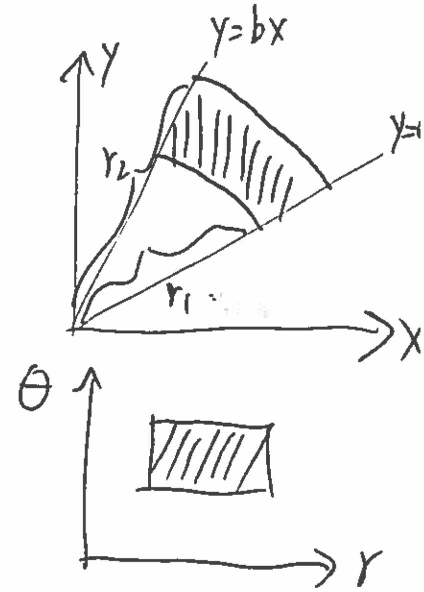
Change of variables

$(x, y) \to (r, \theta)$

$x = r\cos\theta$
$y = r\sin\theta$



$$R_1 = \{(x,y) : r_1^2 \le x^2+y^2 \le r_2^2,\ a \le \frac{y}{x} \le b\}$$

$$\Rightarrow R_2 = \{(r,\theta) : r_1 \le r \le r_2,\ \tan^{-1}(a) \le \theta \le \tan^{-1}(b)\}$$

goal: R_2 is a rectangle !!

why: integral on rectangle is easier !

cost: $\iint_{R_1} f(x,y)\,dA = \iint_{R_2} f(r\cos\theta, r\sin\theta)\, r \, dA$

extra r.

In general $(x,y) \to (u,v)$ $\begin{cases} x = g(u,v) = x(u,v) \\ y = h(u,v) = y(u,v) \end{cases}$

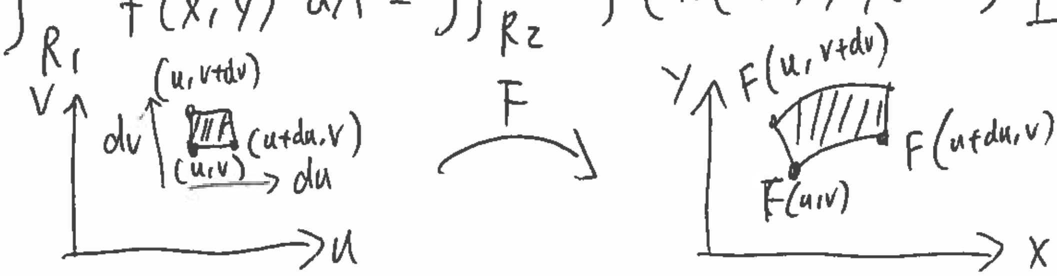
The function $F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$ is one-to-one and onto (bijection)

from $R_2 = \{(u,v)\}$ to $R_1 = \{(x,y)\}$.

(hopefully R_2 is a rectangle!)

$$\iint_{R_1} f(x,y)\,dA = \iint_{R_2} f(x(u,v), y(u,v)) \cdot \boxed{?}\ \underline{dudv}$$

$dA = dudv$



$$F(u\vec{i}+v\vec{j}) = x(u,v)\vec{i} + y(u,v)\vec{j}$$

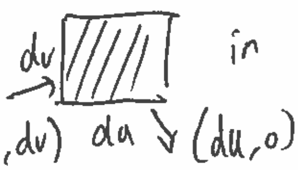
$$F(u+du, v) \cancel{\approx F(u,v) + \frac{\partial F}{\partial u}} = \left(x(u+du,v),\ y(u+du,v)\right)$$

$$\approx \left(x(u,v) + x_u(u,v)\,du,\ y(u,v) + y_v(u,v)\,du\right)$$

$$F(u+du, v) - F(u,v) \approx \left(x_u(u,v)\,du,\ y_u(u,v)\,du\right)$$

Similarly $F(u, v+dv) - F(u,v) \approx \left(x_v(u,v)\,dv,\ y_v(u,v)\,dv\right)$

So a rectangle



in uv-plane is converted into a parallelogram

Original area $|du|\cdot|dv|$.

New Area $\left|(x_u,y_u) \times (x_v,y_v)\right| = \left|(x_u,y_u,0)\times(x_v,y_v,0)\right|$

$$= \left|x_u y_v - x_v y_u\right| = \text{determinant of } \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$\iint_{R_1} f(x,y)\,dA = \iint_{R_2} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

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integral under change of variables.

Ex 12

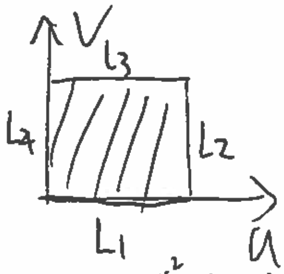
① $x = r\cos\theta,\ y = r\sin\theta$

$$S_1 = \{(r,\theta): r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2\}.$$

$$\tilde{S}_1 = \left\{(x,y): r_1 \le \sqrt{x^2+y^2} \le r_2,\ \theta_1 \le \tan^{-1}\left(\frac{y}{x}\right) \le \theta_2\right\}$$

$$= \left\{(x,y): r_1^2 \le x^2+y^2 \le r_2^2,\ \tan\theta_1 \le \frac{y}{x} \le \tan\theta_2\right\}.$$

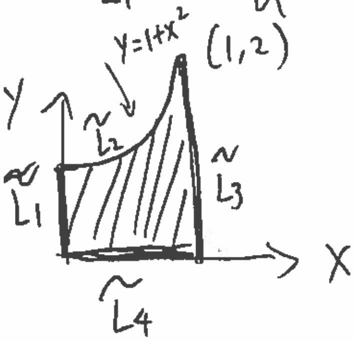
② $x = v,\ y = u(1+v^2)$



$$S_2 = \{(u,v): 0 \le u \le 1,\ 0 \le v \le 1\}.$$

map each boundary:

$L_1: 0 \le u \le 1,\ v = 0 \Rightarrow \begin{cases} x = v = 0 \\ y = u(1+0) = u \Rightarrow 0 \le y \le 1 \end{cases}$



$L_2: u = 1,\ 0 \le v \le 1 \Rightarrow \begin{cases} x = v \quad 0 \le x \le 1 \\ y = u(1+v^2) = 1+v^2 \end{cases}$

$\Rightarrow \{(x, 1+x^2): 0 \le x \le 1\}.$

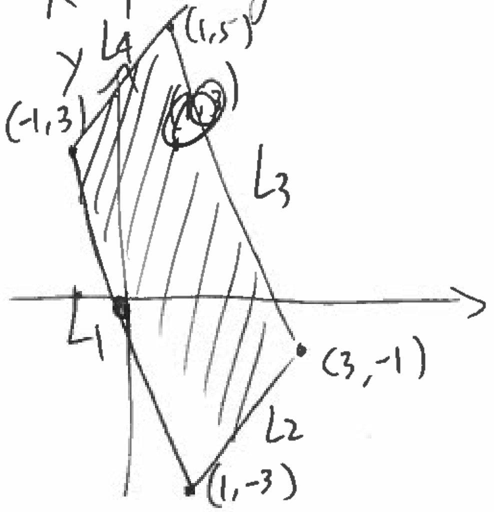
$L_3: 0 \le u \le 1,\ v = 1 \Rightarrow \begin{cases} x = v = 1 \\ y = u(1+1) = 2u \quad 0 \le u \le 1 \end{cases}$

$L_4: u = 0,\ 0 \le v \le 1 \quad \begin{cases} x = v \quad 0 \le x \le 1 \\ y = 0 \end{cases}$

$$\tilde{S}_2 = \{(x,y): 0 \le x \le 1,\ 0 \le y \le x^2+1\}.$$

Ex 13 ① $\iint_R (4x+8y)\,dA$

R parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$



$$\begin{cases} x = \dfrac{u+v}{4} \\ y = \dfrac{v-3u}{4} \end{cases} \Rightarrow \begin{matrix} u+v=4x \\ v-3u=4y \end{matrix} \Rightarrow \begin{cases} u = x-y \\ v = 3x+y \end{cases}$$

L_1: slope $\dfrac{3-(-3)}{-1-1} = \dfrac{6}{-2} = -3$

$y-3 = -3(x+1) \Rightarrow y = -3x$

$3x+y=0 \Rightarrow v=0$.

L_2: slope $\dfrac{-1-(-3)}{3-1} = \dfrac{2}{2} = 1$

$y+3 = 1\cdot(x-1) \Rightarrow y = x-4$

$\Rightarrow x-y=4 \Rightarrow u=4$

L_3: slope $=-3$

$y-5 = -3(x-1)$

$y = -3x+8$

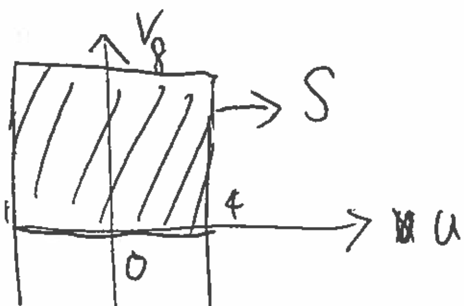
$3x+y=8 \Rightarrow v=8$

L_4: slope $=1$

$y-5 = 1\cdot(x-1)$

$y = x+4$

$x-y=-4 \Rightarrow u=-4$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & +\frac{1}{4} \\ -\frac{3}{4} & +\frac{1}{4} \end{vmatrix}$$

$$= \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$\iint_R (4x+8y)\,dA$$

$$= \iint_S \left(4\cdot\frac{u+v}{4} + 8\cdot\frac{v-3u}{4}\right)\cdot\frac{1}{4}\,du\,dv$$

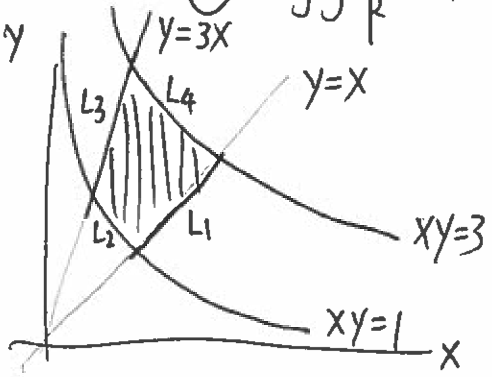
$$= \frac{1}{4}\iint_S (u+v+2v-6u)\,du\,dv$$

$$= \frac{1}{4}\int_{-4}^{4}\int_0^8 (-5u+3v)\,dv\,du$$

$$= \frac{1}{4}\int_{-4}^{4} \left(-5uv+\frac{3}{2}v^2\right)\Big|_0^8\,du$$

$$= \frac{1}{4}\int_{-4}^{4} (-40u+96)\,du = \frac{1}{4}\cdot 96\cdot 8 = 192$$

Ex 13 (2) $\iint_R xy \, dA$ $\quad \begin{cases} x = \frac{u}{v} \\ y = v \end{cases}$



$R_1 = \{ (x,y) : 1 \le \frac{y}{x} \le 3 , 1 \le xy \le 3 \}$

$R_1 = \{ (u,v) : 1 \le \frac{v^2}{u} \le 3 , 1 \le u \le 3 \}$

$\frac{y}{x} = \frac{v}{\frac{u}{v}} = \frac{v^2}{u} , \quad xy = u$

$R_1 = \{ (u,v) : 1 \le u \le 3 , \sqrt{u} \le v \le \sqrt{3u} \}$

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix} \qquad \det J = \frac{1}{v}$$

$$\iint_R xy \, dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \cdot \frac{1}{v} \, dv \, du$$

$$= \int_1^3 u \ln v \Big|_{v=\sqrt{u}}^{v=\sqrt{3u}} du = \int_1^3 u \left(\ln \sqrt{3u} - \ln \sqrt{u} \right) du$$

$$= \int_1^3 \frac{1}{2} u \left(\ln 3 + \ln u - \ln u \right) du = \frac{1}{2} \int_1^3 u \cdot \ln 3 \, du = \frac{\ln 3}{2} \left(\frac{1}{2} u^2 \right) \Big|_1^3$$

$$= \frac{\ln 3}{2} \cdot \frac{1}{2} (9-1) = 2 \ln 3$$

3D – change of variables (similar to 2D)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

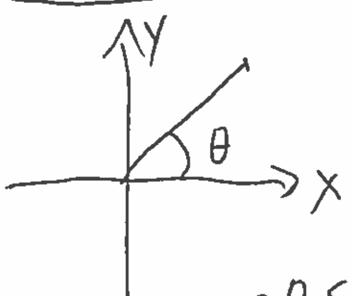
$$\iiint_R f(x,y,z)\,dxdydz = \iiint_S f(\quad)\cdot\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det J = r\cos\theta\cdot\cos\theta + r\sin\theta\cdot\sin\theta = r.$$

spherical coordinates



θ = angle between $\langle x, y, 0\rangle$ and $\langle 1, 0, 0\rangle$

ϕ = angle between $\langle x, y, z\rangle$ and $\langle 0, 0, 1\rangle$

$0 \le \theta \le 2\pi$. $0 \le \phi \le \pi$ $\quad$ ($\phi = 0 \Rightarrow$ north pole) ($\phi = \pi \Rightarrow$ South pole)

(ρ, θ, ϕ)

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2} = \rho\sin\phi \\ z = \rho\cos\theta \end{cases}$$

north, south pole $r=0$, $z=\pm$

equator $r=\rho$, $z=0$

$$x^2+y^2+z^2=\rho^2$$

$$J = \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} = -\rho^2\sin\phi \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \rho^2\sin\phi$$

integral in spherical coordinate

$$\iiint_E f(x,y,z)\,dV = \int_{\phi=\phi_1}^{\phi=\phi_2}\int_{\theta=\theta_1}^{\theta=\phi_2}\int_{\rho=\rho_1}^{\rho=\rho_2} f(\quad)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Ex 14

① $$\iiint_B 1\,dV = \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=R} 1\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{2\pi}1\,d\theta\right)\int_0^R \rho^2\,d\rho$$

$$= 2\cdot 2\pi\cdot\frac{1}{3}R^3 = \frac{4\pi R^3}{3} = \frac{4}{3}\pi R^3$$

② Volume of ice cream cone: above $z=\sqrt{x^2+y^2}$ and below $x^2+y^2+z^2=z$.

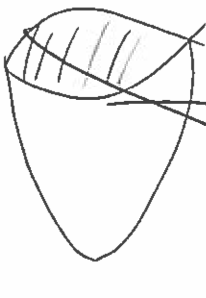
Use spherical coordinate: $z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2}$

$$= \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$$

$$\Rightarrow \cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$$

$$x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\phi \Rightarrow \rho=\cos\phi$$

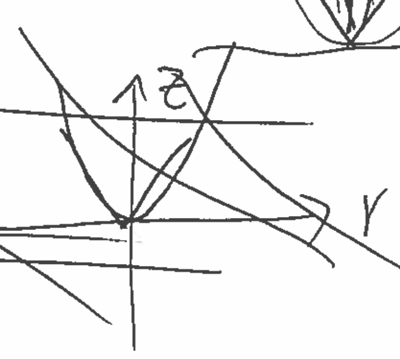
$$R=\{(\rho,\theta,\phi):\ 0\le\phi\le\frac{\pi}{4},\ 0\le\theta\le 2\pi,\ 0\le\rho\le\cos\phi\}$$



~~$\frac{\cos\phi}{\sin^2\phi}=\cos\phi$~~

~~$\Rightarrow \cos\phi=0$ or $\sin\phi=1 \Rightarrow \phi=\frac{\pi}{2}$~~

~~$x^2+y^2=r^2 \Rightarrow \begin{cases} z=r \\ z^2-z+x^2=0\end{cases}$~~



$$\iiint_R 1\,dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\cos\phi} 1\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\right)$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3\sin\phi\Big|_{\rho=0}^{\rho=\cos\phi} d\phi = 2\pi\int_0^{\frac{\pi}{4}}\frac{1}{3}\cos^3\phi\sin\phi\,d\phi$$

$$= \frac{2\pi}{3}\left(-\frac{1}{4}\cos^4\phi\right)\Big|_0^{\frac{\pi}{4}} = \frac{2\pi}{3}\left(-\frac{1}{4}\cdot\frac{1}{4}-\left(-\frac{1}{4}\right)\right) = \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{\pi}{8}$$

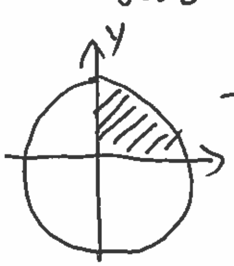
③ $\iiint_E x\,dV \qquad E = \{(x,y,z): 4\le x^2+y^2\le 9,\ 0\le z\le x+y+5\}$.

If use spherical coordinate, $4\le\rho^2\sin^2\phi\le 9 \Rightarrow 2\le\rho\sin\phi\le 3$

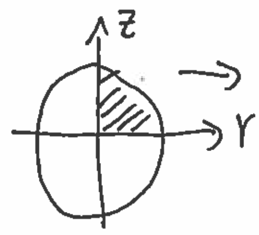
$0\le\rho\cos\phi\le\rho\sin\phi(\cos\theta+\sin\theta)+5$.

not good to use as the region is not "spherical" (not part of a ball) See Lecture 17 where we use cylindrical coordinate

④ $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$ E enclosed by $x^2+y^2+z^2=9$ in the first octant



→ cross-section in xy-plane



→ side-view

$E_1 = \{(x,y,z): x\ge 0, y\ge 0, z\ge 0, x^2+y^2+z^2\le 9\}$.

↓

$E_2 = \{(\rho,\theta,\phi): 0\le\rho\le 3,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}\}$

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 e^\rho\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi\right)\left(\int_0^{\frac{\pi}{2}} 1\,d\theta\right)\left(\int_0^3 e^\rho\rho^2\,d\rho\right)$$

$$= 1\cdot\frac{\pi}{2}\cdot e^\rho(\rho^2-2\rho+2)\Big|_0^3$$

$$= \frac{\pi}{2}\cdot\left[e^3(5)-e^0\cdot 2\right] = \frac{\pi}{2}(5e^3-2)$$

$$\int e^\rho\rho^2\,d\rho = \int\rho^2\,de^\rho$$

$$= \rho^2e^\rho - \int e^\rho 2\rho\,d\rho$$

$$= \rho^2e^\rho - 2\int e^\rho\rho\,d\rho$$

$$= \rho^2e^\rho - 2\int\rho\,de^\rho$$

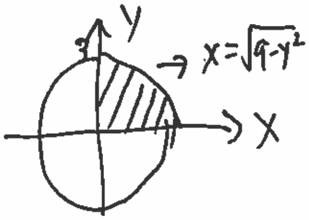
$$= \rho^2e^\rho - 2\left(\rho e^\rho - \int e^\rho\,d\rho\right)$$

$$= \rho^2e^\rho - 2\rho e^\rho + 2e^\rho$$

⑤ $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$

Region $E = \{ (x,y,z) : 0 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2},\ \sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2} \}$.

Use cylindrical $\quad E_2 = \{ (r,\theta,z) : 0 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2},\ r \le z \le \sqrt{18-r^2} \}$.



$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_r^{\sqrt{18-r^2}} (r^2+z^2)\, r\, dz\, dr\, d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} 1\, d\theta\right)\left(\int_0^3 \int_r^{\sqrt{18-r^2}} (r^3+rz^2)\, dz\, dr\right)$$

$$= \frac{\pi}{2} \cdot \int_0^3 \left(r^3 z + \frac{1}{3} r z^3\right)\Big|_{z=r}^{z=\sqrt{18-r^2}} dr$$

$$= \frac{\pi}{2} \cdot \int_0^3 \left(r^3\sqrt{18-r^2} + \frac{1}{3} r (18-r^2)\sqrt{18-r^2} - r^4 - \frac{1}{3} r^4\right) dr$$

$$= \frac{\pi}{2}\left[\int_0^3 \left(\frac{2}{3} r^3 \sqrt{18-r^2} + 6r\sqrt{18-r^2}\right) dr - \frac{4}{3}\int_0^3 r^4\, dr\right]$$

Let $u = 18 - r^2 \quad du = -2r\,dr \quad$ $r=0 \Rightarrow u=18$, $r=3 \Rightarrow u=9$ $\quad$ $r^2 = 18-u$

$$\int_0^3 \left(\frac{2}{3} r^3\sqrt{18-r^2} + 6r\sqrt{18-r^2}\right) dr = \int_{18}^{9} \left(\frac{-2}{3}\cdot\frac{1}{2}\sqrt{u}\,(18-u) + 6\cdot\left(-\frac{1}{2}\right)\sqrt{u}\right) du$$

$$= \int_9^{18} \left(\frac{1}{3}(18-u)\sqrt{u} + 3\sqrt{u}\right) du = \int_9^{18}\left(9\sqrt{u} - \frac{1}{3}u^{\frac{3}{2}}\right) du = \left(9\cdot\frac{2}{3}u^{\frac{3}{2}} - \frac{1}{3}\cdot\frac{2}{5}u^{\frac{5}{2}}\right)\Big|_9^{18}$$

$$= \left(6u^{\frac{3}{2}} - \frac{2}{15}u^{\frac{5}{2}}\right)\Big|_9^{18} = \left(6\cdot 18^{\frac{3}{2}} - \frac{2}{15}18^{\frac{5}{2}}\right) - \left(6\cdot 9^{\frac{3}{2}} - \frac{2}{15}9^{\frac{5}{2}}\right)$$

$$= (\sqrt{8}-1)\,6\cdot 27 - \frac{2}{15}\cdot 243\cdot(\sqrt{32}-1) = (2\sqrt{2}-1)\,81\cdot 2 - \frac{2}{15}\cdot 81\cdot 3\cdot(4\sqrt{2}-1)$$

$$= 81\left[4\sqrt{2} - 2 - \frac{8}{5}\sqrt{2} + \frac{2}{5}\right] = 81\left(\frac{12\sqrt{2}}{5} - \frac{8}{5}\right) = \frac{81\cdot 4}{5}(3\sqrt{2}-2)$$

So

$$= \frac{\pi}{2}\left[\frac{324}{5}(3\sqrt{2}-2) - \frac{4}{3}\cdot\frac{1}{5}\cdot 3^5\right] = \frac{\pi}{2}\cdot\left[\frac{81\cdot 4}{5}(3\sqrt{2}-2) - \frac{4\cdot 81\cdot \cancel{3}}{5\cdot\cancel{3}}\right]$$

$$= \frac{162\pi}{5}(3\sqrt{2}-2-1) = \frac{162\pi}{5}\cdot 3(\sqrt{2}-1) = \frac{486\pi}{5}(\sqrt{2}-1)$$ → somewhere there is a mistake

Use spherical ~~$E_3 = \{(r,\theta,\phi)\}$~~

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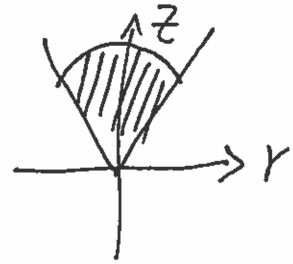
$\rightarrow$ cross-section

$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$
$$r \le z \le \sqrt{18-r^2}$$



$$r = \rho\sin\phi = \sqrt{x^2+y^2} \Rightarrow r \le z \Rightarrow \rho\sin\phi \le \rho\cos\phi$$

$$\Rightarrow \sin\phi \le \cos\phi \quad \text{or} \quad \tan\phi \le 1 \Rightarrow 0 \le \phi \le \frac{\pi}{4}$$

$$z \le \sqrt{18-r^2} \Rightarrow z^2 \le 18 - x^2 - y^2 \Rightarrow x^2+y^2+z^2 \le 18 \Rightarrow \rho^2 \le 18 \Rightarrow \rho \le \sqrt{18} = 3\sqrt{2}$$

we still have $0 \le \theta \le \frac{\pi}{2}$

(why $r \le 3$? since $r \le \sqrt{18-r^2} \Rightarrow r^2 \le 18 - r^2 \Rightarrow r^2 \le 9$).

$$E_3 = \{(\rho,\theta,\phi) : 0 \le \rho \le 3\sqrt{2},\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le \phi \le \tfrac{\pi}{4}\}.$$

$$\text{Integral} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}} \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} 1 \, d\theta\right)\left(\int_0^{\frac{\pi}{4}} \sin\phi \, d\phi\right) \cdot \left(\int_0^{3\sqrt{2}} \rho^4 d\rho\right)$$

$$= \frac{\pi}{2} \cdot (-\cos\phi)\Big|_0^{\frac{\pi}{4}} \cdot \left(\frac{1}{5}\rho^5\right)\Big|_0^{3\sqrt{2}}$$

$$= \frac{\pi}{2} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{1}{5} 3^5 \cdot \sqrt{2}^5 = \frac{\pi}{10}\left(1-\frac{\sqrt{2}}{2}\right) \cdot 81 \cdot 4\sqrt{2} = \frac{\pi}{5}(2-\sqrt{2})\, 81 \cdot \sqrt{2}$$

$$= \frac{\pi}{5} 81 \left(2\sqrt{2} - 2\right) = \frac{162\pi}{5}\left(\sqrt{2}-1\right)$$

Euler

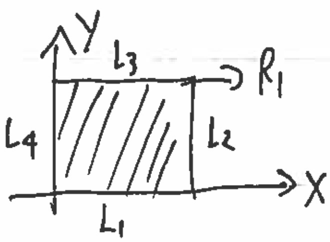
① $\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\frac{1}{1-xy} = \sum_{n=0}^{\infty} (xy)^n$ So $\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n\,dx\,dy$

$= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 x^n y^n\,dx\,dy = \sum_{n=0}^{\infty} \left(\int_0^1 x^n dx\right)\left(\int_0^1 y^n dy\right)$

$= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

② $\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy$



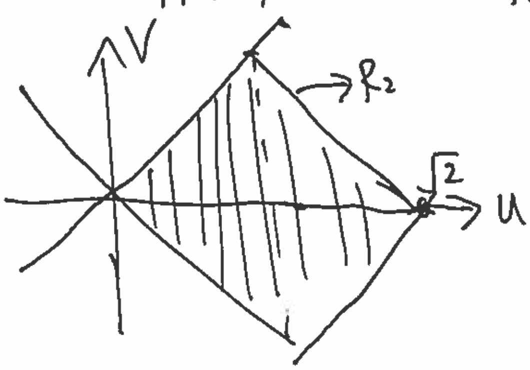
$\begin{cases} x = \frac{u-v}{\sqrt{2}} \\ y = \frac{u+v}{\sqrt{2}} \end{cases}$

$R_1 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$.

$R_2 = \{(u,v) : 0 \le u-v \le \sqrt{2},\ 0 \le u+v \le \sqrt{2}\}$

$J = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $\det J = 1$

So $\iint_{R_1} \frac{1}{1-xy}\,dx\,dy = \iint_{R_2} \frac{1}{1-\frac{u^2-v^2}{2}}\,du\,dv = \iint_{R_2} \frac{2}{2+v^2-u^2}\,du\,dv$



$= \iint_{R_3} \frac{4}{2+v^2-u^2}\,du\,dv$

$R_3 = \{(u,v) : 0 \le v \le u,\ 0 \le u \le \frac{\sqrt{2}}{2}\}$

$\cup \{(u,v) : 0 \le v \le \sqrt{2}-u,\ \frac{\sqrt{2}}{2} \le u \le \sqrt{2}\}$.