# Math 345 Intro to Math Biology
# Lecture 4: Nonlinear Difference Equation Models

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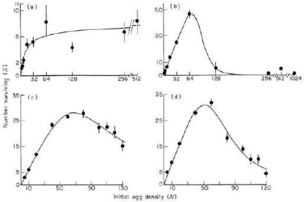
# Nonlinear model of population

$N_{n+1} = \lambda N_n S(N_n)$
$S(N)$ is the survival rate of the population which depends on the size of population, and $\lambda S(N)$ is the per capita reproduction rate

This is a density dependent population growth. Usually unlimited growth is unrealistic, as population grow the amount of resources available per individual decreases.

How to determine $S(N)$?

1. From certain assumptions to derive the mathematical forms
2. Available data suggests the shape of function, then find functions with such shape.




The relationship between number of survivors and density for four stored product beetles. After Bellows, T. S. 1981. The Descriptive Properties of Some Models for Density Dependence. Journal of Animal Ecology, Vol. 50, No. 1. pp. 139-156.

# Some possible growth functions

$N_{n+1} = \lambda N_n S(N_n)$

Bellow's model: $NS(N) = \dfrac{N}{1+(aN)^b}$, $a > 0$, $b \geq 1$.

A similar model:
Hassel's model $NS(N) = \dfrac{N}{(1+aN)^b}$, $a > 0$, $b > 0$.

Parasitoid fly $b = 0.5$ and $\lambda = 3.2$;
Bug: Saccarosydne saccharivora $b = 0.4$ and $\lambda = 13.5$
Mosquito $\beta = 1.9$ and $\lambda = 10.6$
Potato beetle $b = 3.4$ and $\lambda = 75.0$
Blowfly $b = 10$ and $\lambda \approx 100$

Hassell, M.P., Lawton, J.N. & May, R.M. (1976) Patterns of dynamical behaviour in single-species populations. Journal of Animal Ecology, 45, 471-486.

# Simplify the equation

Bellow's equation: $N_{n+1} = \dfrac{\lambda N_n}{1 + (aN_n)^b} = N_n F(N_n) = f(N_n)$.

Let $x_n = aN_n$. Then $x_n = \dfrac{\lambda x_n}{1 + x_n^b}$.

(The number of parameters reduces from 3 to 2.)

or more general $x_{n+1} = f(x_n)$

**Observation 1**: If $\lambda < 1$, then the species is doomed to extinction. (why?)

**Observation 2**: If $b < 1$, then $f(x)$ is an increasing function, then the sequence $\{x_n\}$ is increasing or decreasing.

# Equilibrium

**Equilibrium**: (fixed point, constant solution)
If $x$ satisfies $x = f(x)$, then $x$ is an equilibrium of the difference equation $x_{n+1} = f(x_n)$. An equilibrium is where the graphs of $y = x$ and $y = f(x)$ intersect.

**Stability of an equilibrium**: Suppose that $x_*$ is an equilibrium of $x_{n+1} = f(x_n)$. If we start the iteration sequence from any initial point $x_0$ close to $x_*$, we always have $\lim_{n\to\infty} x_n = x_*$, then $x_*$ is *stable*; otherwise it is unstable.

**Linearization:**
$f(x) \approx f(x_*) + f'(x_*)(x - x_*)$
(linear approximation, first order Taylor expansion)
(**IF** $x$ is close to $x_*$)

So near an equilibrium $x_*$,
$x_{n+1} = f(x_n) \approx f(x_*) + f'(x_*)(x_n - x_*) = x_* + f'(x_*)(x_n - x_*)$
or $x_{n+1} - x_* \approx f'(x_*)(x_n - x_*)$ (A linear equation !)

Let $y_n = x_n - x_*$. Then $y_{n+1} = f'(x_*)y_n$.
If $f'(x_*) > 1$, then exponential growth (unstable)
If $0 < f'(x_*) < 1$, then exponential decay (stable)
If $f'(x_*) < -1$, then exponential growing oscillation (unstable)
If $-1 < f'(x_*) < 0$, then exponential decaying oscillation (stable)

# Stability of Equilibrium

**Condition for stability**: Suppose that $x_*$ is an equilibrium of $x_{n+1} = f(x_n)$. Then $x_*$ is a stable equilibrium if and only if $|f'(x_*)| < 1$ or $-1 < f'(x_*) < 1$.

**Example**: Bellow's equation: $N_{n+1} = \dfrac{\lambda N_n}{1 + N_n^b}$.

Equilibrium: $N_* = 0$ and $N^* = (\lambda - 1)^{1/b}$ (only exists if $\lambda > 1$)

$f(N) = \dfrac{\lambda N}{1 + N^b}$, $f'(N) = \dfrac{\lambda(1 - (1-b)N^b)}{(1+N^b)^2}$.

$f'(0) = \lambda$ (stable is $\lambda < 1$ and unstable if $\lambda > 1$)

$f'(N^*) = 1 - b(1 - \lambda^{-1})$

Case 1: $b \leq 1$: stable with $0 < f'(N^*) < 1$

Case 2: $1 < b \leq 2$: stable with $0 < f'(N^*) < 1$ if $\lambda < \dfrac{b}{b-1}$, and $-1 < f'(N^*) < 0$ if $\lambda > \dfrac{b}{b-1}$

Case 3: $b > 2$: stable when $\lambda < \dfrac{b}{b-2}$, unstable when $\lambda > \dfrac{b}{b-2}$.

The change of dynamics when $b$ and $\lambda$ change is called **bifurcation**.