Math 345 Lecture 10

Models with differential equations

① Malthus model : $\frac{dN}{dt} = KN$

② Logistic model : $\frac{dN}{dt} = N(1-N)$ (after nondimensionalizatio)

③ chemostat model : $\begin{cases} \frac{dN}{dt} = K(C)N - AN \\ \frac{dC}{dt} = -\alpha K(C) N + (C_0 - C) \end{cases}$ $\quad K(C) = aC$ or $K(C) = \frac{aC}{1+C}$

Method : a) solve the solution (calculus)

b) solve the equilibrium, linearziation, stability

c) graph the solutions.

d) dimension analysis.

Scalar model $\quad \frac{dx}{dt} = f(x) \quad$ equilibrium $f(x) = 0$

stability $\quad f'(x) < 0$ (stable)

$f'(x) > 0$ (unstable)

System model $\quad \begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \quad$ equilibrium $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

$\Rightarrow (x_*, y_*)$

$$f(x,y) \approx f(x_*, y_*) + f_x(x_*, y_*)(x - x_*) + f_y(x_*, y_*)(y - y_*)$$

$$g(x,y) \approx g(x_*, y_*) + g_x(x_*, y_*)(x - x_*) + g_y(x_*, y_*)(y - y_*)$$

Example $x' = x(2-x)(1-x)$

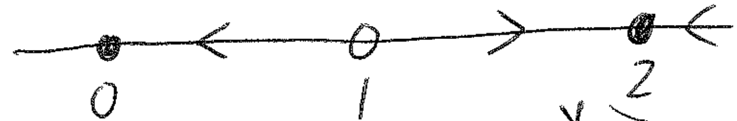
Equilibrium $x=0$, $x=1$, $x=2$

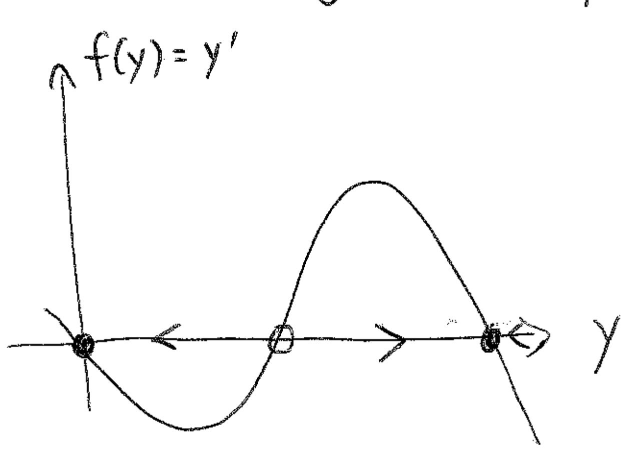
Stability $f(x) = -x^3+3x^2-2x$ $f'(x) = -3x^2+6x-2$

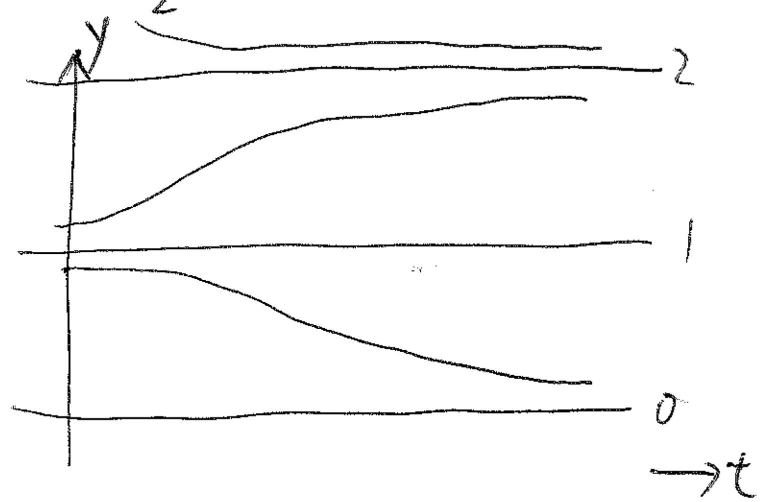
$f'(0) = -2 < 0$ stable $f'(1) = 1 > 0$ unstable

$f'(2) = -2 < 0$ stable

phase line







solution curve

direction field

Linearized Equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} (x_*, y_*)$$

linear system of differential equations

$x_{n+1} = \lambda x_n \Rightarrow x_n = C \lambda^n$

$x' = \lambda x \Rightarrow x(t) = e^{\lambda t}$

How to solve ? $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\lambda t}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \lambda e^{\lambda t} \begin{pmatrix} A \\ B \end{pmatrix} \qquad \text{we use} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = J$$

$$\lambda \cancel{e^{\lambda t}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \cancel{e^{\lambda t}} \qquad \text{So} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \end{pmatrix}$$

Again λ is the eigenvalue of J, and $\begin{pmatrix} A \\ B \end{pmatrix}$ is eigenvector

Assume eigenvalues are λ_1, λ_2, eigenvector $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} e^{\lambda_2 t}$$

Also true for $V' = A \cdot V$ if $V(t) \in \mathbb{R}^n$, A is $n \times n$ matrix

Example $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ eigenvalue $\lambda_1 = -3$, $\lambda_2 = -4$

$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

if $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$c_1 = c_2 = 1$

$x(t) = e^{-3t} + e^{-4t}$

$y(t) = e^{-3t} + 2e^{-4t}$

Example $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Eigenvalues: $(-5-\lambda)(-1-\lambda) - 5\cdot(-4) = 0$

$$\lambda^2 + 6\lambda + 5 + 20 = 0 \qquad \lambda^2 + 6\lambda + 25 = 0$$

$$(\lambda+3)^2 = -16 \qquad \lambda = -3 \pm 4i$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 \\ 1+2i \end{pmatrix} \quad \text{with} \quad \lambda = -3+4i$$

Solution: $e^{(-3+4i)t} \begin{pmatrix} -2 \\ 1+2i \end{pmatrix}$ and $e^{(-3-4i)t} \begin{pmatrix} -2 \\ 1-2i \end{pmatrix}$

$$= e^{-3t} (\cos 4t + i \sin 4t) \begin{pmatrix} -2 \\ 1+2i \end{pmatrix}$$

$$= e^{-3t} \begin{pmatrix} -2\cos 4t - 2i \sin 4t \\ \cos 4t + i \sin 4t + 2i \cos 4t - 2\sin 4t \end{pmatrix}$$

$$= e^{-3t} \begin{pmatrix} -2\cos 4t \\ \cos 4t - 2\sin 4t \end{pmatrix} + i e^{-3t} \begin{pmatrix} -2\sin 4t \\ \sin 4t + 2\cos 4t \end{pmatrix}$$

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} -2\cos 4t \\ \cos 4t - 2\sin 4t \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2\sin 4t \\ \sin 4t + 2\cos 4t \end{pmatrix}$$

The behavior of $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

is determined by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its eigenvalues.

eigenvalue $\quad (a-\lambda)(d-\lambda) - bc = 0$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$T = a+d = \text{Trace}(A) \qquad ad - bc = \text{determinant}(A) = D$

$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - T\lambda + D = 0. \qquad \lambda = \dfrac{T \pm \sqrt{T^2-4D}}{2}$

~~Case 1~~ $\cancel{T^2 - 4D}$ So $T = \lambda_1 + \lambda_2$ and $D = \lambda_1 \lambda_2$.

<u>Case 1</u> $\quad T^2 - 4D > 0 \Rightarrow 2$ real roots.

<u>Case 1a</u> $\quad T > 0$ and $D > 0 \Rightarrow 2$ positive roots $\lambda_1 \geq \lambda_2 > 0$.

<u>Case 1b</u> $\quad T < 0$ and $D > 0 \Rightarrow 2$ negative roots $\lambda_1 \leq \lambda_2 < 0$.

<u>Case 1c</u> $\quad D < 0 \Rightarrow 1$ positive 1 negative $\lambda_1 < 0 < \lambda_2$

<u>Case 2</u> $\quad T^2 - 4D < 0 \Rightarrow 2$ complex roots

$$\lambda = \frac{T}{2} \pm \frac{\sqrt{4D-T^2}}{2} i$$

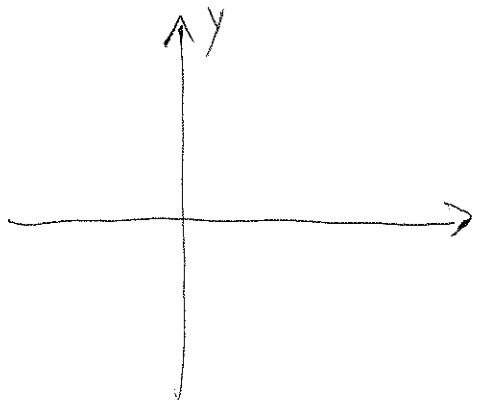
<u>Case 2a</u> $\quad T > 0 \Rightarrow \text{Re}(\lambda) > 0$

<u>Case 2b</u> $\quad T < 0 \Rightarrow \text{Re}(\lambda) < 0$.

Equilibrium $(0,0)$ is <u>stable</u> if $\text{Re}(\lambda_1) < 0$ and $\text{Re}(\lambda_2) < 0$.

1. $\lambda_1 \geq \lambda_2 > 0$ (unstable node) (source in Math 302)
2. $\lambda_1 \leq \lambda_2 < 0$ (stable node) (sink
3. $\lambda_1 < 0 < \lambda_2$ (saddle)
4. $\lambda = \alpha \pm \beta i$, $\alpha > 0$ (unstable spiral) (spiral source)
5. $\lambda = \alpha \pm \beta i$, $\alpha < 0$ (stable spiral) (spiral sink)

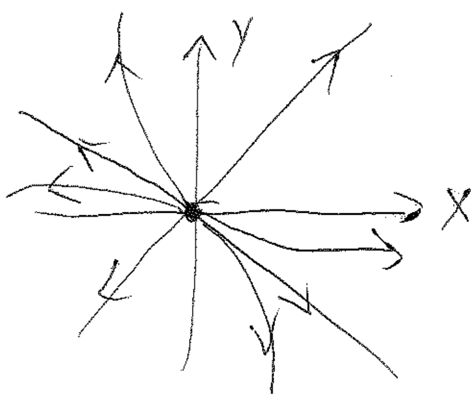
Graphs can be made by using <u>pplane8</u> $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$



phase plane — (x,y) plane

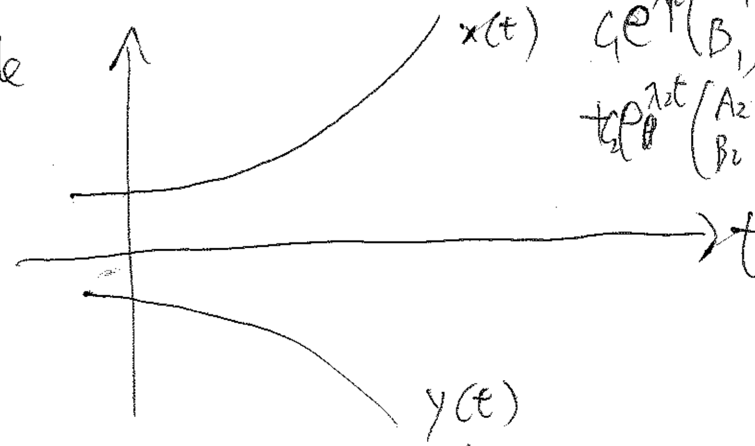
phase portrait — vector field (f, g)

orbit — a solution curve $(x(t), y(t))$ on phase plane.

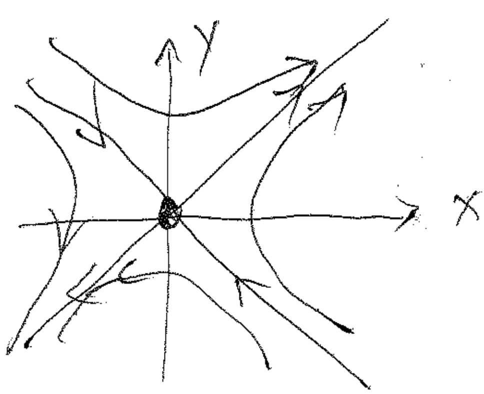


unstable node

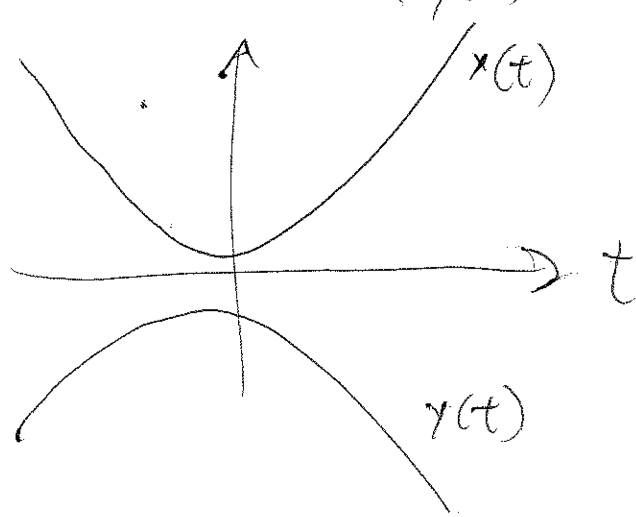
stable node (opposite direction)

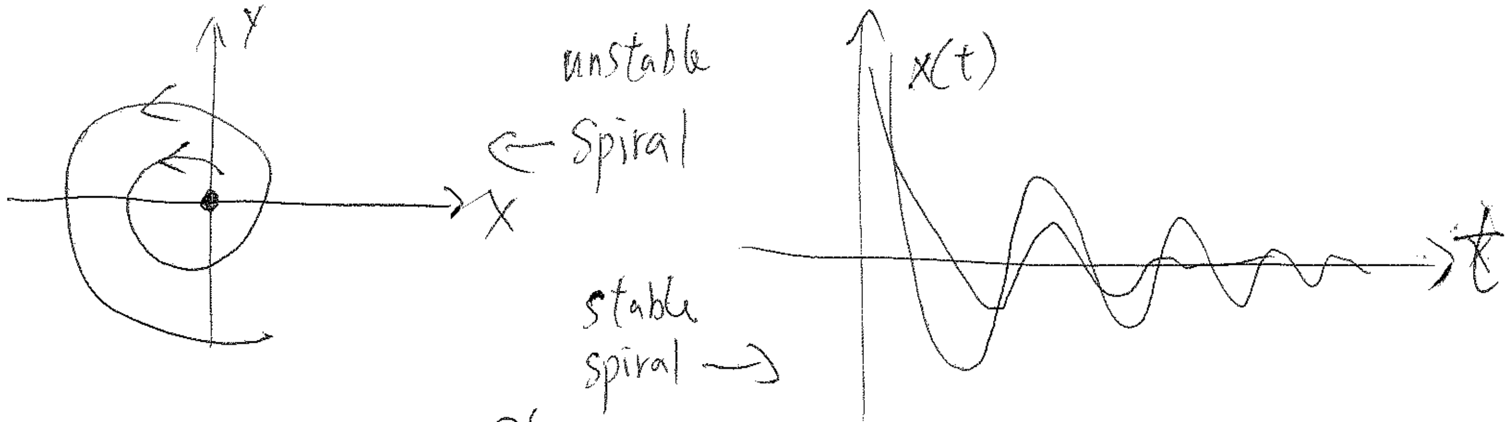


$c_1 e^{\lambda_1 t}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$

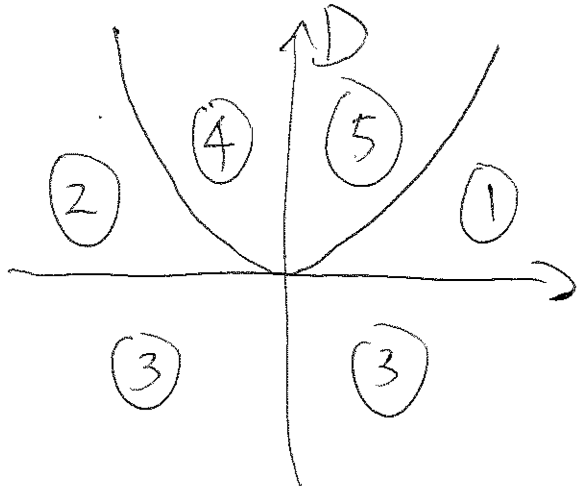


saddle





Trace - Determinant Plane



① $T>0,\ D>0,\ T^2-4D>0$

unstable node

② $T<0,\ D>0,\ T^2-4D>0$

stable node

③ $D<0$ saddle.

④ $T<0\ ,\ T^2-4D>0$

stable spiral

⑤ $T>0\ ,\ T^2-4D>0$

unstable spiral.

If $T=0$ or $D=0$, then nonlinear system cannot be determined by linear system.

Example $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(y^2-y) \\ x-y \end{pmatrix}$

Equilibrium: $x\cdot(y-1)\cdot y=0$

$x=0$ or $y=1$ or $y=0$

$x=0 \Rightarrow y=0$ $(0,0)$

$y=1 \Rightarrow x=1$ $(1,1)$

$y=0 \Rightarrow x=0$

$$J=\begin{pmatrix} y^2-y & 2xy-x \\ 1 & -1 \end{pmatrix}$$

$J(0,0)=\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ $T=-1,\ D=0$ (cannot determine)

$J(1,1)=\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $T=-1,\ D=-1$ (saddle) chemostat

$$N' = ACN - N$$
$$C' = -ACN + 1 - C$$

Equilibrium: $(0,1)$ $\left(1-\frac{1}{A}, \frac{1}{A}\right)$

$$J = \begin{pmatrix} AC-1 & AN \\ -AC & -AN-1 \end{pmatrix} \qquad J(0,1) = \begin{pmatrix} A-1 & 0 \\ -A & -1 \end{pmatrix}$$

$$T = A-2, \quad D = 1-A$$

$0<A<1$, $T<0$, $D>0$ stable (stable node)

$A>1$, $D<0$ unstable (saddle)

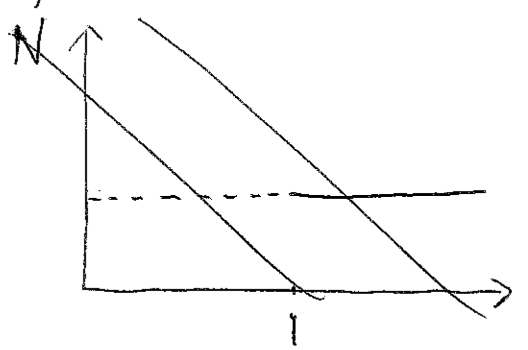
$$J\left(1-\frac{1}{A}, \frac{1}{A}\right) = \begin{pmatrix} A\cdot\frac{1}{A}-1 & A\left(1-\frac{1}{A}\right) \\ -A\cdot\frac{1}{A} & -A\left(1-\frac{1}{A}\right)-1 \end{pmatrix} = \begin{pmatrix} 0 & A-1 \\ -1 & -A \end{pmatrix}$$

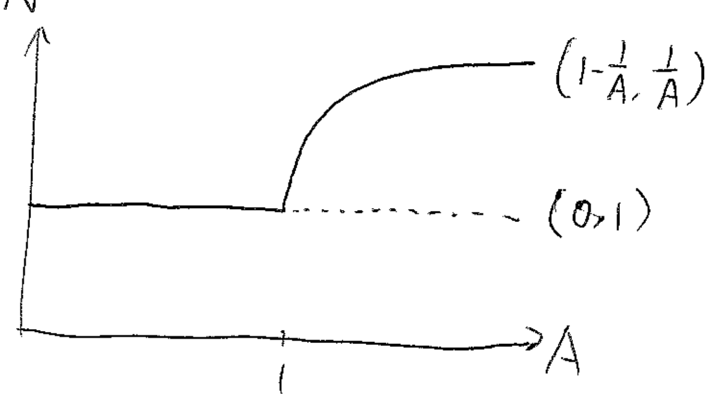
$T=-A$, $D=A-1$ (saddle)

$0<A<1$, $T<0$, $D<0 \Rightarrow$ unstable ($1-\frac{1}{A}<0$ so it is not valid)

$A>1$, $T<0$, $D>0 \Rightarrow$ stable . (stable node)

Bifurcation Diagram





stable node or spiral

$(0,1)$: $T^2-4D = (A-2)^2-4(1-A) = A^2-4A+4-4+4A = A^2>0$ (node!)

$\left(1-\frac{1}{A}, \frac{1}{A}\right)$ $T^2-4D = (-A)^2-4(A-1) = A^2-4A+4 = (A-2)^2>0$ (node!)