Chain rule of multivariable functions

$z = f(x,y)$, $x = x(t)$, $y = y(t)$ $\Rightarrow$ $z(t) = f(x(t), y(t))$

what is $\frac{\partial z}{\partial t}$?

Definition $\frac{z(t+h) - z(t)}{h} = \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}$

$$= \frac{f(x(t+h), y(t+h)) - f(x(t+h), y(t)) + f(x(t+h), y(t)) - f(x(t), y(t))}{h}$$

$$= \frac{f(x(t+h), y(t+h)) - f(x(t+h), y(t))}{y(t+h) - y(t)} \cdot \frac{y(t+h) - y(t)}{h}$$

$$+ \frac{f(x(t+h), y(t)) - f(x(t), y(t))}{x(t+h) - x(t)} \cdot \frac{x(t+h) - x(t)}{h}$$

Taking limit.

$$\lim_{h \to 0} \frac{z(t+h) - z(t)}{h} = f_y(x(t), y(t)) \cdot y'(t) + f_x(x(t), y(t)) \, x'(t)$$

So If $z = f(x(t), y(t))$, then $z_t = f_x \cdot x_t + f_y \cdot y_t$.

If $z = f(x(t,s), y(t,s))$ $\begin{cases} z_t = f_x x_t + f_y y_t \\ z_s = f_x x_s + f_y y_s. \end{cases}$

Example

(1) $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{2t}$ Find z_t

$$z_t = z_x x_t + z_y y_t = \frac{2x}{2\sqrt{x^2+y^2}} \cdot 2e^{2t} + \frac{2y}{2\sqrt{x^2+y^2}} \cdot 2e^{2t} = \frac{2x+2y}{\sqrt{x^2+y^2}} e^{2t}$$

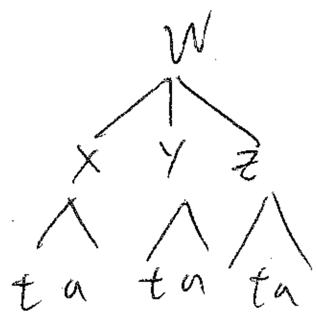
(2) $z = e^{xy} \tan y$; $x = s+2t$, $y = s/t$.

$$z_t = z_x x_t + z_y y_t = y e^{xy} \tan y \cdot 2 + \left(x e^{xy} \tan y + e^{xy} \sec^2 y\right) \cdot -\frac{s}{t^2}$$

$$z_s = z_x x_s + z_y y_t = y e^{xy} \tan y \cdot 1 + \left(x e^{xy} \tan y + e^{xy} \sec^2 y\right) \cdot \frac{1}{t}$$

(3) If $w = f(x,y,z)$, $x = x(t,u)$, $y = y(t,u)$, $z = z(t,u)$

Find w_t and w_u



$$w_t = w_x x_t + w_y y_t + w_z z_t$$

$$w_u = w_x x_u + w_y y_u + w_z z_u$$

(4) If $xyz = \cos(x+y+z)$. Find z_x and z_y

implicit differentiation !

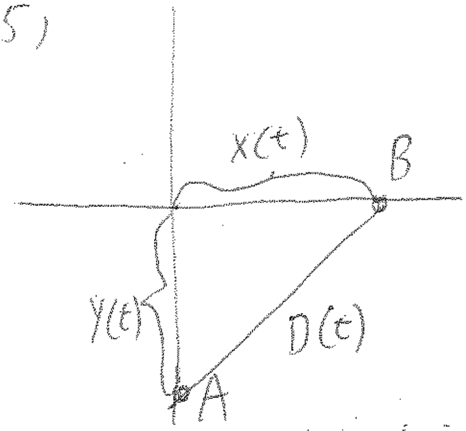
Think $F(x,y) = xyz(x,y) - \cos(x+y+z(x,y)) = 0$

$F_x = 0$ and $F_y = 0$ since $F(x,y) = 0$,

$$F_x = yz + xyz_x + \sin(x+y+z)(1+z_x) = 0 \Rightarrow z_x = -\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$

$$F_y = xz + xyz_y + \sin(x+y+z)(1+z_y) = 0 \Rightarrow z_y = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$

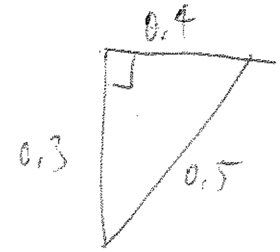
(5)



$x(t) = 0.4 - 80t$

$y(t) = -0.3 + 90t$

$D(x, y) = \sqrt{x^2 + y^2}$



What is $\frac{dD}{dt}$?

$$D_t = D_x X_t + D_y Y_t = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2X \cdot (-80) + \frac{1}{2\sqrt{x^2+y^2}} \cdot 2Y\,(90)$$

$$= \frac{-80X + 90Y}{\sqrt{x^2+y^2}} = \frac{-80 \cdot 0.4 + 90 \cdot (-0.3)}{0.5} = -64 - 54 = -118 \text{ Km/h}$$

So distance is decreasing in a rate 118 Km/h.