**Phase line**

**Autonomous Equation**: \( \frac{dP}{dt} = f(P) \)

The slope field of an autonomous equation does not depend on \( t \), so the slope marks at \((t_1, P)\) and \((t_2, P)\) are same. All solutions are parallel in \( P \) direction in the sense that if \( P(t) \) is a solution, then \( P(t + c) \) is also a solution. So we only need to draw slope marks for a fixed \( t \).

**Phase line**: Phase line is a simplified direction field for autonomous equation only. On a vertical line, we indicate the positive derivative by an arrow pointing up, and negative derivative by an arrow pointing down. For equilibrium point, we use a dot to denote it.

**Example 17**.
(1) Draw the phase line of \( P' = P(1 - P)(2 - P) \);
(2) Determine the long time behavior of the solutions with \( P(0) = 0.3 \) and \( P(0) = 1.4 \).
How to draw the phase line

(1) Draw the $P$-line. (usually vertical, horizontal to save paper)
(2) Find and mark the equilibrium points (solid dot) and undefined points (hollow dot).
(3) Find the intervals of $P$-values for which $f(P) > 0$, and draw arrows pointing up in these intervals.
(4) Find the intervals of $P$-values for which $f(P) < 0$, and draw arrows pointing down in these intervals.

Example 18. (1) Draw the phase line of $P' = \frac{P(1 - P)}{P^2 - 4}$;
(2) Sketch the solutions with $P(0) = 0.4$, $P(0) = 1.2$ and $P(0) = 3$. 
Asymptotic behavior of solutions

Autonomous equation $P' = f(P)$, and $f(P)$ is a differentiable function in its domain.

(1) Approaches to an equilibrium point. \( \lim_{t \to \infty} y(t) = y_0 \)

Example: \( P' = P(1 - P), \ P(0) = 0.5 \)

(2) Goes to infinity in infinite time. \( \lim_{t \to \infty} y(t) = \pm \infty \)

Example: \( P' = P, \ P(0) = 1 \)

(3) Goes to infinity in finite time. (Blow up) \( \lim_{t \to t_0} y(t) = \pm \infty \)

Example: \( P' = P^2, \ P(0) = 1 \)

(4) Stop at an undefined point. (derivative goes to infinity)

\( \lim_{t \to t_0} y(t) = y_0, \ \lim_{t \to t_0} y'(t) = \pm \infty \)

Example: \( P' = -\frac{1}{P}, \ P(0) = 1 \)
Types of equilibrium points:

Sink (stable): each solution nearby initially approaches to this equilibrium point as $t \to \infty$.

Source (unstable): each solution nearby initially gets far away from this equilibrium solution.

Node (unstable): On one side, it is a sink, but on the other side, it is a source.

**Linearization Theorem:** Let $y_0$ be an equilibrium of $y' = f(y)$, i.e. $f(y_0) = 0$.
1. If $f'(y_0) > 0$, then $y_0$ is a source.
2. If $f'(y_0) < 0$, then $y_0$ is a sink.
3. $f'(y_0) = 0$, then we need further information to determine the type of $y_0$.

**Example 19.**
1. Find the equilibrium points of $w' = w^2(w^2 - 8w + 12)$.
2. Determine the types of the equilibrium via linearization.