Math 345 Intro to Math Biology
Lecture 1: Biological Models using Difference Equations

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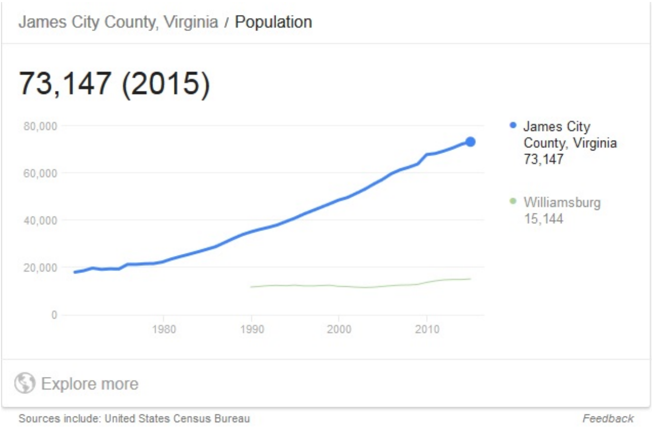
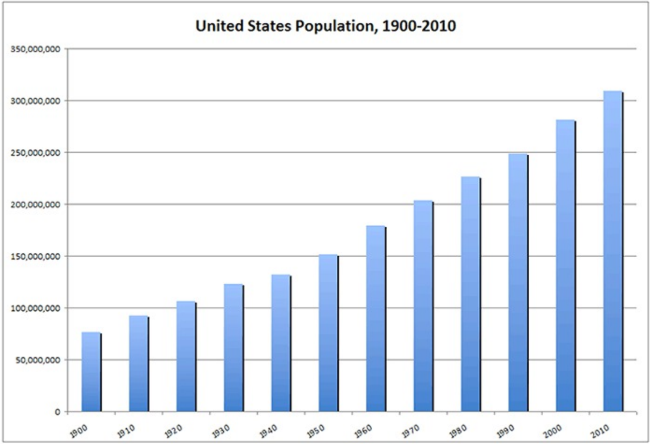
Population of James City County, Virginia

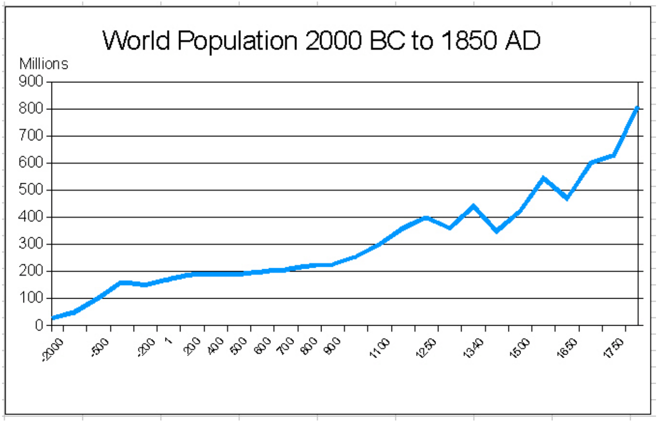
Population is often recorded in a form of data set



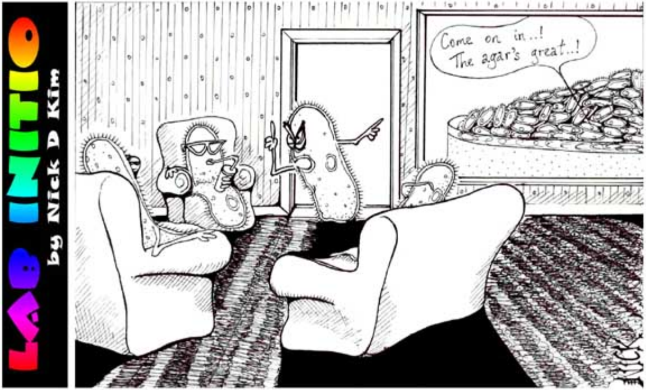
US population 1900-2010



Population of world up to 1850



Population in laboratory: bacteria



"I wish you'd learn to put the lid on your Petri dish, Harry..!! We came here with four kids, and now it looks like we've got twenty million..!!"

Experiment data of yeast cells of G.F. Gause (1934)

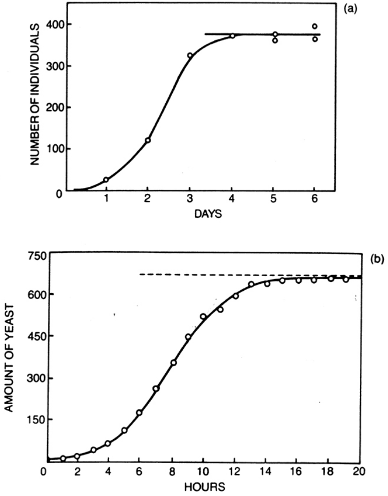


Fig. P2.1. (a) The growth of a laboratory population of *Paramecium caudatum* fitted to the logistic equation. Circles are observed counts; line is the fitted curve. (From Gause.) (b) The logistic growth of a laboratory population of yeast cells. (From Pearl.) (Reproduced, with permission, from *Principles of Animal Ecology*, W.C. Allee, A.E. Emerson, O. Park, T. Park and K.P. Schmidt, W.B. Saunders Co., Philalplina, 1949.

First model

Goal:

1. Find mathematical rules governing the growth
2. Set up mathematical models
3. Analyze the models
4. Fit the model to the original data

Model number 1

Let N_n be the n-th generation of population

$N_{n+1} = N_n + bN_n - dN_n = (1 + d - b)N_n = \lambda N_n$
b =per capita reproduction rate, d =per capita mortality rate

Malthus Model Solution: $N_n = N_0\lambda^n$
exponential growth when $\lambda > 1$,
exponential decay when $0 < \lambda < 1$
It is a homogeneous first-order linear difference equation.

First Matlab program: plotting the time series data (a sequence)

Example 1: Insect population

(page 7-8)

Consider the reproduction of the poplar gall aphid. Adult female aphids produce galls on the leaves of poplars. All the progeny of a single aphid are contained in one gall. Some fraction of these will emerge and survive to adulthood.

Variables:
a_n=number of adult female aphids in the n-th generation
p_n=number of progeny in the n-th generation

Parameters:
m=fractional mortality of the young aphids
f=number of progeny per female aphid
r=ratio of female aphids to the total adult aphids

Model: $p_{n+1} = fa_n$, $a_{n+1} = r(1-m)p_{n+1}$
after substitution: $a_{n+1} = fr(1-m)a_n$ (Malthus model!)

Solution: $a_n = [fr(1-m)]^n a_0$

Fibonacci

Fibonacci is the "greatest European mathematician of the middle ages", his full name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa (Italy), the city with the famous Leaning Tower, about 1175 AD. He was one of the first people to introduce the Hindu-Arabic number system into Europe - the positional system we use today - based on ten digits with its decimal point and a symbol for zero: 1234567890 His book on how to do arithmetic in the decimal system, called Liber abbaci (meaning Book of the Abacus or Book of Calculating) completed in 1202 persuaded many European mathematicians of his day to use this "new" system.

Fibonacci's rabbit problem

In Fibonacci's Liber Abaci (written in 1202), chapter 12, he introduces the following problem:

How Many Pairs of Rabbits Are Created by One Pair in One Year? A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

1. Initially there is one pair of rabbits;
2. Each pair of rabbits can reproduce since they are two months old, (not one month old).

Fibonacci's sequence

Recursive relation: $y_{n+1} = y_n + y_{n-1}$, $y_0 = y_1 = 1$
(y_n is the number of pair of rabbits after n months)
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \cdots$

Guess a solution: exponential solution $y_n = C\lambda^n$
then $\lambda^2 = \lambda + 1$, and $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$,

$$y_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Fibonacci sequence in nature (leaf patterns, pine cone spiral)
`http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html`

Golden ratio: $\dfrac{\sqrt{5}-1}{2} = \dfrac{2}{\sqrt{5}+1} \approx 0.618$

2nd order linear difference equation

$y_{n+1} = \lambda y_n$ solution $y_n = y_0 \lambda^n$ (exponential)
$y_{n+1} = a y_n + b y_{n-1}$ solution $y_n = c_1 \lambda_1^n + c_2 \lambda_2^n$
(c_1 and c_2 are determined by the initial conditions y_0 and y_1

λ_1, λ_2 are solutions of $\lambda^2 = a\lambda + b$, and these two numbers are called eigenvalues of the problem, and $\lambda^2 = a\lambda + b$ is the characteristic equation.

Example: Solve $y_{n+1} = 5y_n - 6y_{n-1}$, $y_0 = 2$ and $y_1 = 5$.

$y_{n+1} = a y_n + b y_{n-1} + c$:
try a solution in form of $y_n = c_1 \lambda_1^n + c_2 \lambda_2^n + e$
(e is the fixed point or equilibrium solution)

Age structured models

Fibonacci's rabbit model not only considers the total number of rabbits, but also the ages of rabbit. We can reformat the model in this way: let M_n be the number of adult pairs of rabbits (at least two months old), and let J_n be the number of juvenile pair (one month old), then

$M_{n+1} = M_n + J_n$, $J_{n+1} = M_n$, with $M_0 = 0$ and $J_n = 1$,
or in matrix notation:
$\begin{pmatrix} J_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} J_n \\ M_n \end{pmatrix}$, and $\begin{pmatrix} J_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

This is a matrix model in discrete time. A quick solution of the equation is
$\begin{pmatrix} J_n \\ M_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} J_0 \\ M_0 \end{pmatrix}$.

Fibonacci's model is not realistic since the rabbits never die! But it is a good approximation of a rapidly growing population in some period of time. We will discuss more realistic (nonlinear) models later.

Propagation of annual plants

(Page 8-11)
Plants produce seeds at the end of their growth season (say August), after which they die. A fraction of seeds survive the winter and some of these germinates at the beginning of the next season (say May), give rise to the new generation of plants. Some seed might remain dormant for a year or more before reviving, and we assume that seeds older than two years are no longer viable.

Variables:
p_n=number of plants in generation n
S_n^1=number of one-year-old seeds in April (before germination),
S_n^2=number of two-year-old seeds in April (before germination),
$\overline{S}_n^1$=number of one-year-old seeds left in May (after germination),
$\overline{S}_n^2$=number of two-year-old seeds left in May (after germination),
S_n^0=number of new seeds produced in August.

Parameters:
γ=number of seeds produced per plant in August
α=fraction of one-year-old seeds that germinate in May
β=fraction of two-year-old seeds that germinate in May
σ=fraction of seeds that survive a given winter

Model for annual plants

Equations:
$p_n = \alpha S_n^1 + \beta S_n^2$
$\overline{S}_n^1 = (1-\alpha)S_n^1$, $\overline{S}_n^2 = (1-\beta)S_n^2$
$S_n^0 = \gamma p_n$
$S_{n+1}^1 = \sigma S_n^0$, $S_{n+1}^2 = \sigma S_n^1$.

After simplification, we get a 2nd order linear difference equation:
$p_{n+1} = \alpha\sigma\gamma p_n + \beta\sigma^2(1-\alpha)\gamma p_{n-1}$
or a system of 1st order linear difference equations:
$p_{n+1} = \alpha\sigma\gamma p_n + \beta\sigma(1-\alpha)S_n^1$ and $S_{n+1}^1 = \sigma\gamma p_n$
or a matrix equation:

$$\begin{pmatrix} p_{n+1} \\ S_{n+1}^1 \end{pmatrix} = \begin{pmatrix} \alpha\sigma\gamma & \beta\sigma(1-\alpha) \\ \sigma\gamma & 0 \end{pmatrix} \begin{pmatrix} p_n \\ S_n^1 \end{pmatrix}.$$

Now we need math ...

More matrix models

(Page 26-28)

Growth of segmental organisms
Red blood cell production
Ventilation volume and blood CO_2 levels

The red blood cells (RBCs) are constantly being destroyed and replaced. Since the cells carry oxygen throughout the body, their number must be maintained at some fixed level. Assume that the spleen filters out and destroys a certain fraction of the cells daily and that the bone marrow produces a number proportional to the number lost on the previous day. What is the cell count on the n-th day?

Variables:
R_n= number of RBCs in circulation on day n
M_n= number of RBCs produced by marrow on day n
Parameters:
f =fraction of RBCs removed by spleen
γ =production constant

$R_{n+1} = (1-f)R_n + M_n$ and $M_{n+1} = \gamma f R_n$
or matrix equation

$$\begin{pmatrix} R_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} 1-f & 1 \\ \gamma f & 0 \end{pmatrix} \begin{pmatrix} R_n \\ M_n \end{pmatrix}.$$