Math 212 Lecture 17

Ex 1. ① $(x,y,z) = (2\sqrt{3}, 2, -1)$ $\quad r = \sqrt{x^2+y^2} = \sqrt{(2\sqrt{3})^2+2^2} = 4$

$$\tan\theta = \frac{y}{x} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

So $(r,\theta,z) = (4, \frac{\pi}{6}, -1)$

② (a) $3x+3y+z=6$

$$3\cdot r\cos\theta + 3\cdot r\sin\theta + z = 6$$

$$z = 6 - 3r(\cos\theta + \sin\theta)$$

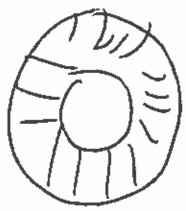
(b) $-x^2-y^2+z^2=1$

$$-(r\cos\theta)^2 - (r\sin\theta)^2 + z^2 = 1$$

$$-r^2+z^2=1 \Rightarrow z^2-r^2=1$$

③ $\iiint_E x\,dV$, E is enclosed by $z=0$, $z=x+y+5$, cylinders $x^2+y^2=4$, $x^2+y^2=9$.

Base:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r(\cos\theta+\sin\theta)} (r\cos\theta \cdot r)\,dz\,dr\,d\theta$$

$D = \{2 \le r \le 3,\ 0 \le \theta \le 2\pi,$
$0 \le z \le x+y+5 = r(\cos\theta+\sin\theta)+5\}$

$$= \int_0^{2\pi}\int_2^3 r^2\cos\theta \cdot z \Big|_{z=0}^{z=r(\cos\theta+\sin\theta)+5} dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 r^2\cos\theta\,\big(r(\cos\theta+\sin\theta)+5\big)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 \left(r^3\cos^2\theta + r^3\cos\theta\sin\theta + 5r^2\cos\theta\right)dr\,d\theta$$

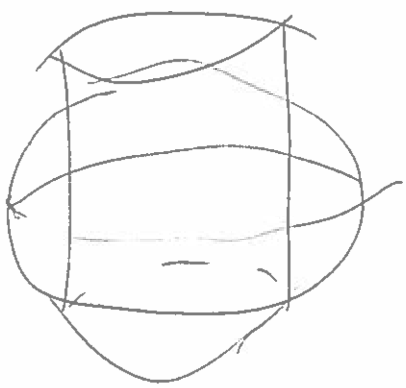
$$= \int_0^{2\pi}\left(\frac{1}{4}r^4\cos^2\theta + \frac{1}{4}r^4\cos\theta\sin\theta + \frac{5}{3}r^3\cos\theta\right)\Big|_{r=2}^{r=3} dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{4}(81-16)(\cos^2\theta+\cos\theta\sin\theta) + \frac{5}{3}(27-8)\cos\theta\right)d\theta$$

$$= \frac{65}{4}\left(\int_0^{2\pi}\cos^2\theta\,d\theta + \int_0^{2\pi}\cos\theta\sin\theta\,d\theta\right) + \frac{95}{3}\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{=0}$$

$$= \frac{65}{4}\left(\int_0^{2\pi}\frac{1-\cos2\theta}{2}\,d\theta + \frac{1}{2}\sin^2\theta\Big|_0^{2\pi}\right) = \frac{65}{4}(\pi) = \frac{65}{4}\pi.$$

(4) Volume of solid ~~between~~ lies in both $x^2+y^2=1$ and $x^2+y^2+z^2=4$



It is inside $x^2+y^2=1$

$$D = \{(x,y,z):\ x^2+y^2\le 1,\ -\sqrt{4-x^2-y^2}\le z\le\sqrt{4-x^2-y^2}\},$$

$$= \{(r,\theta,z):\ r\le 1,\ 0\le\theta\le 2\pi,\ -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\}.$$

$$V=\iiint_D 1\, dV=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\cdot r\, dz\, dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^1\left(rz\Big|_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}}\right)dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^1\left(r\sqrt{4-r^2}-r\cdot\left(-\sqrt{4-r^2}\right)\right)dr\, d\theta=\int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\, dr\, d\theta$$

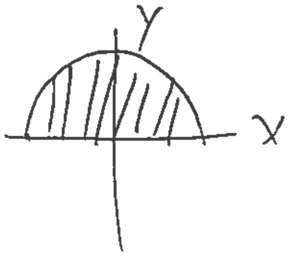
$$=\left(\int_0^{2\pi}1\, d\theta\right)\left(\int_0^1 2r\sqrt{4-r^2}\, dr\right)=2\pi\cdot\left(-\frac{2}{3}(4-r^2)^{\frac{3}{2}}\right)\Big|_0^1$$

$$\boxed{\int 2r\sqrt{4-r^2}\, dr \quad u=4-r^2 \quad du=-2r\, dr \quad \int -\sqrt{u}\, du=-\frac{2}{3}u^{\frac{3}{2}} = -\frac{2}{3}(4-r^2)^{\frac{3}{2}}}$$

$$=2\pi\left(-\frac{2}{3}\cdot\left(3^{\frac{3}{2}}-4^{\frac{3}{2}}\right)\right)=\frac{4\pi}{3}\left(4^{\frac{3}{2}}-3^{\frac{3}{2}}\right)=\frac{4\pi}{3}\left(8-3^{\frac{3}{2}}\right)$$

5 $$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\, dz\, dy\, dx \qquad y=\sqrt{9-x^2}\Rightarrow x^2+y^2=$$

$$D=\{(x,y,z): -3\le x\le 3,\ 0\le y\le\sqrt{9-x^2},\ 0\le z\le 9-x^2-y^2\}$$



$$=\{(r,\theta,z): r\le 3,\ 0\le\theta\le\pi,\ 0\le z\le 9-r^2\}$$

$$=\int_0^{\pi}\int_0^3\int_0^{9-r^2} r\cdot r\, dz\, dr\, d\theta$$

$$=\int_0^{\pi}\int_0^3\left(r^2 z\Big|_{z=0}^{z=9-r^2}\right)dr\, d\theta=\int_0^{\pi}\int_0^3 r^2(9-r^2)\, dr\, d\theta$$

$$=\left(\int_0^{\pi}1\cdot d\theta\right)\int_0^3(9r^2-r^4)\, dr=\pi\cdot\left(\frac{9}{3}r^3-\frac{1}{5}r^5\right)\Big|_0^3$$

$$=\pi\left(81-\frac{243}{5}\right)=81\pi\left(1-\frac{3}{5}\right)=81\pi\cdot\frac{2}{5}=\frac{162\pi}{5}$$