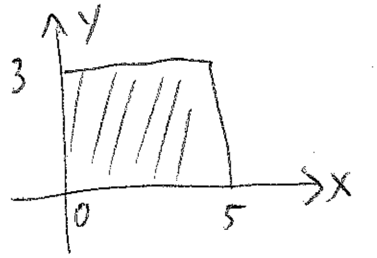
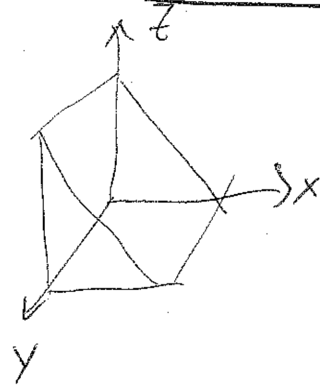
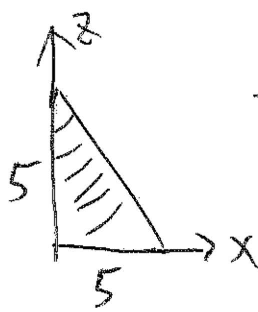
Math 212 Lecture 13

Ex1 $\iint_R (5-x)\,dA$, $R=[0,5]\times[0,3]$







Triangle area

$\frac{1}{2}\cdot 5\cdot 5=\frac{25}{2}$

"height" in y-direction $=3$

So Volume $=\frac{25}{2}\times 3=\frac{75}{2}$

Solving using iterated integral

$$\iint_R (5-x)\,dA=\int_0^3\left(\int_0^5 (5-x)\,dx\right)dy=\int_0^3\left(5x-\tfrac{1}{2}x^2\right)\Big|_{x=0}^{x=5}dy=\int_0^3\left(25-\tfrac{25}{2}\right)dy$$

$$=\frac{25}{2}\cdot y\Big|_{y=0}^{y=3}\Rightarrow\frac{25}{2}\cdot 3=\frac{75}{2}$$

Ex2 ① $\iint_R \frac{xy^2}{x^2+1}\,dA$, $R=[0,1]\times[-3,3]$

$$\int_0^1\int_{-3}^3\frac{xy^2}{x^2+1}\,dy\,dx=\int_0^1\frac{x}{x^2+1}\left(\int_{-3}^3 y^2\,dy\right)dx=\int_0^1\frac{x}{x^2+1}\cdot\left(\frac{1}{3}y^3\Big|_{y=-3}^{y=3}\right)dx$$

$$=\int_0^1\frac{x}{x^2+1}\cdot 18\,dx=18\int_0^1\frac{x}{x^2+1}\,dx$$

$u=x^2+1$
$du=2x\,dx$
$x=0\Rightarrow u=1$
$x=1\Rightarrow u=2$

$$=9\int_1^2\frac{du}{u}=9\ln u\Big|_1^2=9\ln 2$$

If $f(x,y)=g(x)h(y)$ then

$$\iint_R f(x,y)\,dx\,dy=\left(\int_0^1 g(x)\,dx\right)\cdot\left(\int_{-3}^3 h(y)\,dy\right)$$

(separable) Not true if R is not rectangle

② $\iint_R \frac{x}{x^2+y^2}\,dA$, $R=[1,2]\times[0,1]$ x or y first?

y first $\int_1^2 \left(\int_0^1 \frac{x}{x^2+y^2}\,dy\right)dx = \int_1^2 x\left(\int_0^1 \frac{1}{x^2+y^2}\,dy\right)dx = \int_1^2 x\cdot\left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)\Big|_{y=0}^{y=1} dx$

$= \int_1^2 \left(\tan^{-1}\frac{1}{x} - \tan^{-1}0\right)dx = \int_1^2 \tan^{-1}\frac{1}{x}\,dx = x\tan^{-1}\frac{1}{x}\Big|_1^2 - \int_1^2 x\cdot\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)dx$

$= \left(2\tan^{-1}\frac{1}{2} - \tan^{-1}1\right) + \int_1^2 \frac{x}{1+x^2}\,dx = \left(2\tan^{-1}\frac{1}{2} - \frac{\pi}{4}\right) + \frac{1}{2}\int_2^5 \frac{du}{u} = 2\tan^{-1}\frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}(\ln 5 - \ln 2)$

$u = x^2+1 \quad du = 2x\,dx$

$x=1 \Rightarrow u=2,\ x=2 \Rightarrow u=5$

x-first $\int_0^1 \left(\int_1^2 \frac{x}{x^2+y^2}\,dx\right)dy$ $\quad u = x^2+y^2$, $du = 2x\,dx$, $x=1\ u=1+y^2$, $x=2\ u=4+y^2$ $\quad = \int_0^1 \left(\int_{1+y^2}^{4+y^2} \frac{1}{2}\frac{du}{u}\right)dy$

$= \frac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right]dy$ how to solve $\int \ln(a^2+y^2)\,dy$

$\int \ln(a^2+y^2)\,dy = y\ln(a^2+y^2) - \int y\cdot\frac{2y}{a^2+y^2}\,dy = y\ln(a^2+y^2) - 2\int \frac{(a^2+y^2)-a^2}{a^2+y^2}\,dy$

$= y\ln(a^2+y^2) - 2y + 2a\tan^{-1}\left(\frac{y}{a}\right)$

So $\frac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right]dy$

$= \left(\frac{1}{2}y\ln(4+y^2) - y + 2\tan^{-1}\left(\frac{y}{2}\right)\right)\Big|_0^1 - \left(\frac{1}{2}y\ln(1+y^2) - y + \tan^{-1}(y)\right)\Big|_0^1$

$= \frac{1}{2}(\ln 5 - \ln 2) + 2\tan^{-1} - \frac{\pi}{4}$

(3) Volume of solid above $R=[-1,1]\times[-2,2]$

$$\frac{x^2}{4}+\frac{y^2}{9}+z=1 \Rightarrow z=1-\frac{x^2}{4}-\frac{y^2}{9}$$

$$V=\iint_R \left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dA=\int_{-1}^{1}\int_{-2}^{2}\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dy\,dx$$

$$=\int_{-1}^{1}\left[\left(1-\frac{x^2}{4}\right)y-\frac{1}{27}y^3\right]\Big|_{y=-2}^{y=2}dx=\int_{-1}^{1}\left[4\left(1-\frac{x^2}{4}\right)-\frac{16}{27}\right]dx=\left(4x-\frac{1}{3}x^3-\frac{16}{27}x\right)\Big|_{-1}^{1}$$

$$=8-\frac{1}{3}\cdot 2-\frac{16}{27}\cdot 2=8-\frac{2}{3}-\frac{32}{27}=\frac{216-18-32}{27}=\frac{166}{27}$$

(4) Average value $=\frac{1}{A(R)}\iint_R f(x,y)\,dA$ $\quad A(R)=$ area $\quad R=[0,4]\times[0,1]$

$$\iint_R e^y\sqrt{x+e^y}\,dA=\int_0^4\left(\int_0^1 e^y\sqrt{x+e^y}\,dy\right)dx \qquad u=e^y \quad du=e^y dy$$

$y=0 \Rightarrow u=1 \quad y=1 \Rightarrow u=e$

$$=\int_0^4\left(\int_1^e\sqrt{x+u}\,du\right)dx=\int_0^4\frac{2}{3}(x+u)^{\frac{3}{2}}\Big|_{u=1}^{u=e}dx=\int_0^4\frac{2}{3}\left[(x+e)^{\frac{3}{2}}-(x+1)^{\frac{3}{2}}\right]dx$$

$$=\frac{2}{3}\cdot\left(\frac{2}{5}(x+e)^{\frac{5}{2}}-\frac{2}{5}(x+1)^{\frac{5}{2}}\right)\Big|_0^4=\frac{4}{15}\left[\left((4+e)^{\frac{5}{2}}-5^{\frac{5}{2}}\right)-\left(e^{\frac{5}{2}}-1^{\frac{5}{2}}\right)\right]$$

$$=\frac{4}{15}\left[(4+e)^{\frac{5}{2}}+1-5^{\frac{5}{2}}-e^{\frac{5}{2}}\right]$$