Dimensional Analysis

(partly from `http://www.physics.uoguelph.ca/tutorials/dimanaly/`)

Most physical quantities can be expressed in terms of combinations of five basic dimensions. These are mass (M), length (L), time (T), electrical current (I), and temperature, represented by the Greek letter theta (θ). These five dimensions have been chosen as being basic because they are easy to measure in experiments. Dimensions aren't the same as units. For example, the physical quantity, speed, may be measured in units of metres per second, miles per hour etc.; but regardless of the units used, speed is always a length divided a time, so we say that the dimensions of speed are length divided by time, or simply L/T. Similarly, the dimensions of area are L^2 since area can always be calculated as a length times a length.

Now that you can determine the dimensions of physical quantities, it'll be useful to write the SI units for the quantities. SI stands for International System (*Systéme Internationale*). The SI unit for mass is the kilogram, for length the metre, for time the second, for current the ampere, and for temperature the kelvin. Notice that kelvin is abbreviated as just K. The degree symbol, $\circ$, and the word "degree" are not used with kelvin.

Some combinations of SI units are given special names. For example, the unit of energy, $kg \cdot m^2 \cdot s^{-2}$, is given the special name joule, which is abbreviated as J. Study the information presented below.

1. energy: joule (J) $kg \cdot m^2 \cdot s^{-2}$
2. force: newton (N) $kg \cdot m \cdot s^{-2}$
3. frequency: hertz (Hz) $(cycles) \cdot s^{-1}$
4. power: watt (W) J/s $= kg \cdot m^2 \cdot s^{-3}$
5. charge: coulomb (C) $A \cdot s$

Some quantities have no dimensions. For example, the sine of an angle is defined as the ratio of the lengths of two particular sides of a triangle. Thus, the dimensions of the sine are L/L, or 1. Therefore, the sine function is said to be "dimensionless". There are many other examples of "dimensionless" quantities listed in the following table.

1. all trigonometric functions
2. exponential functions
3. logarithms
4. angles
5. quantities which are simply counted, such as the number of people in the room
6. plain old numbers (like 2, π, etc.)

Notice that some quantities which are "dimensionless" have units. For example, angles can be measured in units of radians or degrees, but angles are "dimensionless". Another familiar example is a frequency unit, (cycles) per second. The second, of course, is a time unit but the cycles are "dimensionless". That's the reason for cycles being written in parentheses above. Take a few moments and learn the "dimensionless" quantities above.

The form of a solution of a differential equation can depend critically on the units one chooses for the various quantities involved. Frequently these choices can lead to substantial problems when numerical approximation techniques such as Euler's method are applied. These difficulties can be controlled or avoided by proper scaling. We describe a technique that changes variables so that the new variables are "dimensionless". This technique will lead to a simple form of the equation with fewer parameters. It makes clear that the parameters can interact in the equation and a simpler combined parameter can suffice for more than one parameter. The theoretical foundation of the dimension analysis is the Buckingham π-theorem. The theorem loosely states that if we have a physically meaningful equation involving a certain number, n, of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of $p = n - k$ dimensionless variables constructed from the original variables. This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckinghams theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most "physically meaningful".

We illustrate this technique which is called Nondimensionalization, with some examples.

Example. Consider the following model of an outbreak of the spruce budworm:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{BP^2}{A^2 + P^2}, \quad P(0) = P_0. \tag{1}$$

We give a step by step approach to nondimensionalize this initial value problem.

Step 1. List all of the variables, parameters, and their dimensions. For the dimensions we use T for time, and ρ for population in number of worms.

Variable	Dimension	Parameter	Dimension
t	T	k	$1/T$
P	ρ	N	ρ
		A	ρ
		B	ρ/T
		P_0	ρ

Step 2. Take each variable and create a new variable by dividing by the combination of parameters that has the same dimension in order to create a dimensionless variable. Not that there is not always a unique way to do that, so some experiments may be necessary. Here we create

$$u = \frac{P}{A}, \quad s = \frac{Bt}{A}.$$

We can use our table of dimensions above to check that these new variables are now dimensionless.

Step 3. Now use the chain rule to derive a new differential equation.

$$\frac{dP}{dt} = \frac{dP}{du} \cdot \frac{du}{ds} \cdot \frac{ds}{dt} = A\frac{du}{ds} \cdot \frac{B}{A} = B\frac{du}{ds}$$

The term $kP\left(1-\frac{P}{N}\right)$ becomes $kAu\left(1-\frac{Au}{N}\right)$ and the term $\frac{BP^2}{A^2+P^2}$ simplifies to $\frac{Bu^2}{1+u^2}$. Thus the equation simplifies to

$$B\frac{du}{ds} = kAu\left(1-\frac{Au}{N}\right) - \frac{Bu^2}{1+u^2}$$

Dividing by B gives

$$\frac{du}{ds} = \frac{kA}{B}u\left(1-\frac{u}{N/A}\right) - \frac{u^2}{1+u^2}$$

Noting that the combinations of the parameters that occur above leads us to introduce two new dimensionless parameters

$$\alpha = \frac{kA}{B}, \quad \beta = \frac{N}{A}$$

The equation then becomes

$$\frac{du}{ds} = \alpha u\left(1-\frac{u}{\beta}\right) - \frac{u^2}{1+u^2}$$

At the meantime, the initial condition $P(0) = P_0$ becomes $u(0) = P_0/A$ through the change of variable $u = P/A$. Thus if we introduce another new parameter $\gamma = P_0/A$, then the initial condition becomes

$$u(0) = \gamma$$

Note that the equation has two dimensionless variables s, u and three dimensionless parameters α, β, γ which are combinations of the original parameters. This simplified form of the equation has reduced the number of parameters from 5 to 3, which makes the analysis of the equation simpler.

Exercise:

1. Determine the dimensions of the following quantities: volume, acceleration (velocity/time), density (mass/volume), force (mass × acceleration), charge (current × time).

2. Determine the dimensions of the following quantities: pressure (force/area), electric field (force/charge), work (in 1-D, force × distance), energy (e.g., gravitational potential energy = mgh = mass × gravitational acceleration × height).

3. Determine the corresponding SI units: volume, density, pressure, energy.

4. Consider the equation:

$$\frac{dP}{dt} = kP\left(1-\frac{P}{N}\right) - \frac{BP^2}{A^2+P^2}.$$

   Use the following change of variables:

$$(a)\;\; Q = \frac{P}{N}, s = kt, \qquad (b)\;\; Q = \frac{kP}{B}, s = kt,$$

   to get nondimensionalized equations.