

Problem Set 3

- (1) Obtain the general solutions to recurrence relations using generating function method:

$$a_n = 5a_{n-1} - 6a_{n-2} + n^2 - 2n - 1 + 2^n$$

- (2) Let a_n be the number of nonnegative integer solutions to $4e_1 + 2e_2 + e_3 + 2e_4 = n$. Use the generating function to find a_n .
- (3) In how many ways can one pick 25 coins that are pennies, nickels, or dimes, with at least three nickels, at most five dimes and an even number of pennies?

Use generating function to evaluate the sum below.

- (4)

$$\sum_{k=0}^r (-1)^k \binom{n}{k} \binom{n}{r-k}.$$

- (5)

$$\sum_{k=1}^n k \binom{n}{k}^2.$$

- (6) the central Delannoy number $d_{n,n} = \sum_k \binom{n+k}{2k} \binom{2k}{k}$. Use Snake Oil to obtain the generating function $\sum_{n=0}^{\infty} d_{n,n} x^n$.
- (7) (A6 Putnam 1997) For a positive integer n and any real number c , define x_k recursively by $x_0 = 0, x_1 = 1$, and for $k \geq 0$, $x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}$. Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k , $1 \leq k \leq n$. (hint: let $p(t) = \sum_{i \geq 0} x_{i+1} t^i$.)

- (8) (B2, Putnam 1992) For nonnegative integers n and k , define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^k \binom{n}{j} \binom{n}{k-2j}.$$

- (9) (B4, Putnam 1997) Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i, i} \leq 1.$$

- (10) let $n > 1$, two set of positive integers $\{a_1, a_2, \dots, a_n\} \neq \{b_1, b_2, \dots, b_n\}$ and $\{a_i + a_j : 1 \leq i < j \leq n\} = \{b_i + b_j : 1 \leq i < j \leq n\}$ (note that the equality means that if s appears k times in the left-hand side, then it will also appear k times in the right-hand side). Prove that there is an positive integer h , such that $n = 2^h$. (hint: take $f(x) = x^{a_1} + x^{a_2} + \dots + x^{a_n}$ and $g(x) = x^{b_1} + x^{b_2} + \dots + x^{b_n}$. Then $(f(x))^2 - f(x^2) = (g(x))^2 - g(x^2)$.)