Math 345 Lecture 2

Models ask for such equations

$\longrightarrow \quad X_{n+1} = \lambda X_n$ solution $\boxed{X_n = C\cdot\lambda^n}$ (1st order)

$\longrightarrow \quad X_{n+1} = aX_n + bX_{n-1}$ (2nd order)

Solution $X_n = \lambda^n$

$\lambda^{n+1} = a\lambda^n + b\lambda^{n-1} \Rightarrow \lambda^2 = a\lambda + b$ (characteristic equation)

$\Rightarrow$ two roots λ_1, λ_2 (eigenvalues)

$\boxed{X_n = C_1\lambda_1^n + C_2\lambda_2^n}$ general solution (representing all solutions with arbitrary constants C_1, C_2)

Example $\quad Y_{n+1} = 5Y_n - 6Y_{n-1}, \quad Y_0 = 2, \quad Y_1 = 5$

$\lambda^2 = 5\lambda - 6 \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0$

eigenvalues $\lambda = 2$ or $\lambda = 3$

solution $\quad Y_n = C_1 2^n + C_2 3^n$

$$\begin{aligned} Y_0 &= C_1 + C_2 = 2 \\ Y_1 &= C_1\cdot 2 + C_2\cdot 3 = 5 \end{aligned} \Rightarrow \begin{aligned} C_1 + C_2 &= 2 \\ 2C_1 + 3C_2 &= 5 \end{aligned} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= 1 \end{aligned}$$

So $\quad Y_n = 1\cdot 2^n + 1\cdot 3^n = 2^n + 3^n$

Profile of solution

$\lambda_1 > 0$ and $\lambda_2 > 0 \Rightarrow$ increasing or decreasing

$\lambda_1 < 0$ or $\lambda_2 < 0 \Rightarrow$ oscillating

Which eigenvalue is more important?

$y_n = 2^n + 3^n \qquad y_n \approx 3^n \quad (\text{since } 3^n > 2^n)$

$$\lim_{n\to\infty} \frac{y_n}{3^n} = 1 \qquad \lim_{n\to\infty} \frac{2^n+3^n}{3^n} = \lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n + 1\right] = 1$$

$y_n = 3^n + 5(-4)^n \qquad y_n \approx 5\cdot(-4)^n \qquad \lim_{n\to\infty} \frac{y_n}{(-4)^n} = 5$

If $|\lambda_1| > |\lambda_2|$, then λ_1 is called <u>dominant eigenvalue</u>.

$y_n = c_1\lambda_1^n + c_2\lambda_2^n \approx c_1\cdot\lambda_1^n$ (when n is large)

<u>Behavior of solutions</u>

<u>Case 1</u> $|\lambda_1| < 1 \qquad \lim_{n\to\infty} y_n = 0 \qquad$ Since $|\lambda_1|^n \to 0 \ (n\to\infty)$

<u>Case 2</u> $|\lambda_1| > 1 \qquad \lim_{n\to\infty} |y_n| = \infty$

<u>Complex eigenvalues</u> Case 1 λ_1, λ_2 are real valued

<u>Case 2</u> $\lambda_1, \lambda_2 = \alpha \pm i\beta$

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n$$

but y_n is still real-valued if $y_0, y_1 \in \mathbb{R}$.

Example $\quad X_{n+2} - X_{n+1} + X_n = 0, \quad X_0 = 1, \; X_1 = 2,$

$$\lambda^{n+2} - \lambda^{n+1} + \lambda^n = 0 \Rightarrow \lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Solution: $\quad X_n = C_1 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^n + C_2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^n$

$$X_0 = C_1 + C_2 = 1$$

$$X_1 = C_1 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) + C_2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 2$$

$$\Rightarrow C_1 + C_2 + (C_1 - C_2)\sqrt{3} i = 4$$

$$\Rightarrow (C_1 - C_2)\sqrt{3} i = 3 \Rightarrow C_1 - C_2 = -\sqrt{3} i$$

$$\Rightarrow 2C_1 = 1 - \sqrt{3} i \Rightarrow C_1 = \frac{1}{2} - \frac{\sqrt{3}}{2} i, \quad C_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

So $\quad X_n = \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^n + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^n$

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left[\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n-1} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n-1}\right]$$

$$= \left(\frac{1}{4} + \frac{3}{4}\right) \cdot \left[\qquad \right]$$

So $\quad X_n = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n-1} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n-1}$

Notice that $\quad \frac{1}{2} + \frac{\sqrt{3}}{2} i = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = e^{i\frac{\pi}{3}}$

Euler's formula $\quad e^{i\theta} = \cos\theta + i \sin\theta$

So $\quad \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n-1} = \left(e^{i\frac{\pi}{3}}\right)^{n-1} = e^{i\frac{(n-1)\pi}{3}}$

$$\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n-1} = \left(e^{-i\frac{\pi}{3}}\right)^{n-1} = e^{-i\frac{(n-1)\pi}{3}}$$

$$X_n = \left(\cos\frac{(n-1)\pi}{3} + i\sin\frac{(n-1)\pi}{3}\right) + \left(\cos -\frac{(n-1)\pi}{3} + i\sin -\frac{(n-1)\pi}{3}\right)$$

$\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$X_n = 2\cos\frac{(n-1)\pi}{3}$ $\rightarrow$ That is always real-valued.

Note X_n is periodic (not decaying nor growing)

Now we consider a model of two variables (X_n, Y_n)

$$\begin{cases} X_{n+1} = aX_n + bY_n \\ Y_{n+1} = cX_n + dY_n \end{cases}$$ (system of 1st order equations)

Two approaches ① reduce to 2nd order scalar equation (eliminate X_n or Y_n)

$$X_{n+2} = aX_{n+1} + bY_{n+1}$$
$$= aX_{n+1} + b(cX_n + dY_n)$$
$$= aX_{n+1} + bcX_n + bdY_n$$

$$bY_n = X_{n+1} - aX_n$$

So $X_{n+2} = aX_{n+1} + bcX_n + d(X_{n+1} - aX_n)$

$$= (a+d)X_{n+1} + (bc - ad)X_n$$

$$X_{n+2} - (a+d)X_{n+1} + (ad-bc)X_n = 0$$

Then solve it as before. $\left(X_n = c_1\lambda_1^n + c_2\lambda_2^n\right)$

② linear algebra !

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

solution $\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$

↳ but what is this ?

Try $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lambda^n \begin{pmatrix} A \\ B \end{pmatrix}$

$$\lambda^{n+1} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \lambda^n \begin{pmatrix} A \\ B \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow \lambda$ is an eigenvalue of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} A \\ B \end{pmatrix}$ eigenvector !

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 .$$

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$ (characteristic equation)

Same as the one from approach ①

Solution $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = C_1 \lambda_1^n \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + C_2 \lambda_2^n \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$

Example

$$X_{n+1} = 3X_n + 2Y_n$$
$$Y_{n+1} = X_n + 4Y_n$$

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

Step1 Solve eigenvalues $\det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = 0$

$$(3-\lambda)(4-\lambda) - 2 = 0 \quad \lambda^2 - 7\lambda + 10 = 0$$
$$(\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda_1 = 2, \ \lambda_2 = 5$$

Step2 Solve eigenvectors $\lambda = 2$

$$\begin{pmatrix} 3-2 & 2 \\ 1 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow A_1 + 2B_1 = 0 \Rightarrow A_1 = 2, \ B_1 = -1$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\lambda = 5 \quad \begin{pmatrix} 3-5 & 2 \\ 1 & 4-5 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_2 - B_2 = 0 \Rightarrow A_2 = B_2 = 1$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = C_1 2^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 5^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$