Mass-spring system

\[ my'' + by' + ky = 0 \]

**Example 2**: A mass weighing 4 pounds stretches a spring 4 inches, and it is extended 1 foot from its rest position and released. Find the function of this motion.

**First order system and second order equation**:

\[ y'' + 4y = 0 \] is a second order equation. A second order equation can always be converted to a system of first order equations:

\[ y' = v, \ v' = -4y. \]

So any method which works for 1st order system will also works for 2nd order equation.

**Linear systems**: (1st order linear system with constant coefficients)

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy 
\end{align*}
\]

Examples:
Predator-prey: \( R' = aR - cFR, \quad F' = -bF + dFR \) (nonlinear)
Mixing problem: \( x' = -x/40, \quad y' = x/40 - y/80 \) (linear)
Mass-spring system: \( y' = v, \quad v' = -3y \) (linear)
Example 3: Assume that in the absence of predators, the prey population obeys a logistic rather than an exponential growth model.

\[
\begin{align*}
R' &= 2R \left( 1 - \frac{R}{2} \right) - 1.2FR \\
F' &= -F + 0.9FR \\
R(0) &= R_0 > 0, \quad F(0) = F_0 > 0.
\end{align*}
\]

1. Find the equilibrium points.
2. If there is no predator, what happens to the prey?
3. If there is no prey, what happens to the predator?
4. Suppose that initially both $R$ and $F$ are not zero. Use phase portrait and solution curve to study the behavior of solutions.
5. How would you modify these systems to include the effect of hunting of the predators at a rate proportional to the number of predators?
Epidemic models

Compartamental model (Kermack-Mckendrick, 1927)

1. **Susceptible** population $S(t)$: who are not yet infected
2. **Infective** population $I(t)$: who are infected at time $t$ and are able to spread the disease by contact with susceptible
3. **Removed** population $R(t)$: who have been infected and then removed from the possibility of being infected again or spreading (Methods of removal: isolation or immunization or recovery or death)
4. A average infective makes contact sufficient to transmit infection with $\beta S$ others per unit time
5. A fraction $\alpha$ of infectives leave the infective class per unit time. If a patient is cured in $n$ units of time, then $\alpha = 1/n$.

\[
\begin{align*}
S' &= -\beta SI, \\
I' &= \beta SI - \alpha I, \\
R' &= \alpha I.
\end{align*}
\]
One more example

Reference: Strogatz (1988) *Love affairs and DEs*
*Mathematics Magazine* 61:35.

Romeo is in love with Juliet, but in our version of story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo get discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let $R(t)$ = Romeo’s love/hate for Juliet at time $t$, and let $J(t)$ = Juliet’s love/hate for Romeo at time $t$.

$$R' = aJ, \quad J' = -bR \quad (a, b > 0)$$

Outcome: a neverending cycle of love-hate. (at least they manage to achieve simultaneous love one quarter of the time)