Ramsey Theory

Roughly speaking, Ramsey Theory states that for any k-coloring of the t-subsets of an n-set S, if n is large enough, then there is N-subset, whose elements have the same color. More precisely,

Definition 1. *Given $k, t, p_1, p_2, \ldots, p_k$, there exists an integer N, such that for any k-coloring of the t-subsets of an N-set S, there is a p_i-subset whose elements are colored with i^{th}-color. The smallest integer N is called the* **Ramsey Number** $R(p_1, p_2, \ldots, p_k; t)$.

Examples:

(1) *Happy End Problem* For an integer m, there is an integer $N(m)$ such that every set of at least $N(m)$ points in the plane (no three on a line) contains an m-subset forming a convex m-gon.

Proof. Fact one: Among any five points in the plane, four determine a convex quadrilateral.

Fact two: If very 4-subset of m-points in the plane form a convex quadrilateral, then the m points form a convex m-gon.

Now take $N = R(m, 5; 4)$ and color each 4-set if it forms a convex gon.

(2) *The Schur Theorem* Given $k > 0$, there exists an integer s_k such that every k-coloring of the integers $1, 2, \ldots, s_k$ yields monochromatic x, y, z (not necessarily distinct) satisfying $x + y = z$.

Proof. Let $s_k = R_k(3; 2)+1$. Let f be a k-coloring of the integers $1, 2, \ldots, s_k$. Let f' be a k-coloring of the 2-subsets (edges) of the set $\{1, 2, \ldots, s_k\}$ defined by $f'(\{a, b\}) = f(|a - b|)$.

Then by definition, there are three integers a, b and c (assume $a < b < c$) such that $c - b, b - a, c - a$ are of the same color. Let $x = c - b, y = b - a, z = c - a$, then $f(x) = f(y) = f(z)$ and $x + y = z$.

Problems

(1) A 6 rectangle is tiled by 2 dominoes. Then it has always at least one fault-line, i.e., a line cutting rectangle without cutting any domino.

(2) The points of a plane are colored with three colors. Show that thre exist two points with distance 1 both having the same color.

(3) The points of a plane are colored red or blue. Then one of the two colors contains points with any distance.

(4) Among six persons, there are always three who know each otehr or three who are completely strangers.

(5) Let $R(r, s)$ be the minimum number so that for $n \geq R(r, s)$, a two-coloring of complete graph with n vertices contains a monochromatic r-clique or a monochromatic s-clique. Show that $R(r, s) \leq R(r-1, s) + R(r, s-1)$.

(6) Prove that $R(3, 3, 3) \leq 17$.

(7) Let S be a set of $R(m, m; 3)$ points in the plane no three of which are on a line. Prove that S contains m points that form a convex m-gon. (hint: add a special point to the plane.)

(8) Prove that 2-coloring of the edges of K_6 have at least two monochromatic triangles.