Second order equation with forcing

Linear second order equation without forcing (homogeneous)
y'' + py' + qy = 0

Linear second order equation with forcing (nonhomogeneous)
y'' + py' + qy = f(t)

Physical meaning: In the spring-mass system, \( f(t) \) is an external force acting on the mass

**Example 1:** \( y'' + 3y' + 2y = e^{3t} \)

First observation: \( y(t) = ke^{3t} \) may be a solution of the equation for certain \( k \).

Second observation: If \( y(t) = ke^{3t} \) is a solution, so is \( y(t) = ke^{3t} + e^{-2t} \) since
\[
(e^{-2t})'' + 3(e^{-2t})' + 2(e^{-2t}) = 0.
\]

Guess a general solution:
y(t) = ke^{3t} + c_1 e^{-2t} + c_2 e^{-t}
(c_1 e^{-2t} + c_2 e^{-t} in fact solves \( y'' + 3y' + 2y = 0 \))
Linear Principle for $y'' + py' + qy = 0$

1. If $y_1(t)$ and $y_2(t)$ are two solutions, so is $y_1(t) + y_2(t)$.
2. If $y_1(t)$ is a solution, and $k$ is a constant, then $ky_1(t)$ is also a solution.

General solution: $y(t) = c_1y_1(t) + c_1y_2(t)$

Extended Linear Principle for $z'' + pz' + qz = f(t)$

1. If $z_1(t)$ is a solution of $z'' + pz' + qz = f(t)$, and $y_1(t)$ is a solution of $y'' + py' + qy = 0$, then $z_1(t) + y_1(t)$ is a solution of $z'' + pz' + qz = f(t)$.
2. If $z_1(t)$ and $z_2(t)$ are both solutions of $z'' + pz' + qz = f(t)$, then $z_1(t) - z_2(t)$ is a solution of $y'' + py' + qy = 0$.

General solution: $y(t) = c_1y_1(t) + c_1y_2(t) + y_p(t)$, where $y_1$ and $y_2$ are (linear independent) solutions of $y'' + py' + qy = 0$, and $y_p$ is any particular solution of $z'' + pz' + qz = f(t)$.

Note: this is the same as $y' + ay = 0$ and $y' + ay = f(t)$ discussed in Chapter 1
Undetermined Coefficients

Recipe 5 (for $y'' + py' + qy = f(t)$)

1. Find one particular solution for the nonhomogeneous equation $y_p(t)$.
2. Find the general solution for the homogeneous equation $c_1y_1(t) + c_1y_2(t)$ of $y'' + py' + qy = 0$.
3. General solution: $y(t) = c_1y_1(t) + c_1y_2(t) + y_p(t)$

How to find $y_p(t)$?
A good guess—method of undetermined coefficients

Example 2. $y'' + 8y' + 7y = e^{-2t}$, $y(0) = 2$, $y'(0) = 3$
Example 3. $y'' + 8y' + 7y = e^{-7t}$

Rule of thumb for the guessing game:

Rule 1: If the force is $e^{kt}$, we guess $y_p(t) = ke^{kt}$.
Rule 2: If Rule 1 doesn’t work, try $y_p(t) = kte^{kt}$.
Rule 3: If Rule 2 still doesn’t work, try $y_p(t) = kt^2e^{kt}$ (see one of HW problem.)

Example 4. Solve $y'' + 8y' + 7y = 4$, and explain the solution in term of mass-spring system.
Example 5. Solve $y'' + 4y = e^{-t} + 2t + 1$. 