Bifurcation in 1D differential equation.

$$x' = f(\mu, x) \qquad \mu \text{ is a parameter}$$

Equilibrium x_0 s.t. $f(\mu, x_0) = 0$.

Stable if $\frac{\partial f}{\partial x}(\mu, x_0) < 0$ Unstable if $\frac{\partial f}{\partial x}(\mu, x_0) > 0$

bifurcation $\frac{\partial f}{\partial x}(\mu, x_0) = 0$.

① Fishery model. $u(t)$ = population of fish

$$\frac{du}{dt} = Ku\left(1 - \frac{u}{N}\right) - M \qquad \text{(homework 3)}$$

Nondimensionalized equation $\left(v = \frac{u}{N},\ s = Kt\right)$

$$\frac{dv}{ds} = v(1-v) - \mu \qquad \mu = \frac{M}{KN}$$

Equilibrium $v - v^2 - \mu = 0 \Rightarrow v^2 - v + \mu = 0$

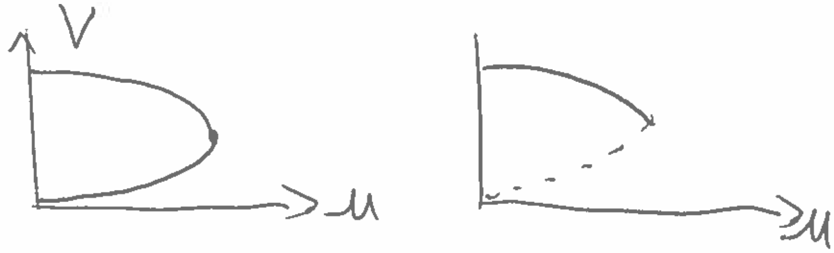
$$v = \frac{1 \pm \sqrt{1-4\mu}}{2} \qquad \text{So } \mu \leq \frac{1}{4}$$

when $0 \leq \mu < \frac{1}{4} \Rightarrow 2$ equilibria $x_+ = \frac{1+\sqrt{1-4\mu}}{2}$ and $x_- = \frac{1-\sqrt{1-4\mu}}{2}$

$\mu = \frac{1}{4} \Rightarrow 1$ equilibrium $x_+ = \frac{1}{2}$

$\mu > \frac{1}{4} \Rightarrow 0$ equilibrium.

$\mu = \frac{1}{4}$ bifurcation point!



$$f(u,v) = v - v^2 - u \qquad \frac{\partial f}{\partial v} = 1-2v$$

$$\frac{\partial f}{\partial v}(u, X_+) = 1 - 2\left(\frac{1+\sqrt{1-4u}}{2}\right) = -\sqrt{1-4u} < 0 \Rightarrow X_+ \text{ is Stable if } 0 \le u < \tfrac{1}{4}$$

$$\frac{\partial f}{\partial v}(u, X_-) = 1 - 2\left(\frac{1-\sqrt{1-4u}}{2}\right) = \sqrt{1-4u} > 0 \Rightarrow X_- \text{ is unstable if } 0 \le u < \tfrac{1}{4}$$

When $u = \frac{1}{4}$ $\quad \frac{\partial f}{\partial v}(u, X_+) = 0$ bifurcation point!

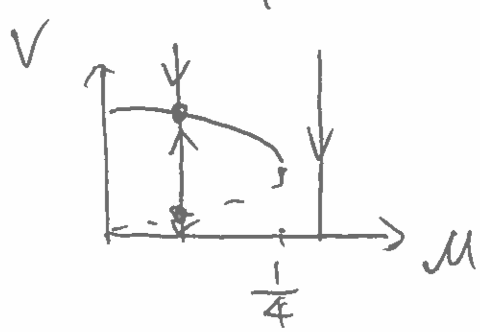
This is a saddle-node bifurcation point $\quad 2 \to 1 \to 0$ # of equilibria

$\xrightarrow{\quad u \quad}$

biological meaning when u (harvesting) $> \frac{1}{4}$, the fish population goes to extinction.

$\Rightarrow$ original model $\quad u = \frac{M}{KN} \Rightarrow \frac{M}{KN} > \frac{1}{4}$

bifurcation value $\boxed{M = \frac{1}{4}KN}$



$M = \frac{1}{4}KN$ is the maximum sustainable yield (MSY).

For fish population to sustain, the harvesting should not be over $M = \frac{1}{4}KN$ (K = growth, N = carrying capacity)

Math 345 Lecture 12.

Nonlinear Differential Equation Models

$$x' = f(x) \quad \text{or} \quad \begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

Growth Model (for one species)

$$\frac{dN}{dt} = N \cdot g(N)$$

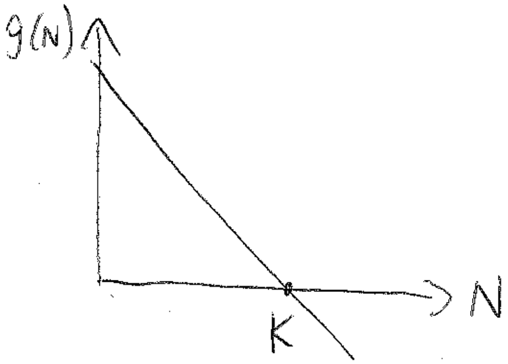
$g(N)$ = growth rate per capita

dimension $(g(N)) = T^{-1}$

(Malthus, 1798) $g(N) = r$ (constant) $\quad N(t) = N_0 e^{rt}$

(Verhulst, 1838) $g(N) = r(1 - \frac{N}{K})$ (~~dep~~ density dependent, logistic)

<u>Negative dependence</u> (Crowding effect) $\quad g(N)$ decreasing



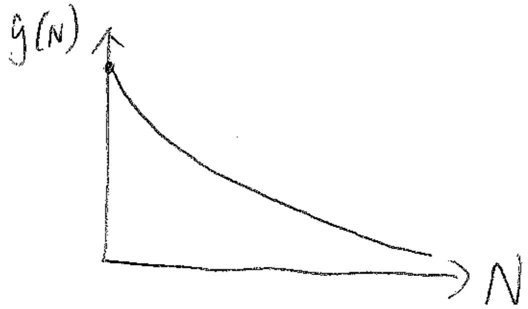
K = carrying capacity

Other negative dependence growth

$g(N) = re^{-\beta N}$ (Ricker model)

$g(N) = \dfrac{r}{\alpha + N}$ (Beverton - Holt)

(do not have a carrying capacity)



<u>Fisheries</u> <u>Ecology</u>

$$\frac{dN}{dt} = N \cdot \frac{r}{\alpha + N} \qquad \int \frac{(\alpha + N)\,dN}{N} = \int r\,dt$$

$$\Rightarrow \alpha \ln N + N = rt + C$$

$$\Rightarrow e^{N} \cdot N^{\alpha} = Ce^{rt}$$

$$\Rightarrow e^{N(t)} \cdot (N(t))^{\alpha} = e^{N_0} N_0^{\alpha} e^{rt}$$

Gompertz growth (Tumor model)

$$g(N) = -k \ln N$$

$$\lim_{N \to 0} g(N) = \infty$$

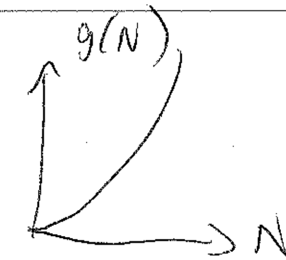
$$\frac{dN}{dt} = -kN \ln N \qquad \lim_{n \to \infty} N \ln N = 0$$

(Solve it homework)

~~another~~ Another explanation of $\dfrac{dN}{dt} = \underbrace{rN}_{\text{growth}} - \underbrace{\dfrac{rN^2}{K}}_{\text{intraspecific competition}}$

probability of encounter between individuals

$N \cdot N$ (mass action)



Positive dependence

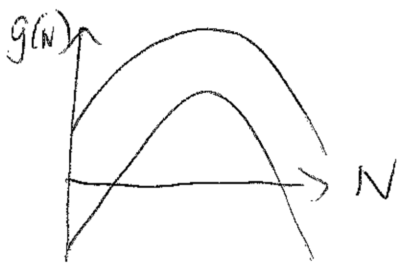
$g(N)$ increasing

rarely happen (von Foerster, 1960) $g(N) = \cancel{a}\, rN$

$\Rightarrow$ Solution blows up at a finite time.

Allee effect

$g(N)$ initially increasing then decreasing



$$g(N) = a_1 + a_2 N + a_3 N^2$$

$$a_2 > 0, \quad a_3 < 0.$$

$a_1 < 0$ strong Allee effect

$a_1 > 0$ weak Allee effect.

 strong Allee effect (bistable)

Interaction between species (2 species)

$$x' = f(x,y) = \underbrace{f_1(x)}_{\text{growth}} + \underbrace{g_1(x,y)}_{\text{interaction}}$$

$$y' = g(x,y) = \underbrace{f_2(y)}_{\text{growth}} + \underbrace{g_2(x,y)}_{\text{interaction}}$$

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

f_x : feedback from x to x | g_x : x to y

f_y : feedback from y to x | g_y : y to y

Competition : $g_x < 0$ and $f_y < 0$ (or $g_1 < 0$, $g_2 < 0$)

mutualism (cooperation): $g_x > 0$ and $f_y > 0$ (or $g_1 > 0$, $g_2 > 0$)

predator-prey (ecology) : $g_x > 0$ and $f_y < 0$ (or $g_1 > 0$, $g_2 < 0$)
Consumer-resource

Competition Model

~~Lotka~~,

$(x(t), y(t))$: ① two biological species competing for same resource

② two companies competing for market
(Coke and Pepsi) (Nike and UA)

③ two armies fighting each other.

possible outcomes: ① one wins, one loses (Competition exclusion)

② Neither wins or loses (coexistence)

Lotka - Volterra Competition Model

Lotka (American chemist / insurance company) 1920 Volterra (Italian mathematician) 1925

Gause (Russian biologist)

$$\frac{dN_1}{dt} = r_1 N_1 \cdot \frac{K_1 - N_1 - \beta_{12} N_2}{K_1}$$

$$\frac{dN_2}{dt} = r_2 N_2 \cdot \frac{K_2 - N_2 - \beta_{21} N_1}{K_2}$$

variable	meaning	dimension
t	time	T
N_1	~~specie~~ species 1	M_1
N_2	species 2	M_2

parameter	meaning	dimension
r_1	growth rate per capita N_1	T^{-1}
r_2	———— N_2	T^{-1}
K_1	Carry capacity N_1	M_1
K_2	———— N_2	M_2
β_{12}	impact of N_2 on N_1	M_1/M_2
β_{21}	———— N_1 on N_2	M_2/M_1

$$s = r_1 t, \quad u = \frac{N_1}{K_1}, \quad v = \frac{N_2}{K_2}$$

$$\frac{dN_1}{dt} = \frac{dN_1}{du}\frac{du}{ds}\frac{ds}{dt} = K_1 \frac{du}{ds} r_1 = r_1 K_1 u \frac{K_1 - K_1 u - \beta_{12} K_2 v}{K_1}$$

$$\Rightarrow \frac{du}{ds} = u \left(1 - u - \frac{\beta_{12} K_2}{K_1} v \right)$$

$$\frac{dN_2}{dt} = \frac{dN_2}{dv}\frac{dv}{ds}\frac{ds}{dt} = K_2 \frac{dv}{ds} r_1 = r_2 K_2 v \frac{K_2 - K_2 v - \beta_{21} K_1 u}{K_2}$$

$$\Rightarrow \frac{dv}{ds} = \frac{r_2}{r_1} v \left(1 - v - \frac{\beta_{21} K_1}{K_2} u \right)$$

Define $a = \dfrac{r_2}{r_1}$, $b = \dfrac{\beta_{12} K_2}{K_1}$, $c = \dfrac{\beta_{21} K_1}{K_2}$

New system $\quad u' = u(1-u-bv)$

$\qquad\qquad\quad v' = av(1-v-cu)$

x-nullcline : $u=0$ or $1-u-bv=0$ $\left(v = \dfrac{1-u}{b}\right)$

y-nullcline : $v=0$ or $1-v-cu=0$ $\left(v = 1-cu\right)$

Equilibrium : $(0,0)$, $(0,1)$, $(1,0)$

$$v = \frac{1-u}{b} \text{ and } v = 1-cu \Rightarrow \frac{1-u}{b} = 1-cu$$

$$\Rightarrow 1-u = b-bcu \Rightarrow (bc-1)u = b-1 \Rightarrow u = \frac{b-1}{bc-1}$$

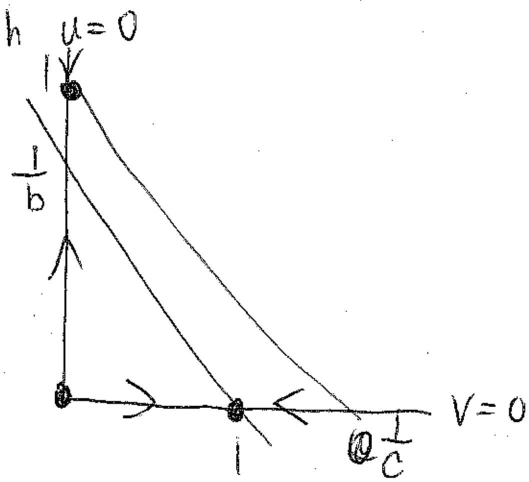
$$v = 1 - c\cdot\frac{b-1}{bc-1} = \frac{(bc-1)-(bc-c)}{bc-1} = \frac{c-1}{bc-1}$$

So $\left(\dfrac{b-1}{bc-1}, \dfrac{c-1}{bc-1}\right) = (u_*, v_*)$

$(0,0)$ extinction $\quad (0,1), (1,0)$ one wins and one loses

$\left(\frac{b-1}{bc-1}, \frac{c-1}{bc-1}\right)$ coexistence.

graph



$b>1$ and $c<1$

$1 > \frac{1}{b}$ and $\frac{1}{c} > 1$ (or $\ >1$)

Case 1 $\quad b<1$ and $c>1$

Case 2 $\quad b>1$ and $c<1$

Case 3 $\quad b<1$ and $c<1$

Case 4 $\quad b>1$ and $c>1$

Jacobian $\quad J = \begin{pmatrix} 1-2u-bv & -bu \\ -acv & a-\cancel{2av}-acu \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ $\quad T = 1+a>0$, $D = a>0$ unstable node

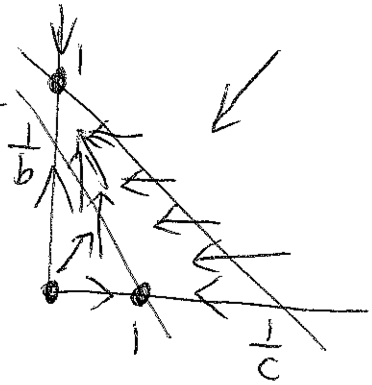
$J(1,0) = \begin{pmatrix} -1 & -b \\ 0 & a(1-c) \end{pmatrix}$ $\quad T = -1+a(1-c)$, $D = -a(1-c)$

$J(0,1) = \begin{pmatrix} 1-b & 0 \\ -ac & -a \end{pmatrix}$ $\quad T = 1-a-b$, $D = -a(1-b)$

Case1

$c<1$

$b>1$



$1-u-bv$: $u'=0$, $v'=av(1-v-cu)$

$1-v-cu$: $v'=0$, $u'=1-u-bv$

$(1,0)$: $D<0$ saddle

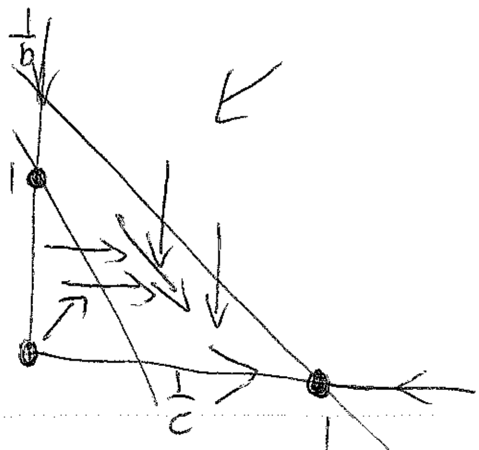
$(0,1)$: $D>0$, $T<0$ stable node

$$\lim_{t\to\infty}(u(t),v(t)) = (0,1)$$

Case2

$c>1$

$b<1$



$(1,0)$: $D>0$, $T<0$ Stable node

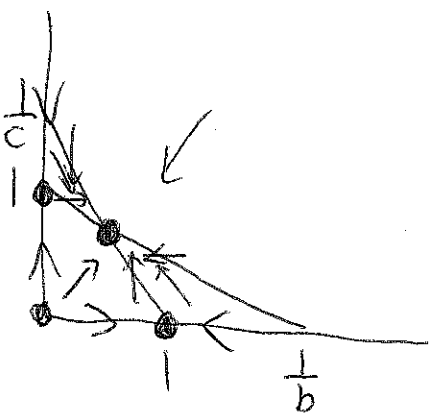
$(0,1)$: $D<0$ saddle

$$\lim_{t\to\infty}(u(t),v(t)) = (1,0)$$

Case3

$b<1$

$c<1$



$(1,0)$ saddle

$(0,1)$ saddle

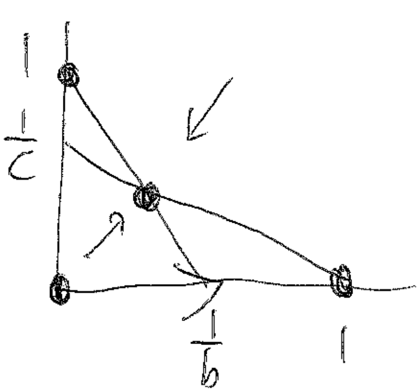
(u_*,v_*) stable node

$$\lim_{t\to\infty}(u(t),v(t)) = (u_*,v_*)$$ Coexistence

Case4

$b>1$

$c>1$



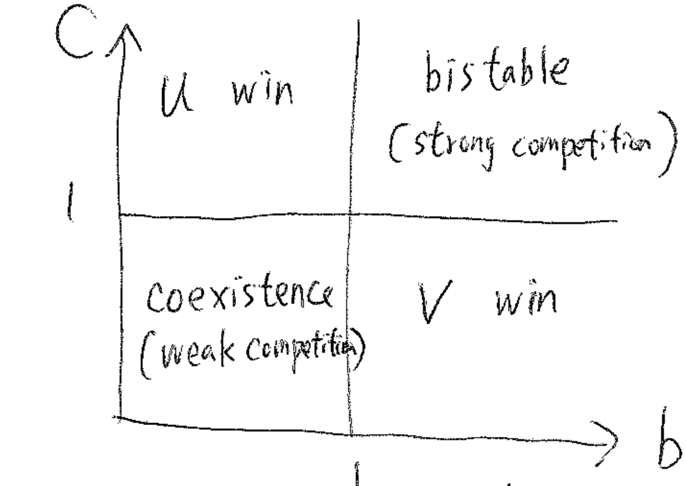
$(1,0)$ $D>0$, $T<0$ stable node

$(0,1)$ $D>0$, $T<0$ stable node

(u_*,v_*) saddle

bistable !

Bifarcation Diagram



① Coexistence is achieved when strength of competition is small ($b<1$ and $C<1$)

② One competitor wins exclusively when the competion does more harm to the other competitor ($b>1>C$ or $C>1>b$)

③ When competition is strong ($b>1, c>1$) the outcome depends on initial condition. One competitor eventually wins.

Principle of Competion exclusion

When competition is intense, only one species will win.

How do species achieve ~~coest~~coexistence in nature?

① Space. Different species occupy different spatial regions. (biological invasion, foreign species invade new territory)

② biodiversity. The result here only work for TWO species. When there are more species, the ecosystem appears to be more stable ~~(less)~~.