] Zou, Henghui, Slow decay and the Harnack inequality for positive solutions of $\Delta u + u^p = 0$ in R^n. *Differential Integral Equations* **8** (1995), no. 6, 1355–1368.

[Z2] Zou, Henghui, Symmetry of positive solutions of $\Delta u + u^p = 0$ in R^n. *J. Differential Equations* **120** (1995), no. 1, 46–88.