Problem Set 3

Discussion Problems Discussion: Sept. 22

1. (a) Four-digit number $S = aabb$ is a square. Find it; (hint: 11 is a factor of $S$)
   (b) If $n$ is a sum of two square, so is $2n$. (hint: simple algebra)

2. (a) If $n$ is an even number, then $323|20^n + 16^n - 3^n - 1$; (hint: factorize 323)
   (b) If $n$ is an integer, then $9|4^n + 15n - 1$. (hint: consider cases when $n$ modulus 3)

3. (a) If $2n + 1$ and $3n + 1$ are squares, then $5n + 3$ is not a prime;
   (b) If $3n + 1$ and $4n + 1$ are squares, then $56|n$. (hint: follow the idea in presentation problem)

4. If $p$ is a prime, then $p^2 \equiv 1 (mod24)$; (hint: prove $24|p^2 - 1$)

5. (a) (VT 1979) Show that for all positive integers $n$, that 14 divides $3^{4n+2} + 5^{2n+1}$;
   (b) (VT 1981) $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?
   (hint: (a) $14 = 2 \cdot 7$, (b) factorizing)

6. (a) (VT 1982) What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer;
   (b) (MIT training 2 star) Let $n$ be an integer greater than one. Show that $n^4 + 4^n$ is not prime. (hint: there is a magic identity due to Sophie Germain: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$)

7. (VT 1988) Let $a$ be a positive integer. Find all positive integers $n$ such that $b = a^n$ satisfying the condition that $a^2 + b^2$ is divisible by $ab + 1$. (hint: prove that $a^m + 1|a^n + 1$, then $m|n$.)

8. (VT 2005) Find the largest positive integer $n$ with the property that $n + 6(p^3 + 1)$ is prime whenever $p$ is a prime number such that $2 \leq p < n$. Justify your answer.

More Problems:

1. (Putnam 1986-A2) What is the units (i.e., rightmost) digit of $\left[ \frac{10^{20000}}{10^{100} + 3} \right]$? Here $[x]$ is the greatest integer $\leq x$.

2. (Putnam 1998-A4) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all $n$ such that 11 divides $A_n$.

3. (Putnam 1998-B6) Prove that, for any integers $a, b, c$, there exists a positive integer $n$ such that $\sqrt{n^2 + an^2 + bn + c}$ is not an integer.

4. (Putnam 1985-A4) Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many $a_i$?

5. (Putnam 1955-B4) Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

6. (Putnam 1956-A2) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

7. (Putnam 1966-B2) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

8. (MIT training 2.5 star) Let $d$ be any divisor of an integer of the form $n^2 + 1$. Prove that $d - 3$ is not divisible by 4.

9. (MIT training 3 star) What is the last nonzero digit of 10000!? 

10. (MIT training 2.5 star) Let $n$ be an integer, and suppose that $n^4 + n^3 + n^2 + n + 1$ is divisible by $k$. Show that either $k$ or $k - 1$ is divisible by 5.