Math 212 Lecture 15

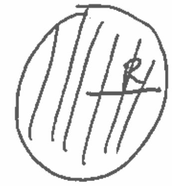
Polar coordinate

$(x, y) \to (r, \theta)$

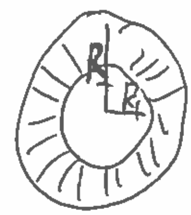
$x = r\cos\theta, \ y = r\sin\theta.$ Inverse: $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$

When to use polar coordinate?

If the region $D \subseteq \mathbb{R}^2$ is "circular"



$D_1 = \{(x,y) : x^2+y^2 \le R^2\}$ $\boxed{\text{disk}}$

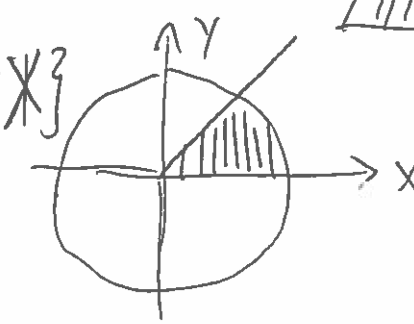


$D_2 = \{(x,y) : R_1^2 \le x^2+y^2 \le R_2^2\}$ $\boxed{\text{annulus}}$ or ring

$D_3 = \{(x,y) : x^2+y^2 \le R^2, \ y \ge 0\}$ $\boxed{\text{half-disk}}$

$D_4 = \{(x,y) : x^2+y^2 \le R^2, \ 0 \le y \le x\}$

"wedge" or "pizza slice"



$D_1 = \{(r,\theta) : 0 \le r \le R, \ 0 \le \theta < 2\pi\}$ rectangle in polar!

$D_2 = \{(r,\theta) : R_1 \le r \le R_2, \ 0 \le \theta \le 2\pi\}$ " "

$D_3 = \{(r,\theta) : 0 \le r \le R, \ 0 \le \theta \le \pi\}$ " "

$D_4 = \{(r,\theta) : 0 \le r \le R, \ 0 \le \theta \le \frac{\pi}{4}\}.$ " "

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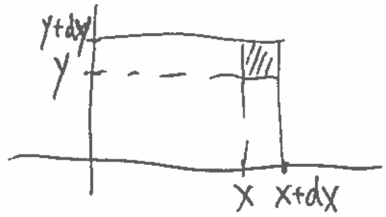
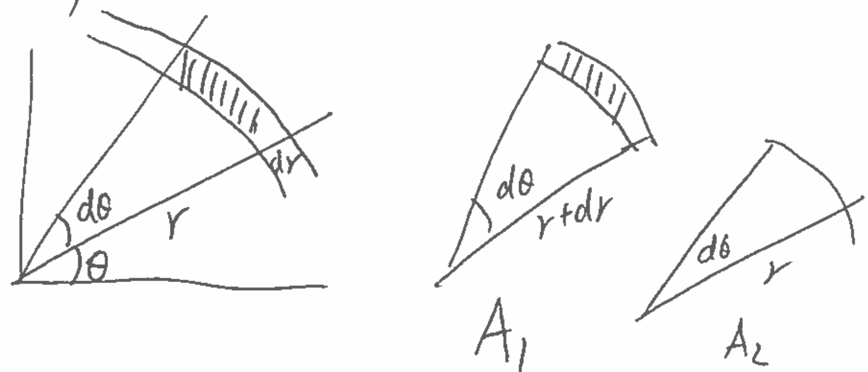
circular regions in (x, y) coordinate $\longrightarrow$ rectangle in polar coordinate!

Integral in xy coordinate



$$dA = dx \cdot dy$$

Integral in polar coordinate



$$A_1 = \frac{1}{2}(r+dr)^2 d\theta \qquad A_2 = \frac{1}{2} r^2 d\theta$$

$$dA \approx A_1 - A_2 = \frac{1}{2}(r^2 + 2r\,dr + dr^2)\,d\theta - \frac{1}{2} r^2 d\theta$$

$$= r\,dr\,d\theta + \frac{1}{2}(dr)^2 d\theta \approx r\,dr\,d\theta$$

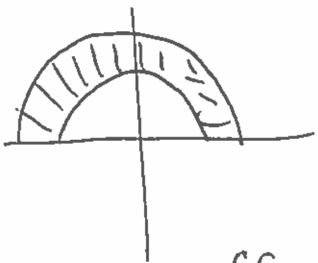
So $\iint_R f(x,y)\,dA \qquad R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$

$$= \int_a^b \left(\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta \right) dr$$

extra r !

Ex 7 $\iint_R (3x+4y)\, dA$

R is the region in the upper half-plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$$R = \{(x,y) : 1 \le x^2+y^2 \le 4,\ y \ge 0\}$$
$$= \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

$$\iint_R (3x+4y)\, dA = \int_1^2 \int_0^\pi (3r\cos\theta + 4r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_1^2 \int_0^\pi (3r^2\cos\theta + 4r^2\sin\theta)\, d\theta\, dr$$

$$= \int_1^2 \left(3r^2\sin\theta - 4r^2\cos\theta\right)\Big|_{\theta=0}^{\theta=\pi} dr = \int_1^2 \left[-4r^2(-1) - (-4r^2)\right] dr$$

$$= \int_1^2 8r^2\, dr = \frac{8}{3}r^3\Big|_{r=1}^{r=2} = \frac{8}{3}(8-1) = \frac{56}{3}$$

Ex 8 ② $\iint_R \tan^{-1}\left(\frac{y}{x}\right) dA$, $R = \{(x,y) : 1 \le x^2+y^2 \le 4,\ 0 \le y \le x\}$

$$= \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{4}\}$$

$$= \int_1^2 \int_0^{\frac{\pi}{4}} \theta\, r\, d\theta\, dr = \left(\int_1^2 r\, dr\right) \cdot \left(\int_0^{\frac{\pi}{4}} \theta\, d\theta\right)$$

$$= \left(\frac{1}{2}r^2\Big|_{r=1}^{r=2}\right) \cdot \left(\frac{1}{2}\theta^2\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}\right) = \frac{1}{2}(2^2-1^2) \cdot \frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2 - 0^2\right)$$

$$= \frac{3}{4} \cdot \left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{64}$$

(3) Find the area of the region enclosed by the curve $r = 4+3\cos\theta$.

$R = \{(r,\theta) : 0 \le r \le 4+3\cos\theta,\ 0 \le \theta \le 2\pi\}$

$$\iint_R 1\, dA = \int_0^{2\pi}\left(\int_0^{4+3\cos\theta} 1\cdot r\, dr\right) d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=4+3\cos\theta} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(4+3\cos\theta)^2\, d\theta = \int_0^{2\pi}\left(8+12\cos\theta+\frac{9}{2}\cos^2\theta\right) d\theta$$

$$= 8\cdot 2\pi + 0 + \frac{9}{2}\int_0^{2\pi}\cos^2\theta\, d\theta = 16\pi + \frac{9}{2}\int_0^{2\pi}\frac{\cos 2\theta + 1}{2}\, d\theta$$

$$= 16\pi + \frac{9}{2}\cdot\pi = \frac{41}{2}\pi$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1}$$

(4) Volume of the solid bounded by $z = 3x^2+3y^2$ and $z = 4-x^2-y^2$

~~$V = \iint_R (3x^2+3y^2) - (4$~~

$3x^2+3y^2 = 4-x^2-y^2$

$4(x^2+y^2) = 4$

$R = \{(x,y) : x^2+y^2 \le 1\}$ $\quad x^2+y^2 = 1$

In R $\quad 4-x^2-y^2 \ge 3x^2+3y^2$

$$V = \iint_R \left(4-x^2-y^2-3x^2-3y^2\right) dA = \iint_R (4-4x^2-y^2)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (4-4r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r-4r^3)\, dr\, d\theta = \int_0^{2\pi}\left(2r^2-r^4\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi$$

⑤ $\int_{-\infty}^{\infty} e^{-x^2} dx$.

We cannot solve it directly

Let $A = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy$.

$$A^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dx dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2} r \, dr \right) d\theta = 2\pi \frac{1}{2} \int_0^{\infty} e^{-r^2} d(r^2)$$

$$= \pi \left(-e^{-r^2} \right) \Big|_{r=0}^{r=\infty} = \pi \cdot 1 = \pi$$

So $A^2 = \pi$ then $A = \sqrt{\pi}$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ and $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ since e^{-x^2} is even

Standard normal distribution $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$ $F(\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds$

$x = \frac{s}{\sqrt{2}}$ $dx = \frac{ds}{\sqrt{2}}$ $F(\infty) = \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{-\infty}^{\infty} e^{-x^2} dx$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

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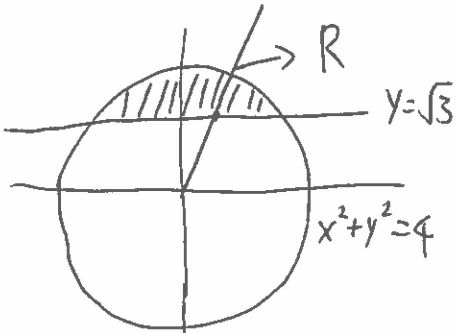

First find boundary points

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$$\begin{cases} x^2+y^2=4 \\ y=\sqrt{3} \end{cases} \Rightarrow x^2+3=4 \Rightarrow x=\pm 1$$

$(x,y)=(-1,\sqrt{3})$, $(x,y)=(1,\sqrt{3})$

In polar coordinate, $x^2+y^2=4 \Rightarrow r=2$ ~~$(-1,\sqrt{3}) \Rightarrow r=2$~~ ~~$\theta=$~~

$y=\sqrt{3} \Rightarrow r\sin\theta=\sqrt{3} \Rightarrow r=\dfrac{\sqrt{3}}{\sin\theta}$

~~$\iint_R \frac{y}{x^2+y^2}\,dA=\int$~~ (Now we find $(\pm 1,\sqrt{3})$ polar coordinate)

$$\begin{cases} r\cos\theta=-1 \\ r\sin\theta=\sqrt{3} \end{cases} \Rightarrow r=2 \Rightarrow \begin{cases} \cos\theta=-\frac{1}{2} \\ \sin\theta=\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \theta=\frac{2\pi}{3} \qquad \begin{cases} r\cos\theta=1 \\ r\sin\theta=\sqrt{3} \end{cases} \Rightarrow \theta=\frac{\pi}{3}$$

$$\iint_R \frac{y}{x^2+y^2}\,dA=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{\frac{\sqrt{3}}{\sin\theta}}^{2} \frac{r\sin\theta}{r^2}\cdot r\,dr\,d\theta=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{\frac{\sqrt{3}}{\sin\theta}}^{2} \sin\theta\,dr\,d\theta$$

$$=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\left(r\sin\theta \Big|_{r=\frac{\sqrt{3}}{\sin\theta}}^{r=2} \right) d\theta=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left(2\sin\theta-\sqrt{3}\right) d\theta$$

$$=\left(-2\cos\theta-\sqrt{3}\,\theta\right)\Big|_{\theta=\frac{\pi}{3}}^{\theta=\frac{2\pi}{3}}=\left(-2\cdot\left(-\tfrac{1}{2}\right)-\sqrt{3}\cdot\tfrac{2\pi}{3}\right)-\left(-2\left(\tfrac{1}{2}\right)-\sqrt{3}\cdot\tfrac{\pi}{3}\right)$$

$$=2-\sqrt{3}\cdot\frac{\pi}{3}=2-\frac{\pi}{\sqrt{3}}=2-\frac{\sqrt{3}\pi}{3}=\frac{6-\sqrt{3}\pi}{3}$$