Problem Set 1

Discussion Problems Discussion: Sept 9

1. A city has 10000 different telephone lines numbered by 4-digit numbers. More than half of the telephone lines are in the downtown. Prove that there are two telephone numbers in the downtown whose sum is again the number of a downtown telephone line. **(Erica)**

2. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals. **(Girikarnika)**

3. (UIUC 2000) Suppose that $a_1, a_2, \cdots, a_n$ are n given integers. Prove that there exist integers r and s with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by n. **(Bailey)**

4. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that for any integer m, there exists a_k such that a_k ends with m zeros. **(?)**

5. (VT 2006) Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

6. (Putnam, 2002-A2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. **(?)**

7. Prove that there exists a multiple of 2005 whose decimal expansion contains only digits 1 and 0.

8. (MIT homework 9) (1.5 star) Find the missing term:

$$10; 11; 12; 13; 14; 15; 16; 17; 20; 22; 24; 31; 100; ?; 10000$$

9. (MIT homework 9) (1.5 star) Explain the rule which generates the following sequence:
$2; 3; 10; 12; 13; 20; 21; 22; 23; 24; 25; 26; 27; 28; 29; 30; 31; 32; 33; 34; 35; 36; 37; 38; 39; 200; 201; 202; \cdots$
(Hint: Don't think mathematically!)

More Challenging Problems:

1. Given any $2n$ integers, show that there are n of them whose sum is divisible by n. (Though superficially similar to some other pigeonhole problems, this problem is much more difficult and does not really involve the pigeonhole principle.)

2. (a) Show that among any 9 points in a triangle of area 1, there are 3 points that form a triangle of area at most 1/4. (b) Show that given any 9 points in a triangle of area 1, there is a triangle of area at least 1/12 that does not contain any of those 9 points in its interior. (Can you improve 1/12?)

3. (Berkeley training) Given an infinite number of points in a plane, prove that if all the distances between them are integers, then the points are all on a straight line.