Test 1 practice  
Math 302

No book, No notes, calculator allowed, 90 minutes time limit, total 100 points (plus possible 10 extra points). In problems 1-3, circle the correct answer; for problems 4-9, you must show necessary work to get full credit.

1. Apply Euler’s method with $\Delta t = 0.25$ to the following initial value problem:

$$\frac{dy}{dt} = y, \quad y(0) = 1.$$  

Then the approximation to $y(1)$ is equal to (or most close to)

(A) 1.22; (B) 2.44; (C) 3.66; (D) 4.88; (E) 6.

2. Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{y(1-y)}, \quad y(0) = \frac{1}{2}.$$  

(A) $\lim_{t \to \infty} y(t) = 1$; (B) There exists $t_0 > 0$ such that $\lim_{t \to t_0} y(t) = 1$;

(C) $\lim_{t \to \infty} y(t) = 0$; (D) There exists $t_0 > 0$ such that $\lim_{t \to t_0} y(t) = 0$.

3. Suppose $y(t)$ is the solution of $y' = y(y - 0.5)(1 - y), y(0) = 0.7$. Then $\lim_{t \to \infty} y(t)$ is

(A) 0; (B) 1; (C) 0.5; (D) does not exist.

4. Given the spring-mass system with mass $m = 2$, damping coefficient $b = 6$, and the spring constant $k = 4$. The corresponding $2 \times 2$ system in matrix form is:

(A) $\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} Y$; (B) $\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} Y$; (C) $\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} Y$; (D) $\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} Y$.

5. Which of the following system has infinite many equilibrium points?

(A) $\frac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} Y$, (B) $\frac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} Y$, (C) $\frac{dY}{dt} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} Y$, (D) $\frac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} Y$.

6. A linear system has one eigenvalue 2 with eigenvector $(1, 1)$ and a second eigenvalue 4 with eigenvector $(0, 1)$. Sketch the phase portrait.

7. A large tank contains 110 gallons of brine in which 10 lbs of salt is dissolved initially. Brine of concentration 1 lb/gal of salt flows into the tank at the constant rate of 2 gal/min. In the meantime, well-mixed brine flows out of the tank at the same rate (2 gal/min).

(a) Setup an initial value problem for the amount of salt in the tank of any time $t$;

(b) Determine the amount of salt in the tank in the long run.
8. Solve the initial value problem:

\[ xy' + y = \sin x, \quad y(1) = 2. \]

9. Find all the equilibrium points of the system

\[
\begin{align*}
\frac{dx}{dt} &= 0.4x \left( 1 - \frac{x}{100} \right) - 0.01xy, \\
\frac{dy}{dt} &= -0.3y + 0.005xy.
\end{align*}
\]

10. Find the solution of \( Y' = \begin{pmatrix} 12 & -15 \\ 4 & -4 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \).

11. Consider the system

\[
\begin{align*}
x' &= 3x + y \\
y' &= x + 3y.
\end{align*}
\]

(a) Find all straight line solutions of the system.

(b) Sketch the phase plane of this system, including all straight line solutions and typical solution curves lying in regions between these straight line solutions.

(c) Find the solution with initial values \( x(0) = 1, \quad y(0) = 0 \). Put its orbit on the phase plane (and label it). As \( t \to \infty \), which one of the straight line solutions is it almost parallel to?

12. Consider the population model

\[
\frac{dP}{dt} = P(4 - P)(P - 1) - hP,
\]

for a species of fish in lake. Here the term \( hP \) represents the quantity of fish removed from the lake by fishing per season, and \( h \) is a positive parameter (we assume that the fishing is proportional to the population of the fish). The unit of fish is in hundreds.

(a) When \( h = 1 \), the equation has three equilibrium points. Find these three equilibrium points and identify their types(sink, source or node).

(b) There is a value \( h_0 \) such that the equation has exactly three equilibrium points when \( 0 < h < h_0 \), exactly two equilibrium points when \( h = h_0 \) and exactly one equilibrium point when \( h > h_0 \). Find the value of \( h_0 \).