Solve the linear system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

Let \( Y = \begin{pmatrix} x \\ y \end{pmatrix} \), \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Linear system is \( \frac{dY}{dt} = A \cdot Y \).

The general solution is

\[
Y(t) = c_1 Y_1(t) + c_2 Y_2(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.
\]

If eigenvalues of \( \lambda_1 \) and \( \lambda_2 \) of \( A \) are two different real numbers. 

**Question**: How do these solutions look like?
Qualitative behavior of solutions (A)

A. Two different positive eigenvalues

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dt}{dy} \\
\frac{dt}{dt}
\end{pmatrix} =
\begin{pmatrix}
2x + 2y \\
x + 3y
\end{pmatrix}
\]

Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$; Eigenvectors: $V_1 = (2, -1)$, $V_2 = (1, 1)$

General solution: $Y = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

1. $(0, 0)$ is the only equilibrium point, and any non-zero solution satisfies $\lim_{t \to \infty} Y(t) = \infty$ and $\lim_{t \to -\infty} Y(t) = (0, 0)$.

2. There are two linear independent straight line solutions:

\[
Y_1 = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t, \quad Y_2 = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}
\]

which are on the lines $y = -0.5x$ and $y = x$ respectively.

3. The non-straight-line solution satisfies

(i) when $t \to -\infty$, the solution is tangent to the straight line solution on $y = -0.5x$,

(ii) when $t \to \infty$, the solution is almost parallel to the straight line solution on $y = x$.

4. This type of equilibrium point is a (2-dimensional) source.
Qualitative behavior of solutions (B)

B. Two different negative eigenvalues

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt} \\
\frac{dt}{dt}
\end{pmatrix} =
\begin{pmatrix}
-2x - 2y \\
-x - 3y
\end{pmatrix}
\]

Eigenvalues: \( \lambda_1 = -1 \) and \( \lambda_2 = -4 \); Eigenvectors: \( V_1 = (2, -1) \), \( V_2 = (1, 1) \)

General solution: \( Y = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} \)

1. \((0, 0)\) is the only equilibrium point, and any non-zero solution satisfies
\( \lim_{t \to -\infty} Y(t) = \infty \) and \( \lim_{t \to \infty} Y(t) = (0, 0) \).

2. There are two linear independent straight line solutions:

\[
Y_1 = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}, \quad Y_2 = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}
\]

which are on the lines \( y = -0.5x \) and \( y = x \) respectively.

3. The non-straight-line solution satisfies
   (i) when \( t \to \infty \), the solution is tangent to the straight line solution on \( y = -0.5x \),
   (ii) when \( t \to -\infty \), the solution is almost parallel to the straight line solution on \( y = x \).

4. This type of equilibrium point is a (2-dimensional) sink.
Qualitative behavior of solutions (C)

C. One negative eigenvalue and one positive eigenvalue

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = \begin{pmatrix}
x + 3y \\
-x - y
\end{pmatrix}
\]

Eigenvalues: \( \lambda_1 = 2 \) and \( \lambda_2 = -2 \); Eigenvectors: \( V_1 = (3, 1) \), \( V_2 = (1, -1) \)

General solution: \( Y = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \)

1. \((0, 0)\) is the only equilibrium points.
2. There are two linear independent straight line solutions:

\[
Y_1 = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t}, \quad Y_2 = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}
\]

which are on the lines \( y = -0.5x \) and \( y = x \) respectively.

3. \( Y_1 \) is the only solution which satisfies \( \lim_{t \to \infty} Y(t) = \infty \) and \( \lim_{t \to -\infty} Y(t) = (0, 0) \).
   (unstable solution)
4. \( Y_2 \) is the only solution which satisfies \( \lim_{t \to -\infty} Y(t) = \infty \) and \( \lim_{t \to \infty} Y(t) = (0, 0) \).
   (stable solution)
5. The non-straight-line solution satisfies
   (i) \( \lim_{t \to \pm \infty} Y(t) = \infty \)
   (ii) when \( t \to \infty \), the solution tends to the unstable solution,
   (ii) when \( t \to -\infty \), the solution tends to the stable solution.
6. This type of equilibrium point is a (2-dimensional) saddle.
   (there is no saddle in 1-dimension)

**Drawing by Hand:**
Example: (Sec. 3.2 (problem 2,6), Sec. 3.3 (problem 2,6))
\[
\frac{dY}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} Y, \quad \frac{dY}{dt} = \begin{pmatrix} 5 & 4 \\ 9 & 0 \end{pmatrix} Y
\]
1. Compute the eigenvalues and eigenvectors, and obtain the general solution;
2. Plot the straight line solutions;
3. Plot the \( x(t) \)- and \( y(t) \)-graphs of the straight line solutions;
4. Sketch other solutions on the phase portrait.