Ordinary Generating Functions

The **ordinary generating function (OGF)** for a series ¡a¿ of complex numbers is the formal power series $\sum_{n=0}^{\infty} a_n x^n$. When a_n counts the objects in a universe A for which an index parameter X has value n, we say that the generating function is **indexed by** X.

- a_n be the number of binary lists of length n. Then the OGF is $\sum_{n=0}^{\infty} 2^n x^n$, which is indexed by length of binary lists.
- Consider k-subsets of an n-set. Then $A_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$, is indexed by size of subsets of $[n]$.

Given two formal power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the sum and product (or convolution) is defined as follows, respectively:

$$sum : \sum_{n=0}^{\infty} (a_n + b_n) x^n, \quad product : \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} a_j b_{n-j} \Big) x^n.$$

Define $A(x)^{-1}$ to be $B(x)$ such that $A(x)B(x) = 1 = 1x^0 + 0x^1 + 0x^2 + \ldots$. Then we may show that $A(x)$ has a multiplication inverse if $[x^0]A(x) \neq 0$.

Examples:

(1) *Multisets from two types of objects.* the OGF is

$$\sum_{k=0}^{\infty} (k+1) x^k = (\sum_{k=0}^{\infty} x^k)^2.$$

(2) *Multisets (selections with repetition.* let d_k be the number of multisets of size k from n types of objects. Then the OGF is

$$\sum_{k \geq 0} d_k x^k = (\sum_{k \geq 0} x^k)^n = \sum_{k \geq 0} \binom{k+n-1}{n-1} x^k.$$

(3) *Multisets with restricted multiplicities.* When S is the set of multiplicities allowed for a type of objects, the generating function for that factor is $\sum_{k \in S} x^k$.

- When no restriction for multiplicity, then the OGF is $\sum_{k \geq 0} x^k = (1-x)^{-1}$.
- For ordinary subsets, $S = \{0, 1\}$, then the OGF is $(1+x)^n$.
- When each type must be used, the OGF is $x + x^2 + x^3 + \ldots = x(1-x)^{-1}$.
- When the usage for a type must be even, the factor for it is $x^2 + x^4 + \ldots = (1-x^2)^{-1}$.

In how many ways can one pick 20 coins that are pennies, nickels, or dimes, with at least three nickels and at most four dimes.

Solution: the number is $[x^{20}](1-x)^{-1}(x^3+x^4+\ldots)(1+x+x^2+x^3+x^4)$.

We may operate on the OGFs and on the expansions without losing equality. This may allow us to use OGF to prove something pleasantly.

Some basic propositions:

(1) *shifting the index.*

$$\sum_{k\geq 0}\binom{k}{r}x^k = \sum_{k\geq r}x^k = \sum_{k\geq 0}\binom{k+r}{r}x^{k+r} = \frac{x^r}{(1-x)^{r+1}}.$$

Let A, B, C be are the OGFs for $< a >, < b >, < c >$, respectively, then

(2) $c_n = a_n + b_n$ for all n if and only if $C(x) = A(x) + B(x)$.

(3) $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for all n iff $C(x) = A(x)B(x)$.

(4) $b_n = \begin{cases} a_{n-k}, & \text{for } n \geq k \\ 0, & \text{for } n < k \end{cases}$ if and only if $B(x) = x^k A(x)$.

(5) $c_n = \sum_{i=0}^{n} a_i$ for all i if and only if $C(x) = \frac{A(x)}{1-x}$.

(6) $b_n = \begin{cases} a_n \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases}$ if and only if $B(x) = 0.5(A(x) + A(-x))$.

(7) $b_n = \begin{cases} 0, & \text{for } n \text{ even} \\ a_n, & \text{for } n \text{ odd} \end{cases}$ if and only if $B(x) = 0.5(A(x) - A(-x))$.

(8) $b_n = \begin{cases} a_k, & \text{for } n = mk \\ 0, & \text{otherwise} \end{cases}$ if and only if $B(x) = A(x^m)$.

(9) $b_n = na_n$ if and only if $B(x) = xA'(x)$.

Examples:

- $\sum k\binom{n}{k} = n2^{n-1}$.
  The sum is the value at $x = 1$ after differentiating $\sum \binom{n}{k}x^k$.
- $\sum \binom{n}{2i}$.
  The sum is the value at $x = 1$ of $\sum \binom{n}{2i}x^{2i}$.
- $\sum_{k\geq 0} k^m$.
  Consider $\sum_{k\geq 0} k^m x^k$.
- $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.
  The convolution of $a_k = \binom{m}{k}$ and $b_k = \binom{n}{k}$.

- $\sum_{k=0}^{n} k(n-k)$.
  The convolution of $a_k = b_k = k$.

Snake Oil method: When n is a parameter in a sum, we can always multiply by x^n to form an OGF $A(x)$. Then interchange the order of summation in the expression for $A(x)$ and perform the new inner sum on n explicitly.

(1) $\sum_{k\geq 0} \binom{k}{n-k}$.
Let $a_n = \sum_{k\geq 0} \binom{k}{n-k}$ with $A(x) = \sum_{n\geq 0} a_n x^n$. Then

$$A(x) = \sum_{n\geq 0}\sum_{k\geq 0} \binom{k}{n-k} x^n = \sum_{k\geq 0} x^k \sum_{n} \binom{k}{n-k} x^{n-k} = \sum_{k\geq 0} x^k (1+x)^k = \frac{1}{1-x-x^2}.$$

We already know that $\frac{1}{1-x-x^2}$ is the OGF for Fibonacci numbers.

(2) $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k$.
Multiply both sides by x^n and sum over n, and interchange the order of the summation.

$$LHS = \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0} \binom{n+k}{m} x^{n+k} = \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{m}{k} x^{-k} = \frac{x^m(1+x^{-1})^m}{(1-x)^{m+1}} = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

$$RHS = \sum_k \binom{m}{k} 2^k \sum_{n\geq 0} \binom{n}{k} x^n = \frac{1}{1-x} \sum_k \binom{m}{k} 2^k (\frac{x}{1-x})^k = \frac{1}{1-x}(1+\frac{2x}{1-x})^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

(3) $\sum_{k=m}^{n} c(n,k)\binom{k}{m} = c(n+1, m+1)$, where $c(n,k)$ denote the number of permutations of $[n]$ with k cycles. The OGF is $C_n(x) = x(x+1)(x+2)\dots(x+n-1) = x^{(n)}$.

$$\sum_{k=m}^{n} c(n,k)\binom{k}{m} x^m = \sum_k c(n,k) \sum_m \binom{k}{m} x^m = \sum_k c(n,k)(1+x)^k = (1+x)^{(n)}.$$

$$\sum_{k=m}^{n} c(n,k)\binom{k}{m} = [x^m](1+x)^{(n)} = [x^{m+1}](1+x)^{(n+1)} = c(n+1, m+1).$$

The generating function method to solve recursive relations

- The **generating function** for a sequence $< a >$ of complex numbers is the *formal power series* $\sum_{n=0}^{\infty} a_n x^n$ or any function with power series expansion $\sum_{n=0}^{\infty} a_n x^n$. The **coefficient operator** $[x^n]$ extracts the coefficient of x^n in a power series in x, so $[x^n]A(x) = a_n$ when $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

- Examples: *Regions among lines, again* $a_n = a_{n-1} + n$ for $n \geq 1$, with $a_0 = 1$.

  From $a_n = a_{n-1} + n$, we have $a_n x^n = a_{n-1}x^n + nx^n$, and
$$\sum_{n\geq 1} a_n x^n = \sum_{n\geq 1} a_{n-1}x^n + \sum_{n\geq 1} nx^n.$$
  Let $A(x) = \sum_{n\geq 0} a_n x^n$, then $\sum_{n\geq 1} a_n x^n = A(x) - 1$ and $\sum_{n\geq 1} a_{n-1}x^n = xA(x)$. Note that $\sum_{n\geq 1} nx^n = x\sum_{n\geq 1} nx^{n-1} = x\frac{d}{dx}(\sum_{n\geq 0} x^n) = x\frac{d}{dx}((1-x)^{-1})$.

  Therefore $A(x) - 1 = xA(x) + x(1-x)^{-2}$, and $A(x) = (1-x)^{-1} + x(1-x)^{-3}$. We get $a_n = [x^n]A(x)$.

- Lemma: $(1-x)^{-n} = \displaystyle\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k.$

- Theorem: Let $\alpha_1, \ldots, \alpha_r$ be distinct numbers satisfying the following equation for complex numbers $c_1, \ldots, c_k$ with $x_k \neq 0$
$$Q(x) = 1 - c_1 x - c_2 x^2 - \ldots - c_k x^k = \prod_{i=1}^{r} (1 - \alpha_i x)^{d_i}.$$
  Then the following are equivalent for a sequence $< a >$.

  (1) $< a >$ satisfies the recurrence $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$ for $n \geq k$.

  (2) $< a >$ has generating function $P(x)/Q(x)$ for some polynomial P of degree less than k.

  (3) $< a >$ has generating function $\sum_{i=1}^{r} F_i(x)(1-\alpha_i x)^{-d_i}$, where each F_i is a polynomial of degree less than d_i.

  (4) a_n for $n \geq 0$ is given by the formula $a_n = \sum_{i=1}^{r} P_i(n)\alpha_i^n$, where each P_i is a polynomial of degree less than d_i.

- More examples:

  (1) Solve $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ for $n \geq 3$, with $(a_0, a_1, a_2) = (2, 4, 7)$.

  (2) Solve $a_n = 5a_{n-1} - 6a_{n-2} + 2n^2 - n + 2^n$ for $n \geq 2$, with $a_0 = a_1 = 1$.

  (3) Solve $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n \geq 1$, with $a_0 = 1$.