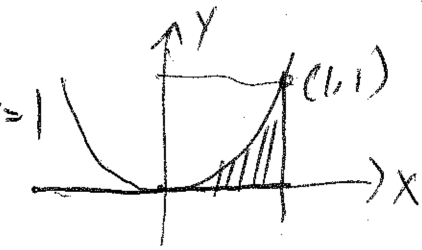
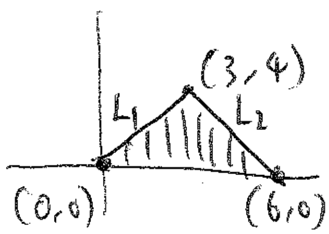
Math 212 Lecture 14

Ex3 ① D bounded by $y=0$, $y=x^2$ and $x=1$



$$D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ 0\le y\le x^2\}$$
$$=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ \sqrt{y}\le x\le 1\}$$

② D triangle with vertices $(0,0)$, $(3,4)$ and $(6,0)$



L_1: $\frac{y}{x}=\frac{4}{3} \Rightarrow y=\frac{4}{3}x$

L_2: slope $\frac{0-4}{6-3}=-\frac{4}{3} \Rightarrow y-0=-\frac{4}{3}(x-6)$

$\Rightarrow y=-\frac{4}{3}x+8$

$$D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 3,\ 0\le y\le \tfrac{4}{3}x\}\cup\{(x,y)\in\mathbb{R}^2 : 3\le x\le 6,\ 0\le y\le -\tfrac{4}{3}x+8\}$$

or $x=\frac{3}{4}y$, $x=6-\frac{3}{4}y$

$$D=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 4,\ \tfrac{3}{4}y\le x\le 6-\tfrac{3}{4}y\}.$$

③ D bounded by a circle with $R=3$, center $(2,5)$

$$D=\{(x,y)\in\mathbb{R}^2 : (x-2)^2+(y-5)^2\le 3^2\}.$$

Ex 4 ① $\iint_D xy\,dA$ D bounded by $y=0$, $y=x^2$, $x=1$

From Ex3 ① $D=\{0\le x\le 1,\ 0\le y\le x^2\}$

$$\iint_D xy\,dA = \int_0^1\left(\int_0^{x^2} xy\,dy\right)dx = \int_0^1 x\cdot\left(\tfrac{1}{2}y^2\right)\Big|_0^{x^2}dx = \int_0^1 x\cdot\tfrac{1}{2}x^4\,dx$$

$$=\frac{1}{2}\cdot\frac{1}{6}x^6\Big|_0^1 = \frac{1}{12}$$

or $$\iint_D xy\,dA = \int_0^1\left(\int_{\sqrt{y}}^1 xy\,dx\right)dy = \int_0^1 y\cdot\left(\tfrac{1}{2}x^2\right)\Big|_{\sqrt{y}}^1 dy = \int_0^1 y\left(\tfrac{1}{2}-\tfrac{1}{2}y\right)dy$$

$$=\left(\frac{1}{4}y^2-\frac{1}{6}y^3\right)\Big|_0^1 = \frac{1}{4}-\frac{1}{6} = \frac{1}{12}$$

② $\iint_D x^2\,dA$, D bounded by $(0,0)$, $(3,4)$ and $(6,0)$

From Ex3 ② $D=\{0\le y\le 4,\ \frac{3}{4}y\le x\le 6-\frac{3}{4}y\}$

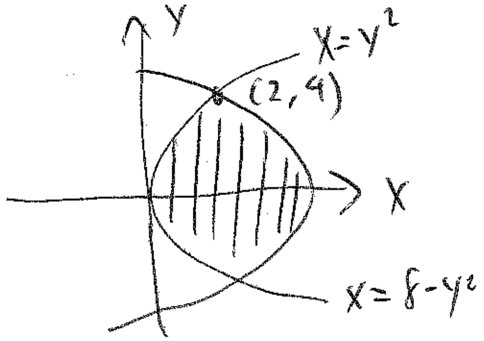
$$\iint_D x^2\,dA = \int_0^4\left(\int_{\frac{3}{4}y}^{6-\frac{3}{4}y} x^2\,dx\right)dy = \int_0^4 \frac{1}{3}x^3\Big|_{\frac{3}{4}y}^{6-\frac{3}{4}y}dy$$

$$=\int_0^4 \frac{1}{3}\left[\left(6-\tfrac{3}{4}y\right)^3-\left(\tfrac{3}{4}y\right)^3\right]dy = \frac{1}{3}\int_0^4\left[6^3-3\cdot 6^2\left(\tfrac{3}{4}y\right)+3\cdot 6\left(\tfrac{3}{4}y\right)^2\right]dy$$

$$=\frac{1}{3}\int_0^4\left[216-81y+\frac{27}{8}y^2\right]dy = \frac{1}{3}\left(216y-\frac{81}{2}y^2+\frac{9}{8}y^3\right)\Big|_0^4$$

$$=\frac{1}{3}\left(864-648+\frac{9}{8}\cdot 64\right) = \frac{1}{3}(864-648+72) = \frac{288}{3} = 96$$

③ $\iint_D y\, dA$ D bounded by $x=y^2$ and $x=8-y^2$ first quadrant



$$\begin{cases} x=y^2 \\ x=8-y^2 \end{cases} \Rightarrow y^2=8-y^2 \Rightarrow 2y^2=8 \Rightarrow y=\pm 2$$

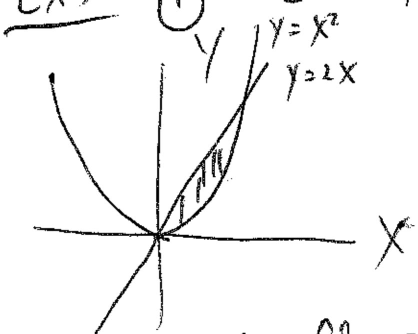
$y=2$ (first quadrant) $x=2^2=4$

$$D=\{0\le y\le 2,\ y^2\le x\le 8-y^2\}$$

$$\int_0^2\left(\int_{y^2}^{8-y^2} y\,dx\right)dy=\int_0^2 y\left(x\Big|_{y^2}^{8-y^2}\right)dy=\int_0^2 y\cdot(8-y^2-y^2)\,dy$$

$$=\int_0^2 (8y-2y^3)\,dy=\left(4y^2-\frac{1}{2}y^4\right)\Big|_0^2=16-8=8$$

Ex5 ① $z=x^2+y^2$ above D = bounded by $y=2x$ and $y=x^2$



$$\begin{cases} y=2x \\ y=x^2 \end{cases} \Rightarrow 2x=x^2 \Rightarrow x=0,\ x=2$$

$$D=\{0\le x\le 2,\ x^2\le y\le 2x\}$$

$$V=\iint_D [z_{top}-z_{bottom}]\,dA=\iint_D (x^2+y^2-0)\,dA$$

$$=\int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx=\int_0^2\left(x^2y+\frac{1}{3}y^3\right)\Big|_{x^2}^{2x}dx=\int_0^2\left(2x^3+\frac{8x^3}{3}-x^4-\frac{1}{3}x^6\right)dx$$

$$=\int_0^2\left(\frac{14}{3}x^3-x^4-\frac{1}{3}x^6\right)dx=\left(\frac{14}{12}x^4-\frac{1}{5}x^5-\frac{1}{21}x^7\right)\Big|_0^2$$

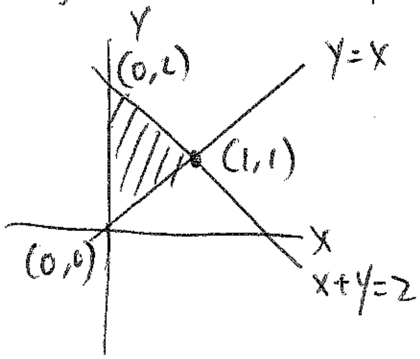
$$=\frac{7}{6}\cdot 16-\frac{1}{5}\cdot 32-\frac{1}{21}128=\frac{56}{3}-\frac{32}{5}-\frac{128}{21}=\frac{392-128}{21}-\frac{32}{5}$$

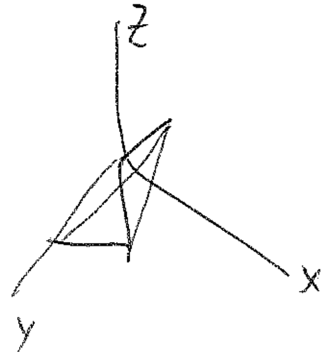
$$=\frac{264}{21}-\frac{32}{5}=\cancel{\frac{1320-672}{105}=\frac{648}{105}}$$

$$=\frac{88}{7}-\frac{32}{5}=\frac{440}{35}-\frac{224}{35}=\frac{216}{35}$$

② Volume of solid bounded by the planes $z=x$, $y=x$, $x+y=2$ and $z=0$ page 58

In general, 4 planes form a tetrahedron





$x+y=2$ and $y=x$ are both "vertical" (perpendicular to xy-plane or // to z-axis)

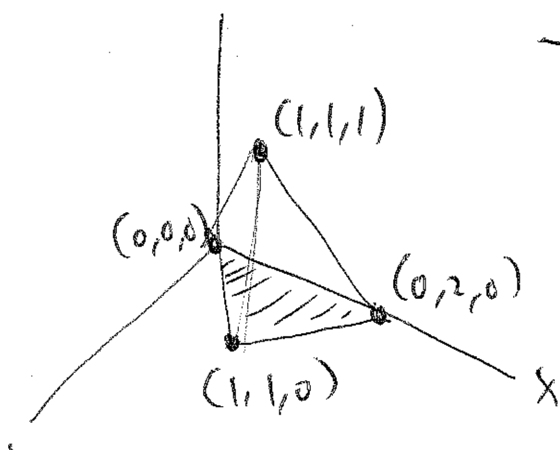
$z=x$ and $z=0$ intersect at $x=0$

$$D = \{(x,y) : 0 \le y \le 2,\ y \le x \le 2-y\}$$

$$V = \iint_D x\, dA = \int_0^2 \left(\int_y^{2-y} x\, dx \right) dy = \int_0^2 \frac{1}{2}x^2 \Big|_y^{2-y} dy$$

$$= \int_0^2 \frac{1}{2}\left[(2-y)^2 - y^2 \right] dy = \frac{1}{2}\int_0^2 (4-4y+2y^2)\, dy = \frac{1}{2}\left(4y-2y^2+\frac{2}{3}y^3\right)\Big|_0^2$$

$$= \frac{1}{2}\left(8-8+\frac{2}{3}\cdot 8\right) = \frac{8}{3}.$$



Vertices where 3 planes intersect

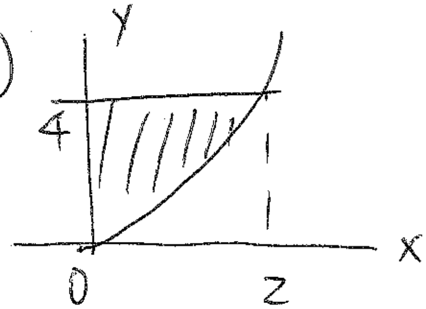
$z=x,\ y=x,\ x+y=2 \Rightarrow (1,1,1)$

$z=x,\ y=x,\ z=0 \Rightarrow (0,0,0)$

$z=x,\ x+y=2,\ z=0 \Rightarrow (0,2,0)$

$y=x,\ x+y=2,\ z=0 \Rightarrow (1,1,0)$

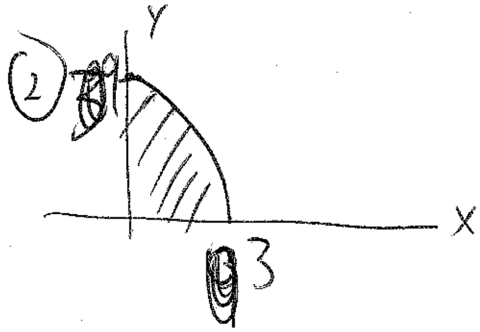
Ex6 ①



$$D = \{0 \le x \le 2,\ x^2 \le y \le 4\}$$
$$= \{0 \le y \le 4,\ 0 \le x \le \sqrt{y}\}$$

$$\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$

②

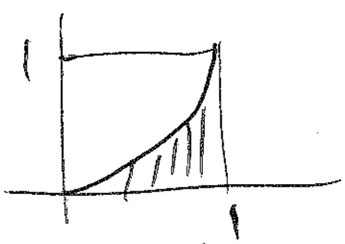


$$D = \{0 \le y \le 9,\quad 0 \le x \le \sqrt{9-y}\}$$
$$= \{0 \le x \le 3,\quad 0 \le y \le 9-x^2\}$$

$$x = \sqrt{9-y} \Rightarrow x^2 = 9-y \Rightarrow y = 9-x^2$$

$$\int_0^9 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy = \int_0^3 \int_0^{9-x^2} f(x,y)\,dy\,dx$$

Ex7 $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$ $\quad \int \sqrt{x^3+1}\,dx$ is not solvable



$$D = \{0 \le y \le 1,\ \sqrt{y} \le x \le 1\} \qquad \sqrt{y} = x \quad y = x^2$$
$$= \{0 \le x \le 1,\ 0 \le y \le x^2\}$$

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^1 \sqrt{x^3+1}\left(y\Big|_0^{x^2}\right)dx$$

$$= \int_0^1 \sqrt{x^3+1}\,x^2\,dx \qquad u = x^3+1 \quad du = 3x^2dx \qquad \begin{matrix} x=0 \Rightarrow u=1 \\ x=1 \Rightarrow u=2 \end{matrix}$$

$$= \int_1^2 \sqrt{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^2 = \frac{2}{9}\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{9}\left(\sqrt{8}-1\right)$$