Laplace transform

It is an integral transformation which can be used to solve differential equations. Integral transformation is a method used very often in applied mathematics. The main idea is to convert an equation to another one which is easier to solve.

Pierre-Simon Laplace (France, 1749-1827)

What is Laplace transform?

Suppose that $y(t)$ is a function defined for $t \geq 0$. Then the Laplace transform of $y(t)$ is $\mathcal{L}[y(t)] = Y(s) = \int_{0}^{\infty} y(t)e^{-st} dt$.

Thus a function $y(t)$ (with variable $t$) is transformed to a new function $Y(s)$ (with variable $s$).
Solving Laplace transform

Let’s solve a few Laplace transform:

\[ y(t) = e^{4t}, \text{ Laplace transform } Y(s) = \frac{1}{s-4}, s > 4. \]

\[ y(t) = e^{kt}, \text{ Laplace transform } Y(s) = \frac{1}{s-k}, s > k. \]

\[ y(t) = 1, \text{ Laplace transform } Y(s) = \frac{1}{s}, s > 0. \]

**Laplace transform of derivatives:** (IF \(|y(t)| < e^{Mt}\))

\[ \mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0) \]

\[ \mathcal{L}[y''(t)] = s^2\mathcal{L}[y(t)] - sy(0) - y'(0) \]

**Linear property:** (k is a constant)

\[ \mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g], \quad \mathcal{L}[kf] = k\mathcal{L}[f] \]
Solve differential equation using Laplace transform

Step 1: Take the Laplace transform of the equation. \((y(t) \rightarrow Y(s))\)
Step 2: Solve \(Y(s)\) in the transformed equation.
Step 3: Find the inverse Laplace transform of \(Y(s)\).
\((Y(s) \rightarrow y(t))\)

Example 1: (let’s solve something we have solved before)

1. \(\frac{dy}{dt} = 4y, \ y(0) = 8.\)
2. \(\frac{dy}{dt} = 4y - e^{-5t}, \ y(0) = 7.\)

Partial fraction revisited:

Find the inverse Laplace transform of \(\frac{2s + 5}{(s - 4)(s - 5)}\).

One more Laplace transform: \(\mathcal{L}[t^n] = ?\)
Summary of formulas

1. $\mathcal{L}[y(t)] = Y(s) = \int_0^{\infty} y(t)e^{-st} dt$

2. $\mathcal{L}[e^{kt}] = \frac{1}{s - k}$

3. $\mathcal{L}[1] = \frac{1}{s}$

4. $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

5. $\mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0)$