Phenomenon that a qualitative change occurs as the parameter changes

What is not a bifurcation?
\[ P' = kP, \quad P(0) = 1, \text{ change } k \text{ from } k = 3 \text{ to } k = 4 \]

What is a qualitative change?
Example: \[ y' = y^2 - k, \quad y(0) = 0 \]
when \( k = 1 \), \( \lim_{t \to \infty} y(t) = -1 \)
when \( k = -1 \), \( y(t) \to \infty \) (blow up)
Thus the asymptotic fate of the solutions are totally different when a parameter is changed
When does a bifurcation occur?

(1) The number of equilibrium points changes;
(2) The stability of equilibrium points change.

Example 20 \[
\frac{dy}{dt} = y^2 - 2y + k
\]
(saddle-node bifurcation, $2 \rightarrow 1 \rightarrow 0$)
(blue-sky bifurcation—a pair of equilibrium points appears out of the clear blue sky)

Example 21 \[
\frac{dy}{dt} = y(y^2 - k)
\]
(pitchfork bifurcation, $3 \rightarrow 1 \rightarrow 1$)

Example 22 \[
\frac{dy}{dt} = y(y - k)
\]
(transcritical bifurcation, $2 \rightarrow 1 \rightarrow 2$)
A real bifurcation problem

**Example 23** Suppose that the population of Atlantic bluefin tuna satisfies a logistic growth without any fishing:
\[
\frac{dP}{dt} = 0.1P \left(1 - \frac{P}{20}\right)
\]
where \(P\) is measured at thousand tons. This kind of fish is valued above all other fish species for sushi and sashimi, thus the fishing activities are regulated by International Commission for the Conservation of Atlantic Tunas (ICCAT). If the annual allowable catch is \(h\) thousand tons, then the equation is
\[
\frac{dP}{dt} = 0.1P \left(1 - \frac{P}{20}\right) - h.
\]
What is the largest \(h\) which will not drive the fish to extinction?

**what does the bifurcation diagram tell us?**

1. There is a maximal allowable catch \(h_0 > 0\).

2. When \(h > h_0\) (fishing too much), there is no equilibrium solutions, the fish population will reach zero in a finite time. (the population become extinct in several years)

3. When \(h < h_0\) (fishing under control), there are two equilibrium solutions \(P_1 > P_2\); \(P_1\) is a sink, and \(P_2\) is source; when the initial population is greater than \(P_2\), then the population will approach to \(P_1\) as \(t \to \infty\); when the initial population is less than \(P_2\), then the population will still become extinct.
How to find bifurcation points and the type of bifurcation?

\[
\frac{dy}{dt} = f_k(y)
\]

1. Find the solutions \((k_0, y_0)\) of \(f_k(y) = 0\) and \(f'_k(y) = 0\).

2. Find the number of equilibrium points when \(k\) is slightly greater than, slightly less than or equal to \(k_0\). (graph \(f_k(y)\) for these \(k\))

**Example 24:** \[
\frac{dy}{dt} = y(1 - y)^2 + k
\]