Lagrange Multiplier Method

Find absolute minimum/maximum of $f(x,y)$ given that $g(x,y) = C$

① Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g = C \end{cases} \Rightarrow (\lambda, x, y)$

② The maximum/minimum among solutions.

Question: ① Does abs. min/max always exist?

② Can we always find abs. min/max using this method?

<u>Theorem 4</u> If f has a local min/max (x_0, y_0) is $S = \{(x,y) : g(x,y) = c\}$, then $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.

If S is a closed and bounded subset of $\mathbb{R}^2$, then abs. min/max always exists.

<u>Example 1</u> $f(x,y) = 3x + 4y$ given $g(x,y) = 5x + 6y = 0$.

$\nabla f = \langle 3, 4 \rangle$ $\nabla g = \langle 5, 6 \rangle$ So $\langle 3, 4 \rangle = \lambda \langle 5, 6 \rangle$

has no solution $\lambda \in \mathbb{R}$! $\Rightarrow$ no local min/min of $f(x,y)$

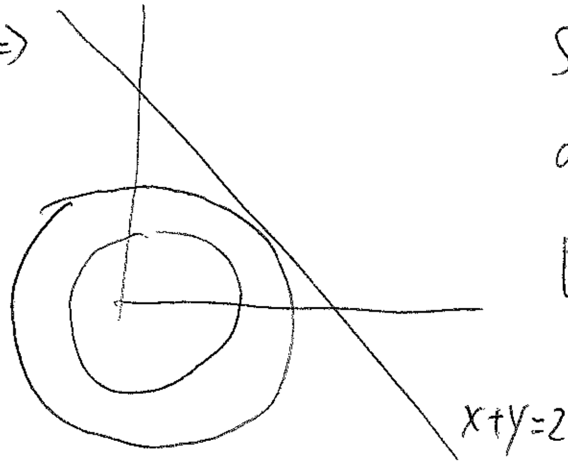
on $S = \{(x,y) \in \mathbb{R}^2 : 5x + 6y = 0\}$,

<u>Example 2</u> $f(x,y) = x^2 + y^2$ given $g(x,y) = x + y = 2$

$\nabla f = \langle 2x, 2y \rangle$ $\nabla g = \langle 1, 1 \rangle$

$\begin{cases} 2x = \lambda \cdot 1 \\ 2y = \lambda \cdot 1 \end{cases} \Rightarrow x = \frac{\lambda}{2},\ y = \frac{\lambda}{2} \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 2 \Rightarrow \lambda = 2$

$x=1,\ y=1 \Rightarrow$



So $(\lambda, x, y) = (2,1,1)$ is a "critical point". $\boxed{\text{page } 61}$

Is it a min or max ?

Take $x=n,\ y=2-n$ Then $n^2+(2-n)^2 \to \infty$ as $n \to \infty$

So max $f(x,y)$ does not exist, Then $(1,1)$ must be a min $f(x,y)$!

Here an abs min exists since $x^2+y^2 \geq 0$ for any (x,y)

Example 3 Find the extremal values of

$$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - \frac{y^2}{2} + 1$$

given $x^2+y^2=4$

$S = \{(x,y) : x^2+y^2=4\}$ is closed, bounded so $\min\limits_{(x,y)\in S} f(x,y)$, $\max\limits_{(x,y)\in S} f(x,y)$

must exist!

$$\begin{cases} f_x = x^2-x & g_x = 2x \\ f_y = y^2-y & g_y = 2y \end{cases} \Rightarrow \begin{cases} x^2-x = \lambda\cdot 2x \Rightarrow x^2-(2\lambda+1)x=0 \Rightarrow x=0 \text{ or } x=2\lambda+1 \\ y^2-y=\lambda\cdot 2y \Rightarrow y^2-(2\lambda+1)y=0 \Rightarrow y=0 \text{ or } y=2\lambda+1 \\ x^2+y^2=4 \end{cases}$$

Case 1 $x=0$ Case 1a $y=0 \Rightarrow (0,0)$ but $x^2+y^2=4 \Rightarrow$ not valid

Case 1b $y=2\lambda+1 \Rightarrow (0, 2\lambda+1)$ $0^2+(2\lambda+1)^2=4 \Rightarrow 2\lambda+1=\pm 2 \Rightarrow (0, \pm 2)$

Case 2 $x=2\lambda+1$ Case 2a $y=0 \Rightarrow (2\lambda+1, 0)$ $(2\lambda+1)^2+0^2=4 \Rightarrow 2\lambda+1=\pm 2 \Rightarrow (\pm 2, 0)$

Case 2b $y=2\lambda+1 \Rightarrow (2\lambda+1, 2\lambda+1)$ $(2\lambda+1)^2+(2\lambda+1)^2=4 \Rightarrow 2\lambda+1=\pm\sqrt{2}$

$\Rightarrow (\sqrt{2}, \sqrt{2}),\ (-\sqrt{2}, -\sqrt{2})$

(x,y)	$(0,2)$	$(2,0)$	$(0,-2)$	$(-2,0)$	$(\sqrt{2},\sqrt{2})$	$(-\sqrt{2},-\sqrt{2})$
$f(x,y)$	$\frac{5}{3}$	$\frac{5}{3}$	$-\frac{11}{3}$	$-\frac{11}{3}$	$\frac{4\sqrt{2}}{3}-1$	$-\frac{4\sqrt{2}}{3}-1$

≈ 1.66 $\downarrow$ abs. max.

≈ -3.66 $\downarrow$ abs. min

≈ 0.8856 $\downarrow$ local min

≈ -2.8856 $\downarrow$ local max