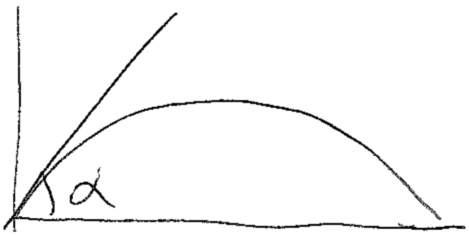
projectile



$$\vec{a}(t) = -g\,\vec{j}$$
$$\vec{V_0} = V_0\cos\alpha\,\vec{i} + V_0\sin\alpha\,\vec{j}$$

$$\vec{v}(t) = \vec{v_0} + \int_0^t \vec{a}\,ds = \vec{v_0} + \vec{a}\,t = \vec{v_0} - gt\vec{j} = V_0\cos\alpha\,\vec{i} + (V_0\sin\alpha - gt)\vec{j}$$

$$\vec{r}(t) = \vec{r_0} + \int_0^t \vec{v}(s)\,ds = \vec{r_0} + \int_0^t (\vec{v_0} - gs\vec{j})\,ds = \vec{r_0} + \vec{v_0}t - \frac{1}{2}gt^2\vec{j}$$

So $\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + t\vec{v_0} = V_0\cos\alpha\, t\,\vec{i} + \left(V_0\sin\alpha\, t - \frac{1}{2}gt^2\right)\vec{j}$

Maximum height when vertical $\vec{v}(t) = 0 \Rightarrow V_0\sin\alpha - gt = 0$

$$\Rightarrow T_1 = \frac{V_0}{g}\sin\alpha$$

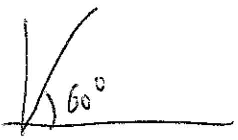
Impact time (time go back to ground) vertical $\vec{r}(t) = 0$

$$\Rightarrow V_0\sin\alpha\, t - \frac{1}{2}gt^2 = 0 \Rightarrow T_2 = \frac{2V_0}{g}\sin\alpha = 2T_1$$

Distance traveled (horizontally) $V_0\cos\alpha\cdot T_2 = \frac{2V_0^2}{g}\sin\alpha\cos\alpha = \frac{V_0^2}{g}\sin(2\alpha)$

When $\alpha = 45°$, $\sin(2\alpha) = 1$ Distance $= \frac{V_0^2}{g}$ (maximized)

Example



$V_0 = 200\ m/s$ (use $g = 10$)

(a) range $\frac{V_0^2}{g}\sin(2\alpha) = \frac{200^2}{10}\sin(120°) = 4000\cdot\frac{\sqrt{3}}{2} = 2000\sqrt{3}$ m

(b) max height $V_0\sin\alpha\cdot T_1 - \frac{1}{2}gT_1^2 = \cancel{V_0\cos\alpha\,\frac{V_0}{g}\sin\alpha = \frac{V_0^2}{g}\sin\alpha\cos\alpha = 1000\sqrt{3}\ m}$

$\frac{V_0^2}{g}\sin^2\alpha - \frac{1}{2}\frac{V_0^2}{g}\sin^2\alpha = \frac{V_0^2}{2g}\sin^2\alpha$
$= 2000\cdot\frac{3}{4} = 1500$ m

(c) speed at impact also $V_0 = 200\ m/s$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ position of planet

$\vec{v}(t)$ = velocity $= \vec{r}'(t)$ $\quad \vec{a}(t)$ = acceleration $= \vec{r}''(t)$ $\quad |\vec{F}| = \dfrac{GMm}{|\vec{r}|^2}$

Newton's 2nd law $\vec{F} = m\vec{a}$ $\quad$ Law of gravitation $\vec{F} = -\dfrac{GMm}{r^3}\vec{r}$

$$\Rightarrow \quad \vec{r}'' = \vec{a} = -\frac{GM}{|\vec{r}|^3}\vec{r} \qquad \text{So } \vec{a} \parallel \vec{r} \Rightarrow \vec{r} \times \vec{a} = \vec{0}$$

Claim 1 $\quad$ The motion is always on a plane.

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r}' \times \vec{v} + \vec{r} \times \vec{v}' = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{0} + \vec{0} = \vec{0}$$

So $\vec{r} \times \vec{v} = \vec{h}$ (a constant vector)

So $\vec{r}(t)$ and $\vec{v}(t)$ are all in a plane perpendicular to $\vec{h}$

Without loss of generality, we assume $\vec{h} = |\vec{h}|\vec{k}$ ($\vec{h}$ is in z direction)

So $\vec{r}(t)$ is in xy-plane So $\vec{r}(t) = \langle x(t), y(t) \rangle$

Differential Equation :

$$\begin{cases} x''(t) = -\dfrac{GM}{(x^2+y^2)^{3/2}}\, x(t) \\ y''(t) = -\dfrac{GM}{(x^2+y^2)^{3/2}}\, y(t) \end{cases}$$

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vector approach $\quad \vec{a} \times \vec{h} = \vec{a} \times (\vec{r} \times \vec{v}) = -\dfrac{GM}{|\vec{r}|^3}\vec{r} \times (\vec{r} \times \vec{r}')$

$$\vec{r} = |\vec{r}| \cdot \frac{\vec{r}}{|\vec{r}|} = R \cdot \vec{u}$$

$$\vec{r}\,' = (R\cdot\vec{u})' = R'\vec{u} + R\vec{u}\,'$$

$$\vec{a}\times\vec{h} = -\frac{GM}{R^3}(R\cdot\vec{u})\times\left(R\cdot\vec{u}\times(R'\vec{u}+R\vec{u}\,')\right)$$

$$= -\frac{GM}{R}\,\vec{u}\times\left(\vec{u}\times(R'\vec{u}+R\vec{u}\,')\right)$$

$$= -\frac{GM}{R}\,\vec{u}\times\left(\vec{u}\times R\vec{u}\,'\right) = -GM\left(\vec{u}\times(\vec{u}\times\vec{u}\,')\right)$$

$$= -GM\left[(\vec{u}\cdot\vec{u}\,')\vec{u} - (\vec{u}\cdot\vec{u})\vec{u}\,'\right] = GM\vec{u}\,'$$

Since $\vec{u}\cdot\vec{u}\,' = \frac{1}{2}(\vec{u}\cdot\vec{u})' = \frac{1}{2}1' = 0$, $\vec{u}\cdot\vec{u} = 1$

So $\vec{a}\times\vec{h} = GM\vec{u}\,' \Rightarrow \vec{v}\times\vec{h} = GM\vec{u} + \vec{c}$

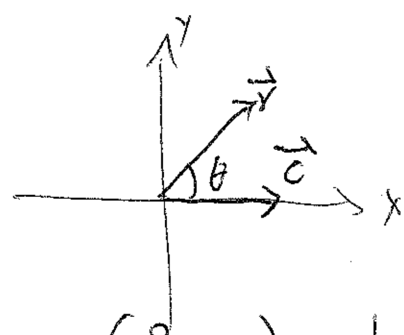
$\vec{u}$ and $\vec{c}$ are both in xy-plane.

$$\vec{r}\cdot(\vec{v}\times\vec{h}) = (\vec{r}\times\vec{v})\cdot\vec{h} = \vec{h}\cdot\vec{h} = |\vec{h}|^2 = \text{constant !} = h^2$$

$$\vec{r}\cdot(\vec{v}\times\vec{h}) = GM(\vec{r}\cdot\vec{u}) + \vec{r}\cdot\vec{c} = h^2$$

$$GM(R\,\vec{u}\cdot\vec{u}) + R\vec{u}\cdot\vec{c} = h^2$$

$$R(GM + \vec{u}\cdot\vec{c}) = h^2$$



Let θ be the angle between $\vec{u}$ and $\vec{c}$ (R,θ) polar coordinate

$$R(GM + |\vec{c}|\cos\theta) = h^2$$

$\Rightarrow$ polar coordinate equation $\quad R(t) = \dfrac{h^2}{GM + |\vec{c}|\cos\theta(t)}$

$$R(t) = \frac{\frac{h^2}{GM}}{1+\frac{|\vec{c}|}{GM}\cos\theta(t)} = \frac{ed}{1+e\cos\theta(t)}, \quad d = \frac{h^2}{|\vec{c}|}$$

What is the carve $(R(t), \theta(t))$ (in polar coordinate) that $R(t) = \dfrac{ed}{1+e\cos\theta(t)}$

$$R + eR\cos\theta = ed \qquad \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}$$

$$\sqrt{x^2+y^2} + ex = ed \qquad e = \frac{|\vec{c}|}{GM}$$

$$\sqrt{x^2+y^2} = e(d-x)$$

$$x^2+y^2 = e^2(d-x)^2$$

$$x^2+y^2 = e^2(d^2-2dx+x^2)$$

$$(1-e^2)x^2 + y^2 + 2de^2x - e^2d^2 = 0$$

$|e|<1 \Rightarrow$ ellipse $\Rightarrow$ Kepler's 1st law

$\Rightarrow$ the orbit of planet is an ellipse