Synchronous Oscillatory Solutions in a Two Patch Predator-Prey Model

Matthew H. Becker[1], Leah Shaw[2], Junping Shi[1]

1. Department of Mathematics, The College of William & Mary, Williamsburg, Virginia, 23187-8795
2. Department of Applied Science, The College of William & Mary, Williamsburg, Virginia, 23187-8795

Abstract

"Populations of adversarial species in nature engaged in predator-prey dynamics may oscillate over time. This is seen best in the example of the lynx and hare. When this occurs, unique patches of species may synchronize such that populations in each patch are equivalent. Oscillatory solutions to a predator-prey model are studied to understand what leads to this synchrony. Here we use Alan Hastings' version of the Rosenzweig-MacArthur model. A correlation coefficient similar to Pearson's Correlation is used as a statistical method to quantify synchrony and we use this as a numerical tool to analyze our data and results."

One Patch Model

$$\frac{du}{dt} = u(1-\alpha u) - \frac{uv}{1+\beta u},$$
$$\frac{dv}{dt} = \frac{uv}{1+\beta u} - \eta v$$

u is the populations of the prey, with v as the population of the predator. α is a non-dimensionalized parameter representing carrying capacity of the prey. $\frac{uv}{1+\beta u}$ measures the amount of prey eaten by the predators when they interact, a Holling Type II functional response. e measures the natural mortality rate of the predator.

This system presents multiple equilibria, but the one we care most about is our coexistence equilibrium (λ, v_λ) for $\lambda = \frac{\eta}{1-\beta\eta}$ and $v_\lambda = (1-\alpha\lambda)(1+\beta\lambda)$. We linearize the system and study the stability using the Jacobian matrix, J, with the characteristic equation $\mu^2 - Tr(J)\mu + Det(J) = 0$ for the eigenvalues of the matrix. Through algebraic manipulation we can find the Hopf Bifurcation and see that when $0 < \lambda < \frac{\beta-\alpha}{2\alpha\beta}$ a periodic orbit will occur. This is the case that we care about for when we go to a two-patch model.

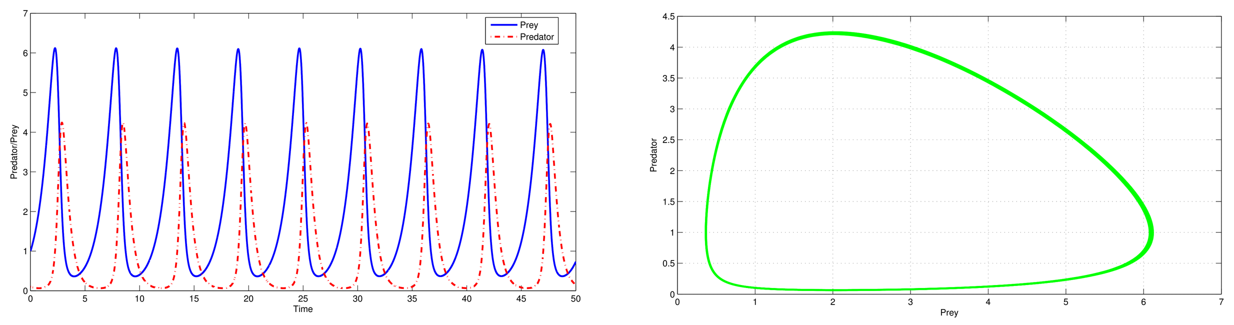

Figure 1: The first plot shows the time series for unstable coexistence in a one patch model, where both populations demonstrate oscillations. The second plots shows the phase space, where predator is mapped against prey and the periodic orbit can be seen.

Two-Patch Model

This is almost identical to the one-patch model, but with two additional terms. Those terms, with a and b as diffusion rates, measure the dispersal between the two patches. We see that the predator and prey may have differing dispersal rates.

$$\frac{du}{dt} = f_1(u,v) + a(w-u),$$
$$\frac{dv}{dt} = g_1(u,v) + b(x-v),$$
$$\frac{dw}{dt} = f_2(w,x) - a(w-u),$$
$$\frac{dx}{dt} = g_2(w,x) - b(x-v),$$

where for $i = 1, 2$,

$$f_i(u,v) = u(1-\alpha_i u) - \frac{uv}{1+\beta_i u}, \quad g_i(u,v) = \frac{uv}{1+\beta_i u} - \eta_i v. \qquad (1)$$

Numerical Analysis

The four sets of plots below show data collected. We assume spatial homogeneity and fix parameters $\alpha = 0.01, \beta = 0.04$, $\eta = 1$, and $T = 1000$. We change diffusion parameters a and b. Initial values are $u(0) = 4, v(0) = 3, w(0) = 0, x(0) = 0$. The far left plot is a time series of predator and prey versus time, the middle plot is the phase portrait of predator in patch 1 versus predator in patch 2, and the far right plot is the time shift τ versus cross correlation (see definition on the right).

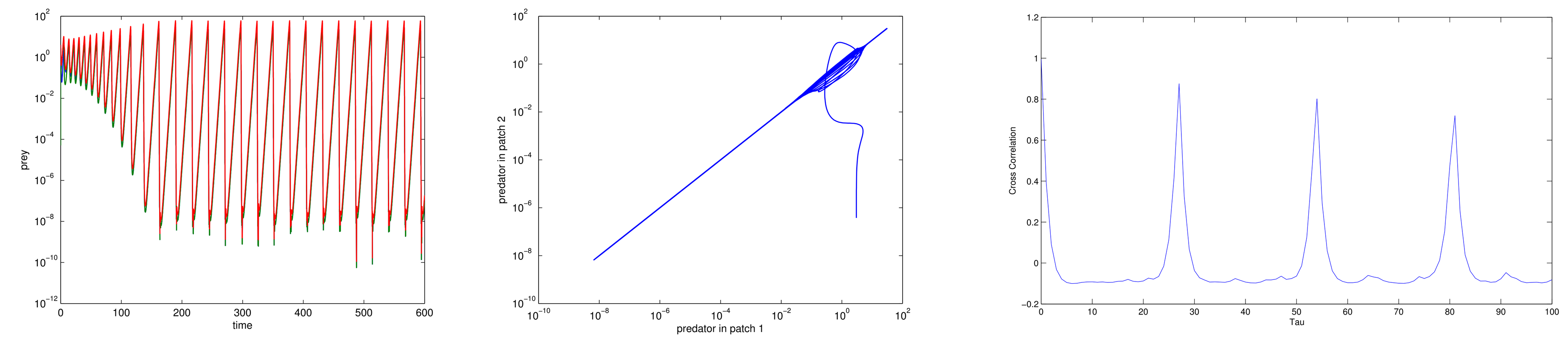

Figure 2: $a = 0.1, b = 0.001.cc = 0.9983$, average $cc(\tau) = -3.9401 \times 10^{-4}$

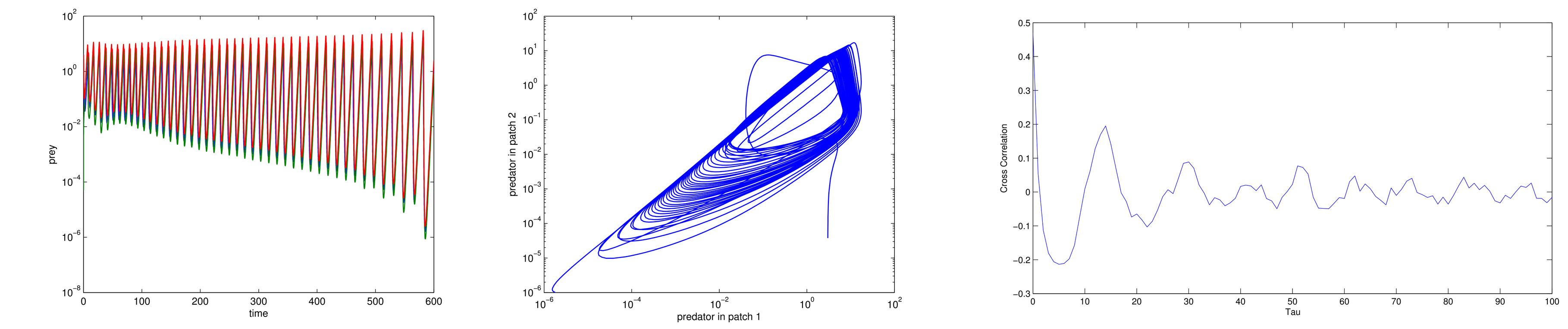

Figure 3: $a = 0.01, b = 0.01.cc = 0.2917$, average $cc(\tau) = -0.0046$.

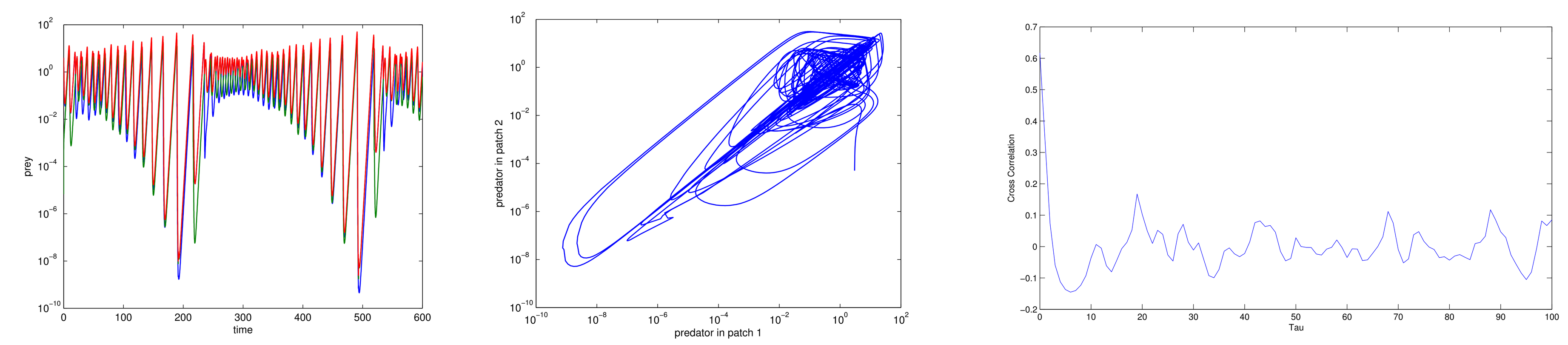

Figure 4: $a = 0.1, b = 0.01.cc = 0.4633$, average $cc(\tau) =$0.0037

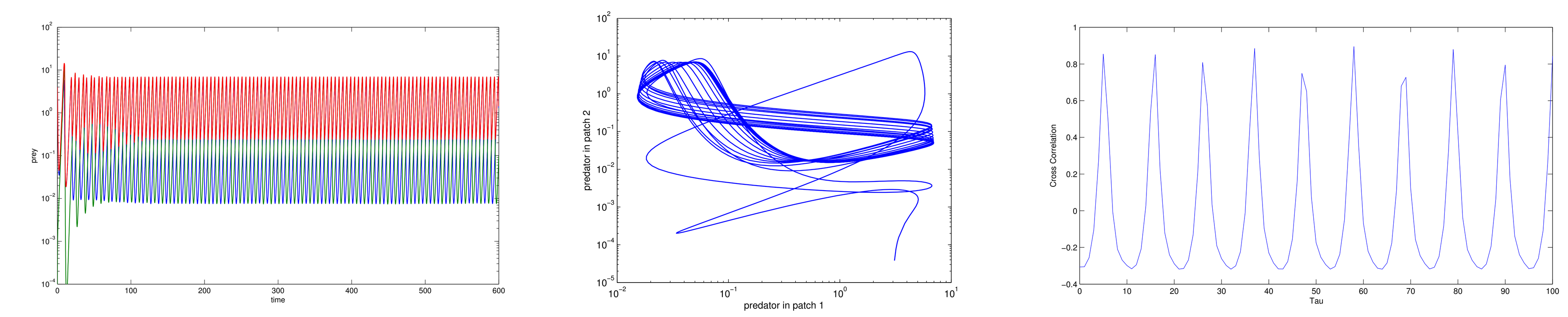

Figure 5: $a = 0.001, b = 0.001.cc = -0.2533$, average $cc(\tau) = 6.6797 \times 10^{-4}$

Conclusions

As the diffusion rates increase, the synchrony of the system also increases, although not necessarily in a linear manner. Additionally, the cross correlation has appears to have a lower bound, situated around -0.3. The system does not achieve antisynchrony. This is corroborated when time is shifted in one patch, where again there is no antisynchrony present. The time shift shows us that there are sets of parameters that will never allow for synchrony (fig3), while others that are simply out of phase (fig4). Finally, we see that the average cross correlation over a number of different time shifts, $cc(\tau)$, remains just around 0.

Cross Correlation

In order to analyze synchrony in the two patches we use a Cross Correlation model. This model is very similar to Pearson's Correlation, one of the most widely used correlation measures in statistical analysis. It is defined $cc(u,w;T) = \frac{\langle u(t)-\bar{u}, w(t)-\bar{w}\rangle}{\sigma_u \sigma_w}$.

The two functions u and w are the prey solutions in the time span $[0,T]$ for $T \in \mathbb{R}$ to the predator-prey model above. $\bar{u}$ and $\bar{w}$ are defined as the average value for each function over $[0,T]$, with $\bar{u} = T^{-1}\int_0^T u(t)dt$. The numerator is akin to a vector inner product on $L^2(0,T)$, which is $\langle f,g\rangle = \int_0^T f(t)\cdot g(t)dt$. σ_u is the standard deviation of function u, and it is defined as follows: $\sigma_u = \sqrt{\int_0^T u^2(t)dt}$.

This equation gives a result between -1 and 1, with 1 meaning perfect synchrony, 0 as asynchrony, and -1 as anti-synchrony.

Cross-Correlation values give a good indication of the synchrony of a solution at a specific time, but can not decipher whether a system might be locked out of phase. In order to analyze more we apply an arbitrary time shift to the correlation. This is defined as $cc_\tau(u,w;T) = \frac{\langle u(t)-\bar{u}, w(t)-\bar{w}\rangle_\tau}{\sigma_u \sigma_w}$
with the numerator being the new vector inner product $\langle f,g\rangle_\tau = \int_\tau^T f(t)\cdot g(t-\tau)dt$.

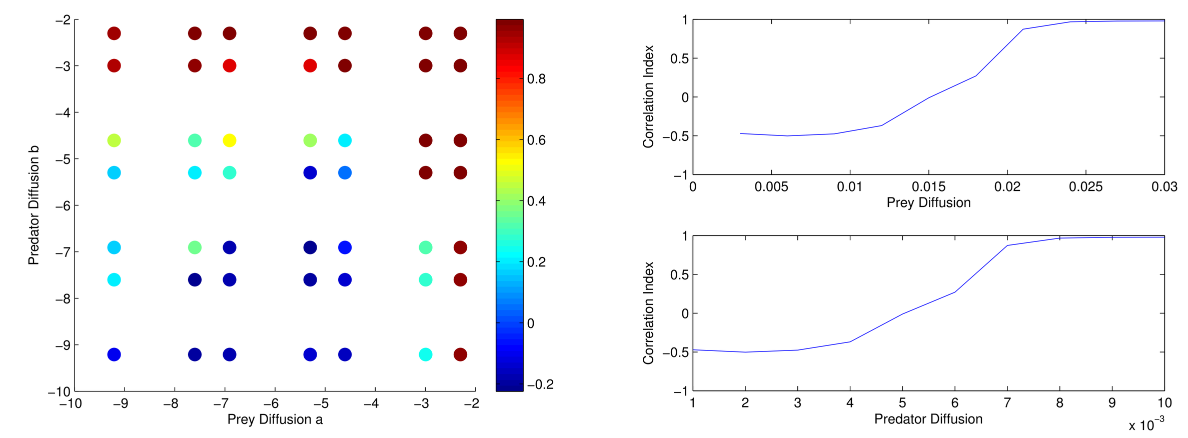

Figure 7: Left: $0.0001 \le a \le 0.1$, $0.0001 \le b \le 0.1$. Right: Cross Correlation vs. Diffusion Coefficients.

Future Work and References

We will continue to work with the correlation coefficient to further understand what leads to synchronous solutions. We have begun analyzing the Floquet multipliers to study the stability of periodic orbits. We are going to combine this with our continued study of synchrony.

References

[1] Glass, L.; Synchronization and Rhythmic Processes in Physiology. *Nature*, **410** (2001), 277-284.

[2] Hastings, A.; Transient Dynamics and Persistence of Ecological Systems. *Ecology Letters* **4** (2001), 215-220.

[3] Rosenzweig, M.L. and MacArthur, R.H.; Graphical Representation and Stability Conditions of Predator-Prey Interactions. *American Naturalist* (1963), 209-223.

[4] Rodgers, J. L.; Nicewander, W. A., Thirteen Ways to Look at the Correlation Coefficient. *The American Statistician*, **42** (1988), 59-66.

Acknowledgments

This research is supported by NSF CSUMS Grant DMS-0703532 and NSF Grant DMS-1022648.