Ex 9. ① $\int_0^1 \int_0^3 \int_0^{\sqrt{1-z^2}} ze^y \, dx\, dy\, dz$

$= \int_0^1 \int_0^3 (ze^y x)\Big|_{x=0}^{x=\sqrt{1-z^2}} dy\,dz = \int_0^3 \int_0^1 z\sqrt{1-z^2}\, e^y \, dy\, dz.$

$= \left(\int_0^1 z\sqrt{1-z^2}\, dz\right) \cdot \left(\int_0^3 e^y \, dy\right) = \left(-\frac{1}{3}(1-z^2)^{3/2}\right)\Big|_{z=0}^{z=1} \cdot (e^y)\Big|_{y=0}^{y=3}$

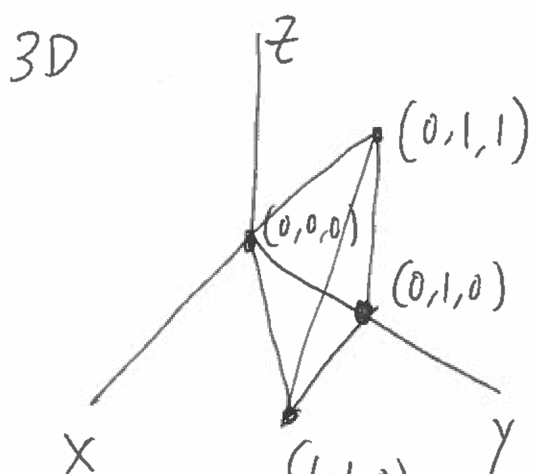
$\int z\sqrt{1-z^2}\, dz \quad U = 1-z^2 \quad dU = -2z\,dz \quad = -\frac{1}{2}\int \sqrt{U}\, dU$

$= -\frac{1}{2}\cdot\frac{2}{3} U^{\frac{3}{2}} = -\frac{1}{3}(1-z^2)^{3/2}$

$= \frac{1}{3}\cdot(e^3 - 1)$

② $\iiint_E xz\, dV$. E = tetrahedron with vertices (0, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1)

3D



Base: $z=0$. (triangle (0,0,0) − (0,1,0) − (1,1,0)

"Side": $x=0$ (triangle (0,0,0) − (0,1,1) − (0,1,0)

$y=1$ (triangle (0,1,1) − (0,1,0) − (1,1,0)

how to find last side? triangle (0,0,0) − (0,1,1) − (1,1,0)

plane: $ax+by+cz+d=0$.

$\Rightarrow ax+by+z+d=0$

(0,0,0) $\Rightarrow d=0$

(0,1,1) $\Rightarrow b+1+d=0 \Rightarrow b=-1$

(1,1,0) $\Rightarrow a+b+d=0 \Rightarrow a=1$

$x-y+z=0 \Rightarrow z = y-x$

We set up it as

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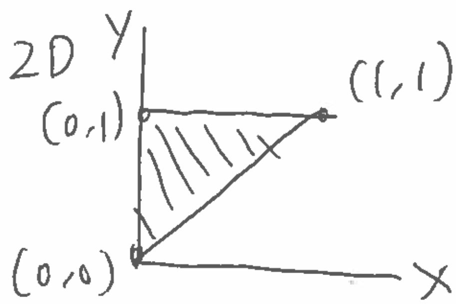
$$\iint_R \left(\int_{z=z_1(x,y)}^{z=z_2(x,y)} xz \, dz \right) dA$$

$R =$ triangle on $z=0$. $\quad z_1(x,y) = 0, \quad z_2(x,y) = y-x.$



$R = \{(x,y) : 0 \le x \le 1, \ x \le y \le 1\}$

$$\int_0^1 \int_x^1 \int_0^{y-x} xz \, dz \, dy \, dx$$

$$= \int_0^1 \int_x^1 \left(\frac{1}{2} x z^2 \Big|_{z=0}^{z=y-x} \right) dy \, dx$$

$$= \int_0^1 \int_x^1 \frac{1}{2} x (y-x)^2 \, dy \, dx$$

$$= \int_0^1 \frac{1}{2} x \cdot \left(\frac{1}{3} (y-x)^3 \Big|_{y=x}^{y=1} \right) dx$$

$$= \int_0^1 \frac{1}{6} x \left[(1-x)^3 - 0 \right] dx$$

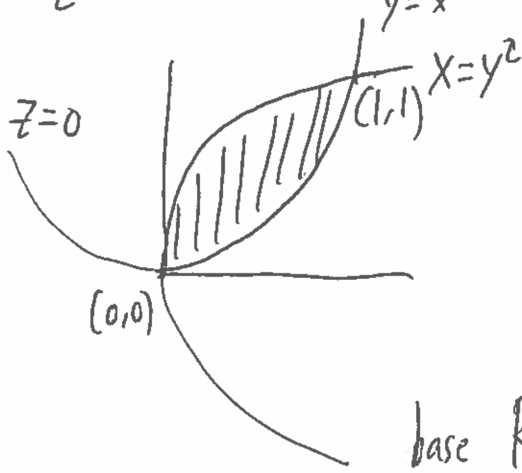
$$= \frac{1}{6} \int_0^1 x(1-x)^3 \, dx = \frac{1}{6} \int_0^1 \left(x - 3x^2 + 3x^3 - x^4 \right) dx$$

$$= \frac{1}{6} \left(\frac{1}{2} x^2 - x^3 + \frac{3}{4} x^4 - \frac{1}{5} x^5 \right) \Big|_0^1$$

$$= \frac{1}{6} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right)$$

$$= \frac{1}{6} \left(\frac{10}{20} - \frac{20}{20} + \frac{15}{20} - \frac{4}{20} \right) = \frac{1}{6} \cdot \frac{1}{20} = \frac{1}{120}$$

③ $\iiint_E xy\,dV$ E bounded by $y=x^2$, $x=y^2$ and $z=0$ and $z=x+y$



~~base~~ ~~R={(x,y)~~ :

$$\begin{cases} y=x^2 \\ x=y^2 \end{cases} \Rightarrow y=y^4 \Rightarrow y^4-y=0 \Rightarrow y(y^3-1)=0$$

$\Rightarrow y=0$ or $y=1$ $\quad (0,0),\ (1,1)$

base $R=\{(x,y): 0\le x\le 1,\ x^2\le y\le\sqrt{x}\}$

for $0\le x\le 1$, $\sqrt{x}\ge x^2$ for example $\sqrt{\frac{1}{2}}>\left(\frac{1}{2}\right)^2$

Top: $z=x+y > z=0$ (bottom.

$$\iiint_E xy\,dV=\iint_R\left(\int_0^{x+y} xy\,dz\right)dA=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy\,dz\,dy\,dx$$

$$=\int_0^1\int_{x^2}^{\sqrt{x}}(xyz)\Big|_{z=0}^{z=x+y}dy\,dx=\int_0^1\int_{x^2}^{\sqrt{x}}xy(x+y)\,dy\,dx=\int_0^1\int_{x^2}^{\sqrt{x}}(x^2y+xy^2)\,dy\,dx$$

$$=\int_0^1\left(\frac{1}{2}x^2y^2+\frac{1}{3}xy^3\right)\Big|_{y=x^2}^{y=\sqrt{x}}dx=\int_0^1\left[\left(\frac{1}{2}x^3+\frac{1}{3}\underbrace{x^2\sqrt{x}}_{x^{\frac{5}{2}}}\right)-\left(\frac{1}{2}x^6+\frac{1}{3}x^7\right)\right]dx$$

$$=\left(\frac{1}{8}x^4+\frac{1}{3}\cdot\frac{2}{7}x^{\frac{7}{2}}-\frac{1}{14}x^7-\frac{1}{24}x^8\right)\Big|_0^1$$

$$=\frac{1}{8}+\frac{2}{21}-\frac{1}{14}-\frac{1}{24}=\left(\frac{3}{24}-\frac{1}{24}\right)+\left(\frac{4}{42}-\frac{3}{42}\right)=\frac{1}{12}+\frac{1}{42}$$

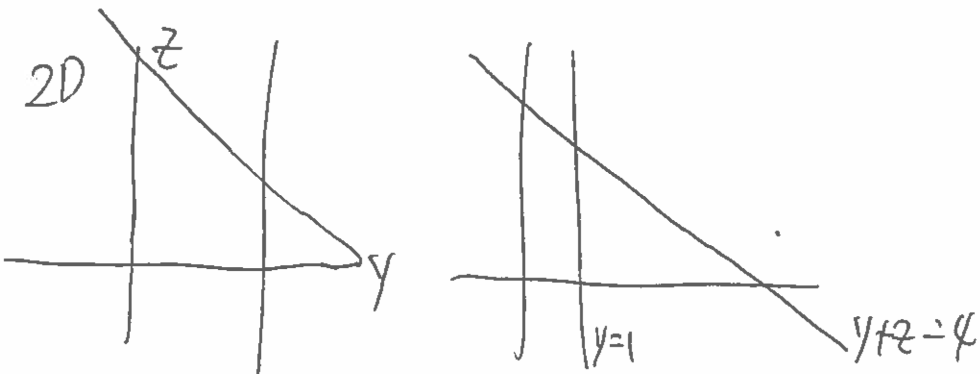
$$=\frac{7}{84}+\frac{2}{84}=\frac{9}{84}=\frac{3}{28}$$

④ Volume of solid enclosed by $x^2+z^2=4$ and $y=-1$ and $y+z=4$

$$V = \iiint_E 1 \, dV$$ but what is E ?

$y=1$ and $y+z=4$ are planes perpendicular to yz-plane $(x=0)$

2D



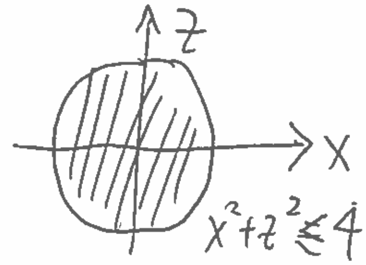
~~$x^2+z^2=4 \Rightarrow -2\le z\le 2$ $z=-2 \Rightarrow y+z=4$~~

Find boundary

Strategy : base: xz-direction.

top: $y=4-z$

bottom $y=-1$



In $x^2+z^2\le 4$, $|z|\le 2 \Rightarrow -2\le z\le 2$, $y=4-z\ge 2>-1$

$$V=\iint_{x^2+z^2\le 4}\left(\int_{y=-1}^{y=4-z} dy\right) dA = \iint_{x^2+z^2\le 4} (5-z)\, dA.$$

polar
$x=r\cos\theta$
$z=r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 (5-r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (5r-r^2\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{5}{2}r^2-\frac{1}{3}r^3\sin\theta\right)\Bigg|_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(10-\frac{8}{3}\sin\theta\right) d\theta = \left(10\theta+\frac{8}{3}\cos\theta\right)\Bigg|_0^{2\pi}$$

$$= 20\pi.$$

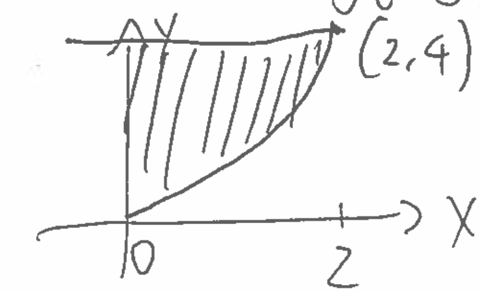
⑤ $\iiint_B (z^3 + \sin y + 3)\,dV \quad B = x^2+y^2+z^2 \le 1$

z^3 is odd function $(-z)^3 = -z^3$ B is symmetric in z

So $\iiint_B z^3 = 0$ $\quad$ $\sin y$ is odd So $\iiint_B \sin y \, dV = 0$

$\iiint_B 3\,dV = 3 \cdot \text{Volume} = 3 \cdot \cancel{4\pi}\,\frac{4}{3} r^3 = \cancel{4\pi} = 4r^3 = 4$

Ex10 ① $\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$



$y = x^2$

$\Rightarrow x = \sqrt{y}$

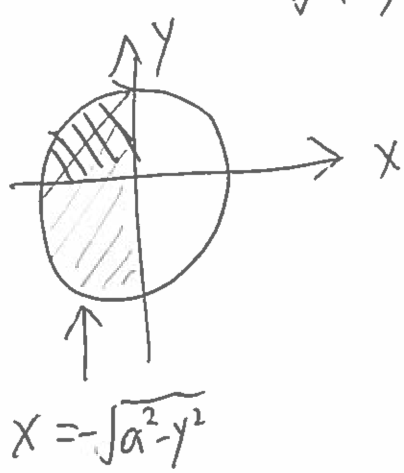
② $\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y\,dx\,dy$

$x = -\sqrt{a^2-y^2}$ to $x = 0$

$\Rightarrow x^2 = a^2 - y^2 \Rightarrow x^2 + y^2 = a^2$

y from 0 to a

$\Rightarrow y > 0$



$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^a (r\cos\theta)^2\, r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^a r^4 \cos^2\theta \sin\theta\,dr\,d\theta = \left(\int_{\frac{\pi}{2}}^{\pi} \cos^2\theta \sin\theta\,d\theta\right)\left(\int_0^a r^4\,dr\right)$$

$$= \left(-\frac{1}{3}\cos^3\theta\right)\Big|_{\theta=\frac{\pi}{2}}^{\theta=\pi} \cdot \left(\frac{1}{5}r^5\right)\Big|_{r=0}^{r=a} = \frac{1}{3} \cdot \frac{1}{5} a^5 = \frac{1}{15} a^5$$

③ $\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy$

Base: in yz-plane $\int_0^1 \int_1^y dz\,dy$

bottom: $x=0$, top: $x=z$

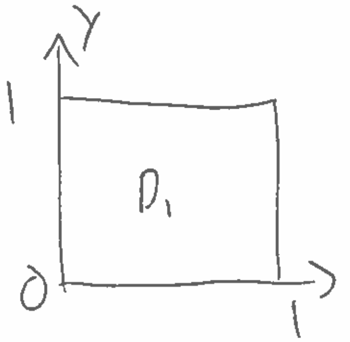
$$D = \{ (x,y,z) : 0 \le y \le 1,\ y \le z \le 1,\ 0 \le x \le z \}$$

$\max\{x,y\} \le z \le 1$

base in xy-plane $D_1 = \{ (x,y) : 0 \le x \le z \le 1,\ 0 \le y \le 1 \}$

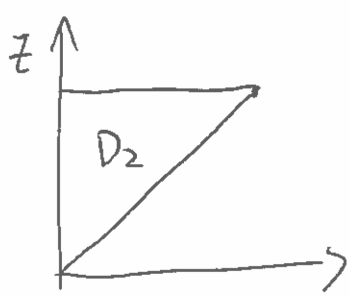
yz-plane $D_2 = \{ (y,z) : 0 \le y \le 1,\ y \le z \le 1 \}$ $0 \le x \le z$

xz-plane $D_3 = \{ (x,z) : 0 \le x \le z,\ 0 \le z \le 1 \}$ $0 \le y \le z$



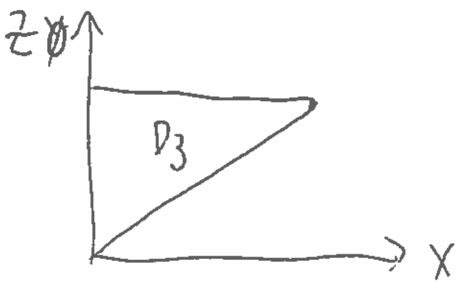
a) $\int_0^1 \int_0^1 \int_{\max\{x,y\}}^1 f(x,y,z)\,dz\,dx\,dy$

b) $\int_0^1 \int_0^1 \int_{\max\{x,y\}}^1 f(x,y,z)\,dz\,dy\,dx$



c) $\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy$

d) $\int_0^1 \int_0^z \int_0^z f(x,y,z)\,dx\,dy\,dz$



e) $\int_0^1 \int_0^z \int_0^z f(x,y,z)\,dy\,dx\,dz$

f) $\int_0^1 \int_x^1 \int_0^z f(x,y,z)\,dy\,dz\,dx$