**Matrix**: numbers in a table (spreadsheet)
(A vector is $1 \times n$ matrix or $n \times 1$ matrix)

A $2 \times 2$ matrix: \[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\]  
A $2 \times 3$ matrix: \[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\]

$x' = ax + by$
$y' = cx + dy$

These can be written as
\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \cdot \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

**Matrices multiplication**

Multiplying a $k \times n$ matrix and $n \times m$ matrix, you get a $k \times m$ matrix. Number of columns of the first matrix is the same as the number of rows of the second matrix.
(You can multiply a $3 \times 4$ matrix and $4 \times 2$ matrix, but not a $4 \times 2$ matrix and a $3 \times 4$ matrix.)
Short form of linear system

Let $Y = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Linear system is $\frac{dY}{dt} = A \cdot Y$.

**Equilibrium points**: Solve $0 = ax + by$
$0 = cx + dy$.

$(0, 0)$ is always a solution, any more?

**Example 1**: (A) $\frac{dY}{dt} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} Y$, $2x + 3y = 0$, $4x + 5y = 0$, $x = 0$.

(B) $\frac{dY}{dt} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} Y$, $2x + 3y = 0$, $4x + 6y = 0$, $x = 0$.

Solution: any $(x, y)$ satisfying $2x + 3y = 0$. For example, $(3, -2)$, $(-6, 4)$.

**Theorem**: $(0, 0)$ is the only equilibrium point if $ad - bc \neq 0$, ($\frac{a}{c} \neq \frac{b}{d}$), and if $ad - bc = 0$, ($\frac{a}{c} = \frac{b}{d}$), then any point on the line $ax + by = 0$ is an equilibrium point.

**Determinant**: for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is the number $ad - bc$. (The difference between the product of diagonal line entries.)
Linear principle

Consider \( \frac{dY}{dt} = A \cdot Y \) (nonhomogeneous system)

1. If \( Y_1 \) is a solution and \( c \) is a constant, then \( c \cdot Y_1 \) is also a solution.
2. If \( Y_1 \) and \( Y_2 \) are both solutions, then \( Y_1 + Y_2 \) is also a solution.
3. (Consequently) If \( Y_1 \) and \( Y_2 \) are both solutions, and \( c_1, c_2 \) are two constants, then \( c_1 Y_1 + c_2 Y_2 \) is also a solution.

**General solution:** \( Y = c_1 Y_1 + c_2 Y_2 \), where \( Y_1 \) and \( Y_2 \) are two linear independent solutions (that means \( Y_1 \not= kY_2 \)).

**Strategy of solving equation:** Find two solutions linear independent solutions.

**Example 2:** \( \frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} Y \).

1. Verify that \( Y_1(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} \) and \( Y_2(t) = \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} \) are solutions of the system. Thus the general solution of the system is \( Y(t) = c_1 Y_1(t) + c_2 Y_2(t) \).

2. Find the solution of the system with \( Y(0) = (2, 3) \).
Solve the linear system

\[
\begin{align*}
  x' &= ax + by, \\
  y' &= cx + dy.
\end{align*}
\]

Let \( Y = \begin{pmatrix} x \\ y \end{pmatrix} \), \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Linear system is \( \frac{dY}{dt} = A \cdot Y \).

**Motivation:** How is \( y' = ky \) is solved? \( y(t) = y(0)e^{kt} \)

We try \( x(t) = k_1e^{rt}, \ y(t) = k_2e^{rt} \).

If \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{rt} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \) is a solution of \( Y' = A \cdot Y \), then

\[
\det \left( \begin{array}{cc} a - r & b \\ c & d - r \end{array} \right) = 0, \ \text{or} \ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = r \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.
\]

The determinant equation: \( r^2 - (a + d)r + ad - bc = 0 \), or use traditional notation:

\[
\lambda^2 - (a + d)\lambda + ad - bc = 0, \ \text{characteristic equation}
\]

The roots of the characteristic equation are the **eigenvalues** of matrix \( A \), and also the exponents of exponential solution. There are two eigenvalues (it is a quadratic equation.) If \( \lambda_1 \) is one of the eigenvalues, then the corresponding \( V = (k_1, k_2) \) is an **eigenvector**.
General idea of solving the linear system

1. Write the characteristic equation: \( \lambda^2 - (a + d)\lambda + ad - bc = 0 \) and solve the eigenvalues \( \lambda_1, \lambda_2 \).

2. Find the eigenvectors \( V_1 \) associated with \( \lambda_1 \), and \( V_2 \) associated with \( \lambda_2 \). Then, \( Y_1(t) = e^{\lambda_1 t}V_1 \) and \( Y_2(t) = e^{\lambda_2 t}V_2 \) are two solutions of the equation.

3. The general solution is \( Y(t) = c_1 Y_1(t) + c_2 Y_2(t) = c_1 e^{\lambda_1 t}V_1 + c_2 e^{\lambda_2 t}V_2 \).

But:

(a) what if \( \lambda_1 = \lambda_2 \) ?
(b) what if eigenvalues are complex numbers?
(Sections 3.4 and 3.5)

Example 3: Solve the following equations:

\[
\frac{dY}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} Y, \ Y(0) = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.
\]

\[
\frac{dY}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} Y, \ Y(0) = \begin{pmatrix} 100 \\ \pi \end{pmatrix}.
\]