Math 345 Lecture 13

predator-prey system

Lotka-Volterra

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

or

$$\frac{dx}{dt} = ax \cdot \frac{K-x}{K} - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

~~$s = at$~~

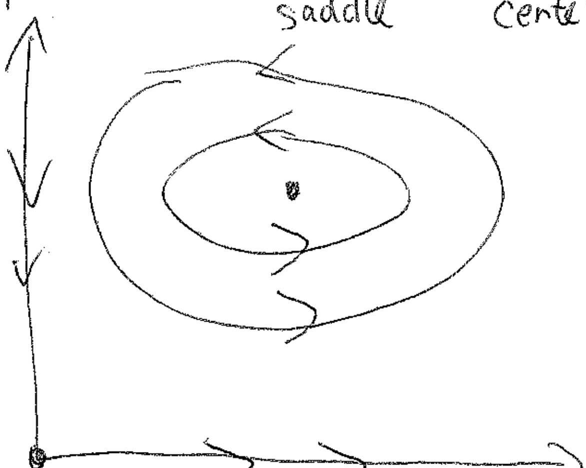
t	T	a	T^{-1}
x	M_1	b	$T^{-1}M_1^{-1}$
y	M_2	c	T^{-1}
		d	$T^{-1}M_2^{-1}$
		K	M_1

$$s = at, \quad u = \frac{bx}{a}, \quad v = \frac{dy}{c}$$

New Equations

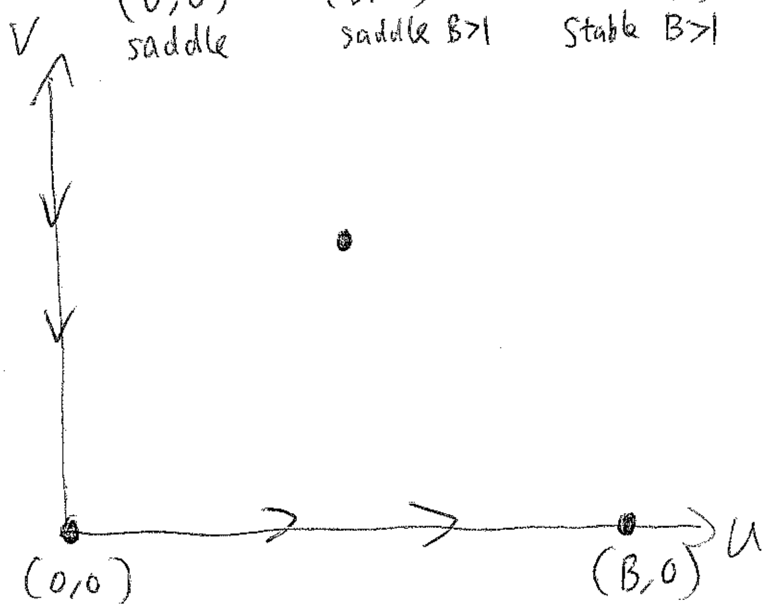
$$\frac{du}{ds} = u - uv$$
$$\frac{dv}{ds} = A(-v + uv)$$

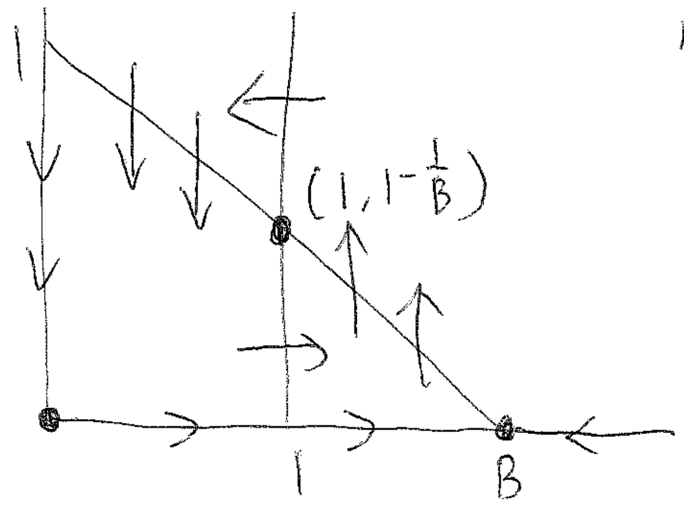
Equilibrium: $(0,0)$ saddle, $(1,1)$ center



$$\frac{du}{ds} = u\left(1 - \frac{u}{B}\right) - uv$$
$$\frac{dv}{ds} = A(-v + uv)$$

$(0,0)$ saddle $\quad (B,0)$ saddle $B>1$ $\quad \left(1, 1-\frac{1}{B}\right)$ Stable $B>1$





$(B>1)$

nullcline: $u=0$; $u'=0$, $v'=-Av$

$v=0$; $v'=0$, $u'=u(1-\frac{u}{B})$

$v=1-\frac{u}{B}$; $u'=0$, $v'=Av(u-1)$

$u=1$; $v'=0$, $u'=1-\frac{1}{B}-v$

$$J=\begin{pmatrix} 1-\frac{2u}{B}-v & -u \\ Av & -A+Au \end{pmatrix}$$

$$J(0,0)=\begin{pmatrix} 1 & 0 \\ 0 & -A \end{pmatrix} \quad \text{saddle}$$

$$J(B,0)=\begin{pmatrix} -1 & -B \\ 0 & A(B-1) \end{pmatrix} \quad \begin{matrix}\text{saddle} \\ B>1 \\ \text{stable node} \\ B<1\end{matrix}$$

$\lambda_1=-1$, $\lambda_2=A(B-1)$

$$J(1, 1-\frac{1}{B}) = \begin{pmatrix} 1-\frac{2}{B}-(1-\frac{1}{B}) & -1 \\ A(1-\frac{1}{B}) & A(1-1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{B} & -1 \\ A(1-\frac{1}{B}) & 0 \end{pmatrix}$$

Trace $= -\frac{1}{B} < 0$

$D = A(1-\frac{1}{B}) > 0$ if $B>1$ $\Rightarrow$ stable

$$T^2-4D = \frac{1}{B^2} - 4A(1-\frac{1}{B}) = \frac{1-4AB^2+4AB}{B^2}$$

$$4AB^2-4AB-1=0 \Rightarrow B=\frac{4A \pm \sqrt{16A^2+16A}}{8A} = \frac{A\pm\sqrt{A^2+A}}{2A}$$

$B_* = \dfrac{A+\sqrt{A^2+A}}{2A}$ when $B>B_*$ stable spiral

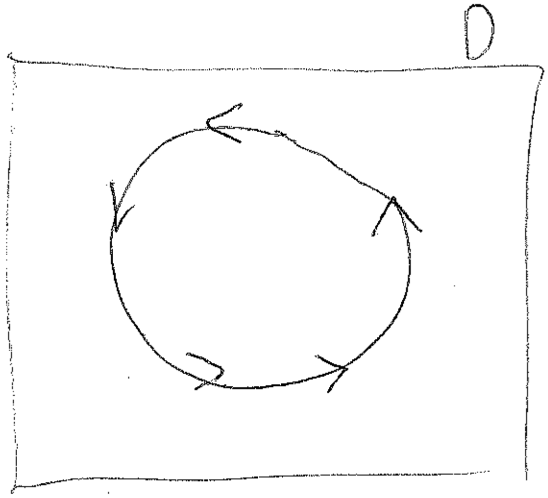
$B_* > B > 1$ stable node

Cycle or no cycle?

$$x' = F(x,y)$$
$$y' = G(x,y)$$

Bendixson's criterion Suppose D is a simply connected region in $\mathbb{R}^2$, and $\frac{\partial F}{\partial x} + \frac{\partial g}{\partial y} \neq 0$ for $(x,y) \in D$. Then there is no closed orbit in D.

proof



Green's Theorem (Math 212/213)

$$\int_C F(x,y)\,dy - G(x,y)\,dx$$
$$= \iint_S \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dx\,dy$$

But $\int_C F(x,y)\,dy - G(x,y)\,dx$

$$= \int_C \left(x'\,dy - y'\,dx\right) = 0.$$

So $\iint_S \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dx\,dy = 0$. if $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} > 0$ (or <0)

That is a contradiction. ▨

$$\boxed{\begin{array}{llll} u' = u - \frac{u^2}{B} - uv & = F(u,v) & \frac{\partial F}{\partial u} & = 1 - \frac{2u}{B} - v \\ v' = -Av + Auv & = G(u,v) & \frac{\partial G}{\partial v} & = -A + Au \end{array}}$$

not working.

Dulac's criterion: $\exists$ function $H(x,y)$ s.t

$$\frac{\partial(HF)}{\partial x} + \frac{\partial(HG)}{\partial y} > 0 \quad (\text{or} < 0)$$

$$\begin{cases} Hx' = H \cdot F \\ Hy' = H \cdot G \end{cases} \qquad H(u,v) = \frac{1}{uv}$$

$$HF = \frac{1}{v} - \frac{u^2}{Bv} - 1 \qquad \frac{\partial(HF)}{\partial u} = -\frac{2u}{Bv}$$

$$HG = -\frac{A}{u} + A \qquad \frac{\partial(HG)}{\partial v} = 0$$

$$\Rightarrow \frac{\partial(HF)}{\partial u} + \frac{\partial(HG)}{\partial v} = -\frac{2u}{Bv} < 0$$

$\Rightarrow$ So there is no periodic orbit for this system.

Lynx-Hare data.